TESTABLE IMPLICATIONS OF PARAMETRIC ASSUMPTIONS IN ORDERED RANDOM UTILITY MODELS

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Abstract. We study the implications of commonly-used parametric assumptions on the empirical content of ordered random utility models. After characterizing these models in a continuous setting and in the absence of fundamental parametric restrictions, we show that assumptions on the type distribution alone are immaterial, while assumptions on the map linking types to utilities are relevant only insofar as this restricts the class of utilities at stake. Importantly, the joint presence of such parametric assumptions, as per common practice, restricts further the empirical content of the model. We then provide a characterization of commonly-used parametric ordered-logit models. We conclude the paper applying our results to economic settings of relevance.

Keywords: Ordered random utility model; empirical content; parametric restrictions; ordered logit; cumulative logit.
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1. Introduction

In many settings, alternatives have a natural order and choices are driven by an ordered latent variable. This simple structure is a fundamental instrument for empirical research, spanning diverse economic areas such as health, finance, labor, welfare, management, insurance, political economy, networks and gender.[1] The usual economic

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[1]See, respectively, Barsky et al. (1997), Kaplan and Zingales (1997), Blau and Hagy (1998), Campante and Yanagizawa-Drott (2015), Cummings, (2004), Cohen and Einav (2007), Besley and Persson (2011), Bailey et al. (2018) and Carlana (2019).

modeling in these cases is by way of Ordered Random Utility Models (ORUMs).[2] These models have two main components. To introduce them, let the latent variable be on the reals and refer to each possible value of this variable as a type. The first component of an ORUM is a type-utility map where higher types are associated with utilities that produce higher choices. The second component is a type distribution that describes the prevalence of each type.

The empirical application of ORUMs often relies on parametric assumptions. To illustrate, regarding the first component, applications to political or insurance choices are often built upon specific parametric models such as the ones in which type t has Euclidean preference centered at t or CRRA coefficient equal to t, respectively.[3] Regarding the second component, the most common parametric assumptions involve the use of logistic or Gaussian distributions over the set of types, that lead to the so-called ordered-logit or ordered-probit models, respectively. Therefore, it is critical to carefully examine when and how parametric assumptions bear down on the space of datasets that can be explained by ORUMs. This is the purpose of this paper.

We work in the context of cumulative choice data from an arbitrary collection of continuous decision problems. We start with the fully non-parametric case in which neither the type-utility map nor the type distribution is restricted. For this case, we show that the property characterizing data can be conveniently derived from the standard, deterministic, notion of rationalizability. In a nutshell, suppose that, for each probability value $p \in (0,1)$, we construct the hypothetical deterministic choice function c^p selecting, in each decision problem, the alternative that attains first cumulative choice above p.[4] Theorem 1 shows that data is explained by an ORUM if and only if every c^p is rationalizable. We note, in passing, that this constitutes the first characterization result of ORUMs in continuous settings.

Fundamentally for our purposes, the proof of Theorem 1 is instrumental to study the effects of imposing parametric restrictions. Building upon it we show in Proposition 1 that, if the type-utility map remains unrestricted, parametric assumptions over

[2]For early modeling see Small (1987); see also Train (2009) and Greene and Hensher (2010). For micro-foundations, see Apesteguia Ballester and Lu (2017), Filiz-Ozbay and Masatlioglu (2023), Petri (2023) and Apesteguia and Ballester (2023).

[3]This type of parametric restrictions is often referred as semi non-parametric (see, e.g., Barseghyan, Molinari and Thirkettle (2021)).

[4]This corresponds to the observed p-quantile alternative in each problem.

the type distribution have no empirical content beyond the one described by Theorem 1. To emphasize, in this case the analyst can be certain that fixing ex-ante her preferred statistical distribution has no consequences; if the data can be rationalized by an ORUM, then it can be rationalized by another ORUM using the assumed type distribution, and the appropriate transformation of the type-utility map. Second, fixing ex-ante a type-utility map that operates over a sub-class of utilities has obvious implications, because this parametric assumption constrains the model to use only the specific sub-class of utilities. However, modulo these utilities, the assumption comes at no further cost. Again building on the proof of Theorem 1, we show in Proposition 2 that, if the type distribution remains unrestricted, fixing the type-utility map has no further empirical content beyond the rationalizability property described in Theorem 1, restricted to the sub-class of utilities used by the given map. Third, we show that fixing both the type-utility map and the type distribution comes at a cost. We argue by way of example that parametric assumptions constrain the set of observed datasets that can be rationalized, and they do so beyond the obvious restriction imposed by the sub-class of utilities at stake.

To illustrate these results, consider the case in which an analyst is trying to explain a collection of observed distributions of choices on a number of linear budget sets involving lotteries.[5] A natural starting point of this analysis is to consider the class of expected utilities. Theorem 1 describes the empirical content of ORUMs based upon an ordered family of expected utilities: each p-quantile must be rationalizable by an expected utility. Let us now consider the empirical implications of different parametric assumptions. First, we may consider specific type distributions, such as the logistic. Proposition 1 establishes that, when considered in isolation, this parametric assumption incurs no additional costs. Second, we may consider instead fixing a specific type-utility map operating over a restricted sub-class of expected utilities, such as the one in which type t is assigned the expected utility with CRRA coefficient equal to t. Naturally, this may have empirical consequences, since there may be behavioral patterns that can be accommodated by other expected utilities than CRRA. However, these are the only empirical consequences. Proposition 2 establishes that the specific assignment of types to CRRA expected utilities is without loss of generality. Finally, if the analyst has parametrically shaped the model both by assuming the CRRA type-utility map and the logistic type distribution, the empirical content is restricted. To see why, notice

[5]See, e.g., Choi et al. (2014), for an experimental design using such a setting.

that the logistic distribution must now operate over the CRRA coefficient, imposing a good deal of structure on admissible patterns of lottery choices.

It is then evident that understanding the empirical content of commonly-used parametric models requires of tailored analysis and results beyond the content of Theorem 1. Given the preponderance of parametric ordered-logit models in the empirical literature, in this paper we study them in detail.[6] Suppose then that we fix a given, yet generic, type-utility map, providing specific meaning to the latent variable, and restrict attention to the logistic distribution. What are the exact properties of data generated by such a model? In Theorem 2, we show that two simple properties, that we call corner extremeness and cumulative logit additivity, characterize parametric ordered-logit models. Corner extremeness imposes that a corner alternative receives non-null choice probability if and only if this alternative is the maximizer for at least one of the utilities at stake. Cumulative logit additivity uses the well-known cumulative logit notion, i.e., the log-ratio of masses below and above a given alternative, and states that equal sums of types must lead to equal sums of cumulative logits. Theorem 2 establishes the first characterization result in the literature of parametric ordered-logit models, giving foundations to a popular tool in the empirical literature.

All our results are built to help with portability. Theorems 1 and 2 can be particularized to a variety of economic settings, thus providing foundations for specific ORUMs and parametric ordered-logit models within these settings. We illustrate this idea by developing three different applications involving political economy, consumption/altruism, and risk choices. For each of them, Theorem 1 is particularized by using the standard classes of utilities in each of these three settings, such as single-peaked, strictly monotone and convex, and expected utilities, respectively. Meanwhile, Theorem 2 is particularized by assuming that the parametric ordered-logit model operates over the peak of Euclidean utilities, the coefficient of Cobb-Douglas utilities and the coefficient of CRRA expected utilities, respectively, showing the specific empirical content of corner extremeness and cumulative logit additivity in these cases.

We close this introduction with a brief comment on the links of this paper with other strands of literature. First, the paper contributes to the study of choice-based foundations of various stochastic choice models. The classic works are those of Luce (1959)

[6]The majority of works cited in footnote 1 adopt the ordered-logit format, which gives a good sense of the popularity of the model.

and Block and Marshak (1960).[7] Second, it is also worth mentioning the connection to recent papers seeking to bridge the gap between the choice-based foundations and the econometric implementation of stochastic models, such as Kitamura and Stoye, (2018), Dardanoni, Manzini, Mariotti, and Tyson (2020), Aguiar and Kashaev (2021), Barseghyan, Molinari, and Thirkettle (2021), Apesteguia and Ballester (2021), and Kovach and Tserenjigmid (2022).

2. Non-parametric ordered RUMs

We focus on a setting involving linear decision problems for three key reasons. First, while ubiquitous in applications, the theoretical foundations of this general continuous model remain underexplored. Second, the continuous structure facilitates the theoretical treatment. Third, as demonstrated in Section 5, our results apply directly to a variety of classic economic applications modeled through linear budget sets. Moreover, our analysis readily extends to other settings involving non-linear or discrete menus.

Let $X \subseteq \mathbb{R}^K$ be a convex space of alternatives. There is a collection of decision problems (or menus) $\{A_j\}_{j=1}^J$, that are ordered line segments of X. Each menu is composed by two corner alternatives and their convex combinations, i.e., $A_j = \{(1-a)\underline{x}_j + a\overline{x}_j : \underline{x}_j, \overline{x}_j \in X \text{ and } a \in [0,1]\}$, from which one alternative must be chosen. Thus, any alternative $x_j \in A_j$ is determined by its relative position in the line segment, i.e. by the unique value $a(x_j) \in [0,1]$ such that $x_j = (1-a(x_j))\underline{x}_j + a(x_j)\overline{x}_j$.

Choice data F corresponds to the observed distribution of choices in each menu. That is, $F = \{F_j\}_{j=1}^J$ is a collection of cumulative distribution functions (CDFs), with the value $F_j(x_j) \in [0,1]$ representing the probability of observing the choice of any alternative $y_j \in A_j$ such that $a(y_j) \leq a(x_j)$. As it is usual in most applications, we assume that each of the CDFs is continuous and strictly increasing in $A_j \setminus \{\underline{x}_j, \overline{x}_j\}$.

We now discuss the rationalization of data. Let $\mathcal{U}$ be a generic collection of utility functions on X, all of them having a unique maximizer in each menu. Denote by x_j^U

[7] Other recent contributions are Gul and Pesendorfer (2006), Manzini and Mariotti (2014), Caplin and Dean (2015), Fudenberg, Iijima and Strzalecki (2015), Matejka and McKay (2015), Brady and Rehbeck (2016), Cerreia-Vioglio, Dillenberger, Ortoleva, and Riella (2019), Frick, Iijima, and Strzalecki (2019), Natenzon (2019), Cattaneo, Ma, Masatlioglu and Suleymanov (2020), Alós-Ferrer, Fehr and Netzer (2021), or He and Natenzon (2023).

the unique alternative that is maximal according to $U \in \mathcal{U}$ in menu A_j. There are two components in ORUMs:

(1) Ordered-choice: consider an ordered set of latent types (or simply types, for short) represented by $\mathbb{R}$, and assign utilities to types in a way that higher types select higher alternatives. Formally, there is a *type-utility map* $\gamma : \mathbb{R} \to \mathcal{U}$ such that, for every pair of types $t \leq t'$, $a(x_j^{\gamma(t)}) \leq a(x_j^{\gamma(t')})$ holds for every menu A_j. To emphasize, the maximal alternative in A_j according to the utility associated to type t, namely $\gamma(t)$, is lower than the corresponding one of type t' in the same menu. We also assume that, in every menu, every non-corner alternative is maximal for some utility $\gamma(t)$.

(2) Stochasticity: each type is assigned a probability describing its prevalence. Formally, there is a *type distribution* g over the reals, that we assume has full density.

We then say that F is rationalizable by an ORUM (from now on, ORUM-rationalizable), with type-utility map γ and type distribution g, whenever for every menu A_j and alternative $x_j \in A_j$, $F_j(x_j) = \int_{t: a(x_j^{\gamma(t)}) \leq a(x_j)} g(t)dt$. That is, for every menu and alternative, the observed cumulative choice mass and the mass of types that maximize below such alternative coincide.

Deterministic rationalizability can be seen as the limit case of ORUM-rationalizability. Consider the case in which a type-utility map is fixed and take a sequence of type distributions converging to a degenerate one. In the limit, all the mass concentrates on a single utility and the standard notion of rationalizability is recovered. Formally, we say that a deterministic choice function $c : \{1, \ldots, J\} \to X$, with $c(A_j) \in A_j$, is rationalizable whenever there exists a utility function $U \in \mathcal{U}$ (or, equivalently, a unique relevant type with mass one) such that, for every menu A_j, $c(A_j) = x_j^U$, i.e., the observed choice and the alternative providing maximal utility coincide.

We now provide a characterization of ORUM-rationalizability that builds upon the deterministic notion of rationalizability. Consider any value $p \in (0, 1)$. For each menu A_j, denote by c_j^p the first alternative in the menu attaining cumulative choice mass p, i.e.,

$$c_j^p = \{x_j \in A_j : F_j(x_j) \geq p \text{ and } a(y_j) < a(x_j) \Rightarrow F_j(y_j) < p\}.$$

Given our basic assumptions on F_j, alternative c_j^p is well-defined. Notice that c_j^p simply describes the p-quantile obtained from the choice data in menu A_j. Denote by $c^p = \{c_j^p\}_{j=1}^J$ the (quantile) choice function that is defined in such a way.

Theorem 1. *F is ORUM-rationalizable if and only if c^p is rationalizable for every p.*

ORUM-rationalizability is equivalent to every quantile choice function being rationalizable in the standard, deterministic, sense. The intuition for the sufficiency part is as follows. For every $p \in (0,1)$, the rationalizability of c^p allows to identify a utility function U^p that rationalizes c^p. Choices must be increasing across different levels of p due to the quantile definition of c^p. Then, data can be explained by uniformly randomizing over these quantiles. Hence, in order to construct the type-utility map γ and the type distribution g, we need to project the interval $(0,1)$ into the real line by means of a bijection, and consider the corresponding induced distribution over the reals. We do so by means of the standard logistic transformation.

It is important to stress that the notion of ORUM-rationalizability is unrestricted in terms of which type-utility maps and type distributions can be used. Continuing with the discussion in the previous paragraph, one could be using other projections from $(0,1)$ into the real line together with the appropriate distributions to rationalize the same choice dataset.

3. Implications of parametric restrictions in ordered RUMs

In the previous section, both the type-utility map and the type distribution were unrestricted, a feature that we refer as non-parametric. Empirical work usually comes with a variety of parametric restrictions, either by adopting specific type-utility maps, or specific type distributions, or both. This may, e.g., facilitate the computational estimation exercise or the interpretation of the results. In this section, we discuss the potential implications of these parametric restrictions.

We first consider the case in which the type distribution is assumed to belong to a specific family, as in the ordered-logit or ordered-probit models. In our next result, we build upon the discussion of Theorem 1 and formally show that this is without loss of generality. Indeed, a stronger result can be proved; if data is ORUM-rationalizable, it is always possible to find an alternative rationalization in which any given type distribution g^* is used.

Proposition 1. *Let g^* be any given type distribution. If F is ORUM-rationalizable, then F is ORUM-rationalizable with type distribution g^*.*

That is, parametric restrictions on the type distribution have no empirical content per se. The reason for this is that an appropriate relabelling of the utilities allows to freely modify the structure of the type distribution. The following example illustrates this idea.

Example 1. Let $X = \mathbb{R}$, $J = 1$ with $A_1 = [-\log 2, +\log 2]$, F such that $F_1(-\log 2) = \frac{1}{3}$, $F_1(0) = 0.5$ and $\lim_{x \to +\log 2} = \frac{4}{5}$, and consider $\mathcal{U}$ to be the class of Euclidean utilities.[8] Since there is just one menu of alternatives, data is trivially ORUM-rationalizable.

Now, we argue that forcing g^* to adopt a specific structure, such as being logistic, is not a relevant restriction. For instance, we can rationalize F by defining $\gamma(t) = -(x - f(t))^2$ with $f(t) = \frac{t}{2}$ whenever $t \geq 0$ and $f(t) = t$ whenever $t < 0$, allowing us to use the logistic distribution with location and scale parameters equal to 0 and 1, respectively. $\square$

Second, we consider the case in which a specific type-utility map is fixed ex-ante. This is common practice in empirical work, and is usually referred to as a semi non-parametric restriction. As an example, consider the study of decisions under risk, and let $\mathcal{U}$ be the class of all expected utilities. Suppose that the analyst is interested in restricting further the set of admissible preferences using, e.g., the map γ^* in which type t corresponds to the CRRA expected utility with risk aversion coefficient equal to t. It is clear this restricts the model's flexibility, as CRRA utilities may not capture choice patterns explained by other expected utilities. However, we show below that conditional on this constrained set of allowed utilities, Proposition 1's logic still applies. To formalize this, we adapt the notion of rationalizability; when ORUM-rationalizability is obtained using utilities from a sub-class $\mathcal{U}^* \subset \mathcal{U}$, we speak of ORUM-rationalizability*.

Proposition 2. *Let γ^* be any given type-utility map with image $\mathcal{U}^*$. If F is ORUM-rationalizable*, then F is ORUM-rationalizable with type-utility map γ^*.*

Proposition 2 establishes the exact bite of adopting a specific type-utility map in a semi non-parametric exercise. Essentially, this restricts the analysis to a sub-class of utilities, but has no further effect. Modulo the restricted set of admissible utilities, the

[8]For simplicity, we assume there is just one menu, and we describe F in a few points.

specific assignment of types to utilities is without loss of generality because, again, we can adapt to a specific labelling of these utilities by finding the exact type distribution that suits this labelling. We can continue with Example 1 above in order to illustrate this case.

Example 1 (continued). Suppose now that we impose a particular type-utility map. For example, consider the type-utility map assigning to type t the Euclidean utility centered at t, that is $\gamma^*(t) = -(x-t)^2$. In order to rationalize F, we simply need a type distribution with mass $\frac{1}{3}$ (resp. $\frac{1}{5}$) distributed across types with peak below $-\log 2$ (resp. above $\log 2$), and a mass of $\frac{1}{6}$ (resp. $\frac{3}{10}$) distributed across types with peak between $-\log 2$ and 0 (resp. between 0 and $+\log 2$). The combination of γ^* and the constructed type distribution rationalizes the data. □

Finally, empirical work often comes with restrictions on both components of the model, i.e., the analyst often fixes a specific type-utility map γ^* and some structural properties of the type distribution g^*. In this case, relabelling utilities and transforming the distribution may fail to rationalize the data and hence the restriction may come at a cost. We show this by means of our example.

Example 1 (continued). Imposing both restrictions above, namely the particular type-utility map $\gamma^*(t) = -(x-t)^2$ and the particular logistic distribution, ORUM-rationalization of the data is no longer possible. From $F_1(0) = 0.5$, it must be that the location of the logistic is the peak of type 0, which is 0. Since types $-\log 2$ and $\log 2$ have peaks that are symmetric with respect to 0, the logistic assumption would require that $F_1(-\log 2) = 1 - F_1(+\log 2)$, a contradiction. □

When parametric restrictions are jointly imposed on the type-utility map and the type distribution, the model may be significantly restricted. It is then evident that the foundations of parametric models often used in empirical work, such as ordered-logit models over a specific latent parameter, may not be well understood throughout the general result contained in Theorem 1. In the next section, we take a first step in order to fill this gap and show that it is also possible to provide tailored characterization results for such parametric models.

4. Parametric ordered logit

We exemplify our approach to the study of the foundations of fully parametric models using the logistic distribution. The ordered-logit model is a popular model not only in economics, but in other disciplines such as political science, sociology or biology.[9] Yet, despite this widespread interest in the model, the literature has not yet provided choice-based foundations.

To simplify the analysis, we consider the following richness assumption: for every two menus A_j and $A_{j'}$, there exists a sequence of menus, $A_j = A_{j^0}, \ldots, A_{j^k} \ldots, A_{j^K} = A_{j'}$ such that, for every $k \in \{0, \ldots, K-1\}$, there exists an interval of types that produce non-corner maximizers in both A_{j^k} and $A_{j^{k+1}}$. To motivate this richness condition, notice that in our setting there always exists a non-empty interval of types for which the maximizer is non-corner. Hence, if corner alternatives were never optimal, the richness assumption would be trivially met.[10] Similarly, the richness assumption is also trivially met when only one of the corner alternatives can be the result of utility optimization.[11] In some occasions, though, optimization may lead to choice in both corners and, since the interval of utilities generating non-corner maximizers may vary across decision problems, the condition may not be trivially met. As an example, this may possibly be the result of two consumption sets having disparate prices. However, notice that, under these circumstances, the presence of another (or a chain of other) consumption sets where prices are intermediate would produce the desired linkage expressed in our richness assumption.

For the study of the parametric ordered-logit model, fix a generic type-utility map $\gamma^* : \mathbb{R} \to \mathcal{U}$ and restrict the type distribution to the logistic family $\mathcal{G}^L$. Recall that a logistic type distribution (that we denote by g^L) has a CDF of the form $G^L(t) = \frac{1}{1+e^{-(t-\tau)/\sigma}}$, where $\tau \in \mathbb{R}$ and $\sigma > 0$ are the location and scale parameters.

Given the discussion in the previous section, the foundations of the parametric ordered-logit model do not follow from the rationalizability of all the quantiles, and

[9] See the discussion around footnote 1 in the Introduction for references to a wide diversity of economic settings using the ordered-logit model.

[10] For example, in a standard consumption setting with two goods, this is the case for Cobb-Douglas utilities with strictly positive weights.

[11] For example, in the maximization of strictly convex quasi-linear utility functions in a consumption setting with two goods.

we need to provide tailored properties. These properties aim to capture the specific patterns of data that the logistic distribution creates whenever a given type-utility map γ^* is fixed. First, as with any RUM, notice that a necessary condition for any alternative to have strictly positive mass is the existence of at least one type for which this alternative is optimal. Importantly, notice also that for the case of corner alternatives and the logistic model, the existence of one such type always implies the existence of an unbounded interval of types with the same property, and must result into a strictly positive mass for this corner alternative.[12]

Corner extremeness (CE). $F_j(\underline{x}_j) > 0$ (respectively, $\lim_{x \to \overline{x}_j} F_j(x) < 1$) if and only if there exists $t \in \mathbb{R}$ such that $x_j^{\gamma^*(t)} = \underline{x}_j$ (respectively, $x_j^{\gamma^*(t)} = \overline{x}_j$).

To formulate our second property, we use the notion of cumulative logit. Given a non-corner alternative x_j, its cumulative logit is the value $\ell(x_j) = \log \frac{F_j(x_j)}{1-F_j(x_j)}$, i.e., the logarithm of the ratio between the cumulative mass below and above alternative x_j. When the data is generated by a logistic type distribution, it turns to be the case that the cumulative logit $\ell(x_j)$ corresponds to the standardized type that has x_j as maximizer.[13] Consider then two menus and a pair of non-corner alternatives in each menu. If the sum of the pair of types associated to the first pair of alternatives coincides with the sum of the pair of types associated to the second pair of alternatives, the sum of cumulative logits of both pairs of alternatives should coincide.

Cumulative Logit Additivity (CLA). Suppose that the types t_1, t_2, t'_1, t'_2 are such that: (i) $x_j^{\gamma^*(t_1)}, x_j^{\gamma^*(t_2)}$ and $x_{j'}^{\gamma^*(t'_1)}, x_{j'}^{\gamma^*(t'_2)}$ are non-corner alternatives, and (ii) $t_1 + t_2 = t'_1 + t'_2$. Then, $\ell(x_j^{\gamma^*(t_1)}) + \ell(x_j^{\gamma^*(t_2)}) = \ell(x_{j'}^{\gamma^*(t'_1)}) + \ell(x_{j'}^{\gamma^*(t'_2)})$.

Conditions CE and CLA are not only necessary but also sufficient for data to be ORUM-rationalizable with type-utility map γ^* and a logistic type distribution g^L.

Theorem 2. *Let γ^* be any type-utility map. F is ORUM-rationalizable with type-utility map γ^* and a type distribution in $\mathcal{G}^L$ if and only if F satisfies CE and CLA.*

The sufficiency part of the proof of Theorem 2 comprises a number of steps. First, when considering the non-corner alternatives in a given menu, the data immediately induces a specific CDF over the corresponding interval of types that is associated to

[12]This is in the spirit of Gul and Pesendorfer (2006) extremeness property, and hence our name.

[13]That is, $\ell(x_j) = \frac{t-\tau}{\sigma}$ whenever $x_j = x_j^{\gamma^*(t)}$.

these non-corner alternatives. When relevant, we need to account for the censoring produced by corner choices, that will be optimal for an unbounded interval of types. Hence, the masses observed at the corners must be appropriately distributed among all their rationalizing types, in such a way as to ensure that the constructed CDF over all types satisfies the same additivity requirement that CLA imposes over the non-corner alternatives. We address this requirement by using a recursive construction. Second, the ordered-logit functional form requires us to build upon Galambos and Kotz's (1978) Theorem 2.1.5. This classical, statistical, result provides a necessary and sufficient condition over triplets of real numbers for a single CDF over the reals, assumed to be symmetric with respect to the origin, to be logistic. We naturally need to extend this result to our revealed preference setting, where: (i) distributions may have any mean and have not yet been proven to be symmetric and, (ii) we have not one, but a collection of menu-dependent distributions, that may potentially differ. Our CLA property using quadruplets proves sufficient to show that our menu-dependent distributions are all logistic and, in fact, share the same location and scale parameters. Finally, note also that the parameters (τ, σ) of the logistic type distribution that rationalizes the data must be unique.

5. Applications

An advantage of our results, both non-parametric and parametric, is that they are portable to the analysis of specific economic settings of interest. We illustrate now with three examples involving political choices, consumption or altruistic decisions, and choices over lotteries. In each of them, we first particularize Theorem 1 providing non-parametric characterizations, and then we particularize Theorem 2 to provide parametric characterizations of commonly used ordered-logit models in these settings.

5.1. **Political Economy.** Let $X = \mathbb{R}$ represent the space of possible policies. Following Moulin (1984), a menu is described by a closed interval $A_j = [\underline{x}_j, \overline{x}_j]$ capturing the subset of feasible policies at a given situation. As it is common in political models, let $\mathcal{U}$ be the collection of single-peaked utilities over the real line. That is, a utility U belongs to $\mathcal{U}$ whenever there exists an alternative $z \in \mathbb{R}$, the peak of the utility, such that for every pair satisfying $x' < x \leq z$ or $z \leq x < x'$, we have $U(x) > U(x')$.

We start by presenting a characterization of ORUM-rationalizability within this setting.[14] Denote by I_j^c the interval set of peak locations that are compatible with the choice observation in menu j. That is, $I_j^c = (-\infty, \underline{x}_j]$ whenever $c(A_j) = \underline{x}_j$, $I_j^c = c(A_j)$ whenever $c(A_j) \neq \{\underline{x}_j, \overline{x}_j\}$, and $I_j^c = [\overline{x}_j, +\infty)$ whenever $c(A_j) = \overline{x}_j$. We say that the choice function c satisfies the intersection property whenever $I_j^c \cap I_{j'}^c \neq \emptyset$ holds for every pair of menus A_j and $A_{j'}$.

Theorem 3. *In the political economy domain, F is ORUM-rationalizable if and only if c^p satisfies the intersection property for every p.*

We now discuss an intuitive ordered-logit parametric model involving Euclidean utilities, that is a special class of single-peaked utilities. Consider the type-utility map $E(t) = -(x-t)^2$ in which type t is assigned the Euclidean preference with peak at t. Notice that the following two features are true: (i) for any given menu, corner solutions are always related to an interval of Euclidean utilities, and (ii) the type that maximizes at the non-corner alternative x_j is the one with Euclidean peak at x_j. A straightforward adaptation of Theorem 2 allows us to obtain the following immediate characterization of this parametric model.

Theorem 4. *In the political economy domain, F is ORUM-rationalizable with type-utility map E and a type distribution in $\mathcal{G}^L$ if and only if F satisfies: (i) $F_j(\underline{x}_j) > 0$ and $\lim_{x \to \overline{x}_j} F_j(x) < 1$ and (ii) if $x_j, y_j, x_{j'}, y_{j'}$ are non-corner alternatives such that $x_j + y_j = x_{j'} + y_{j'}$, then $\ell(x_j) + \ell(y_j) = \ell(x_{j'}) + \ell(y_{j'})$.*

5.2. **Altruism.** Let $X = \mathbb{R}_+^2$ represent the space of all possible monetary allocations in which the payments of a dictator, x^1, and a second person, x^2, are described.[15] As it is usual in experimental exercises, subjects are offered budget sets of the form $B_j = \{x \in X : p_j^1 x^1 + p_j^2 x^2 \leq 1\}$, where p_j^i denotes the price of a money allocation to agent i, with both prices being strictly positive. Notice that monotonicity of preferences allows us to focus attention on the line segment $A_j = \{x_j \in X : p_j^1 x_j^1 + p_j^2 x_j^2 = 1\}$, i.e.,

[14] Remark 1 in Moulin (1984) presents a deterministic characterization of single-peaked rationalizability based upon the classical property of independence of irrelevant alternatives and a continuity requirement (see also Bossert and Peters (2009), who analyze further the problem). These results require the choice function to be observed across all possible menus. Since these properties are not sufficient when data is arbitrary, as in our setting, we need to introduce a novel deterministic property, the intersection property.

[15] The same results can be presented for the classical consumption setting with two commodities.

a menu is formed by all convex combinations of the corner allocations $\underline{x}_j = (\frac{1}{p_j^1}, 0)$ and $\overline{x}_j = (0, \frac{1}{p_j^2})$, where the allocations within a menu are ordered by the amount of money donated (which, allegedly, is a proxy for altruism). We first provide a characterization of ORUM-rationalizability in the altruism domain where, for simplicity, we consider utilities $\mathcal{U}$ that are strictly quasi-concave and strictly monotone in both components.[16] We say that the choice function c satisfies WARP if $c(A_j) \in B_{j'}$ and $c(A_{j'}) \in B_j$ implies $c(A_j) = c(A_{j'})$.

Theorem 5. *In the altruism domain, F is ORUM-rationalizable if and only if every c^p satisfies WARP.*

We now discuss a simple parametric result involving the logistic distribution, and the sub-class of $\mathcal{U}$ involving Cobb-Douglas utilities. Namely, we fix the type-utility map $CD(t) = x^1$ whenever $t \leq 0$, $CD(t) = (x^1)^t(x^2)^{1-t}$ whenever $t \in (0, 1)$ and $CD(t) = x^2$ whenever $t \geq 1$. Notice that, for the case of such Cobb-Douglas utilities: (i) for any given menu, corner solutions are always related to an interval of types, and (ii) the type that maximizes at the non-corner alternative x_j is that with parameter $p_j^1 x_j^1$ (since this is the fraction of income allocated to the dictator). We can then provide the following characterization result.

Theorem 6. *In the altruism domain, F is ORUM-rationalizable with type-utility map CD and a type distribution in $\mathcal{G}^L$ if and only if F satisfies: (i) $F_j(\underline{x}_j) > 0$ and $\lim_{x \to \overline{x}_j} F_j(x) < 1$ and (ii) if $x_j, y_j, x_{j'}, y_{j'}$ are non-corner alternatives such that $p_j^1(x_j^1 + y_j^1) = p_{j'}^1(x_{j'}^1 + y_{j'}^1)$, then $\ell(x_j) + \ell(y_j) = \ell(x_{j'}) + \ell(y_{j'})$.*

5.3. **Risk.** Let $X = \mathbb{R}_+^2 \times [0, 1]$ represent the set of all possible (two) state-contingent lotteries, with x^1 and x^2 representing the payouts in the two states and $q_j^1 \in [0, 1]$ describing the probability of the first state (with $q_j^2 = 1 - q_j^1$ describing the probability of the second state). A menu is a linear budget set that can be described exactly as in the altruism case. To simplify the exposition, we assume that state 1 always pays more in expectation, i.e., $\frac{p_j^2}{p_j^1} > \frac{q_j^2}{q_j^1}$.[17] Also, in the present setting, it is typically assumed that the states are symmetric in terms of their utility evaluation, and hence we can restrict attention to the interval of alternatives between the corners $\underline{x}_j = (\frac{1}{p_j^1}, 0)$

[16]This is done for the ease of exposition, since this assumption allows us to use classical, deterministic, results (Rose (1958) and Matzkin and Richter (1991)).

[17]Notice that if this were not the case, we could relabel the states in each menu to guarantee it.

and $\overline{x}_j = (\frac{1}{p_j^1+p_j^2}, \frac{1}{p_j^1+p_j^2})$. Notice that higher levels of x_2 are associated with more risk aversion.

Kubler, Selden and Wei (2014) characterize, within this type of data, deterministic rationalizability for the class $\mathcal{U}$ of expected utilities with continuous, strictly increasing and strictly concave monetary utility functions. In order to be able to use their deterministic result, we consider the same class of utilities. To formalize their property, denote the (menu-dependent) ratio of prices-to-probabilities between states by $\rho_j^i = \frac{p_j^i}{q_j^i}$. Given two menus A_j and $A_{j'}$, we can define the operator $L(j,j') = 0$ whenever $x_{j'}^{s'} > x_j^s$ for all $s, s' \in \{1,2\}$ and $L(j,j') = \max_{s,s':x_j^s > x_{j'}^{s'}} \frac{\rho_j^s}{\rho_{j'}s'}$ otherwise. We then say that the choice function c satisfies SAREU whenever for every sequence of menus $j_1, \ldots, j_K$, it is $\prod_{k=1}^{K-1} L(j_k, j_{k+1}) < 1$.[18] The following is a direct implication of Theorem 1.

Theorem 7. *In the risk domain, F is ORUM-rationalizable if and only if every c^p satisfies SAREU.*

We conclude with a parametric application in which we consider the logistic distribution and the restriction of $\mathcal{U}$ to CRRA expected utilities. Let EU_{CRRA} be the type-utility map assigning to type t the CRRA expected utility with risk aversion coefficient equal to t, i.e., with monetary utility function $\frac{x^{1-t}}{1-t}$.[19] Denote by $\kappa(x_j) = \log \frac{x_j^1}{x_j^2} / (\log \frac{q_j^1}{q_j^2} - \log \frac{p_j^1}{p_j^2})$ the expression that normalizes the log-consumption ratio by the log-ratio of probabilities and the log-ratio of prices in the menu. Notice that: (i) for any given menu, the lower corner solution is always related to an interval of CRRA expected utilities, and (ii) the type that maximizes at the non-corner alternative x_j is that with CRRA coefficient equal to $\kappa(x_j)$ (an immediate implication of the first-order condition).

Theorem 8. *In the risk domain, F is ORUM-rationalizable with type-utility map EU_{CRRA} and a type distribution in $\mathcal{G}^L$ if and only if F satisfies: (i) $F_j(\underline{x}_j) > 0$ and $\lim_{x \to \overline{x}_j} F_j(x) = 1$ and (ii) if $x_j, y_j, x_{j'}, y_{j'}$ are non-corner alternatives such that $\kappa(x_j) + \kappa(y_j) = \kappa(x_{j'}) + \kappa(y_{j'})$, then $\ell(x_j) + \ell(y_j) = \ell(x_{j'}) + \ell(y_{j'})$.*

[18] SAREU stands for strong axiom of revealed expected utility. In line with Kubler, Selden and Wei (2014), we simplify the exposition by assuming that, in c^p, each of the state consumptions in each of the observations is different.

[19] This holds for $t \in \mathbb{R} \setminus \{1\}$ while $\gamma(1)$ is the expected utility with logarithmic monetary utility function.

6. Conclusions

This paper has explored the empirical implications of commonly imposed parametric assumptions in ordered random utility models (ORUMs). We have begun by characterizing the non-parametric content of ORUMs, showing data is rationalizable if and only if each quantile choice function is rationalizable. We have then demonstrated how parametric restrictions on the type-utility map and type distribution can further shape the empirical content of the model. We have shown that when either parametric type distributions or semi-parametric type-utility maps are made alone, the model retains full flexibility. However, jointly restricting both components, as is typical in empirical work, significantly limits the space of observable behaviors consistent with the model. We have formalized this insight by characterizing the precise restrictions created by parametric ordered-logit models. Finally, we have demonstrated how our results provide a rigorous theoretical foundation for commonly used specifications across political economy, altruism, and risk choice settings.

Several avenues for future work are worth exploring. An obvious one would entail expanding beyond the logistic to examine other often-used distributions in empirical research, such as the Gaussian one, to provide fully parametric foundations for ordered-probit type of models.

Appendix A. Proofs

Proof of Theorem 1: We start by proving the necessity part. Suppose that F is ORUM-rationalizable with type-utility map γ and type distribution g. For every $p \in (0,1)$, denote by t^p the first type such that $G(t^p) \geq p$, where G is the CDF associated to g. We claim that, for every j, the utility function $\gamma(t^p)$ produces a maximizer that coincides with c^p_j, proving the deterministic rationalizability of c^p. We proceed by contradiction. Suppose there is a menu j such that $x_j^{\gamma(t^p)} \neq c^p_j$. If $a(x_j^{\gamma(t^p)}) < a(c^p_j)$, the ordered choice structure guarantees that for every type $t \leq t^p$, the maximizer of utility function $\gamma(t)$ lies below $x_j^{\gamma(t^p)}$. By the fact that the pair (γ, g) rationalizes the data, it must be that $F_j(x_j^{\gamma(t^p)}) \geq G(t^p) \geq p$, contradicting the definition of c^p_j. If $a(x_j^{\gamma(t^p)}) > a(c^p_j)$, it must be that $c^p_j \in A_j \setminus \{\overline{x}_j\}$. If $c^p_j = \underline{x}_j$, the fact that $F_j(c^p_j) \geq p > 0$, together with (γ, g) rationalizing the data, guarantees that there must exist a type t for which c^p_j is maximal. Similarly, whenever $c^p_j \neq \underline{x}_j$, the continuous and strictly increasing nature of F_j away from the corners, together with the rationalization of the data, guarantees that there must exist a type t for which c_j^p is maximal. In both cases, the ordered-choice structure guarantees that this type is such that $t < t^p$. Given the rationalization of the data, it must be $G(t) \geq p$, which contradicts the definition of t^p and proves the claim.

We now prove the sufficiency part. For every $p \in (0,1)$, c^p is rationalizable and hence, there exists $U^p \in \mathcal{U}$ rationalizing c^p. For every type $t \in \mathbb{R}$, define $\gamma(t)$ as the utility $U^{\frac{1}{1+e^{-t}}}$. We first claim that, for every menu, the maximizers induced by γ are non-decreasing in t. To see this, take any menu A_j. Given rationalizability, we merely need to prove that choices $c_j^{\frac{1}{1+e^{-t}}}$ are non-decreasing in t. This follows immediately from the definition of such alternatives and the fact that F_j is non-decreasing. Now, consider g to be the logistic distribution with location 0 and variance 1. We claim that, for every menu j and every $x_j \in A_j$ such that $0 < F_j(x_j) < 1$, $F_j(x_j)$ coincides with the mass of utilities with a maximizer below x_j. Given the non-decreasing nature of the maximizing alternatives, we need to prove that the utility function $U^{F_j(x_j)}$ is the last utility with maximizer below x_j. First, consider $p > F_j(x_j)$. Since x_j has not reached cumulative probability p, it must be that $a(c_j^p) > a(x_j)$, and since utility U^p rationalizes c_j^p, the maximizer of U^p lies strictly above x_j. Second, consider the utility function $U^{F_j(x_j)}$. By construction, $c_j^{F_j(x_j)}$ lies below x_j and since $U^{F_j(x_j)}$ rationalizes $c_j^{F_j(x_j)}$, the maximizer of $U^{F_j(x_j)}$ lies below x. Finally, if $F(\underline{x}_j) = 0$ it is straightforward that no utility can produce a maximizer below $\underline{x}_j$, and the mass of these utilities is zero. Trivially, since all utilities have a maximizer below $\overline{x}_j$, $F(\overline{x}_j) = 1$ is also rationalized. This concludes the sufficiency part and the proof. ■

Proof of Proposition 1: For every $p \in (0,1)$, define U^p as in the proof of Theorem 1. Then, for every type $t \in \mathbb{R}$, define $\gamma^*(t)$ as the utility $U^{G^*(t)}$, where G^* is the CDF of g^*. That is, type t is assigned the utility that corresponds exactly to the quantile, according to g^*, of this type. The ordered-choice structure of type-utility map γ^* is immediate and ORUM-rationalizability follows from reproducing the proof of Theorem 1 with the pair (γ^*, g^*). ■

Proof of Proposition 2: Suppose that F is ORUM-rationalizable*. Then, we can use the same logic of Theorem 1 to show that c^p, $p \in (0,1)$, must be rationalizable by some utility U^p in $\mathcal{U}^*$. Given $\mathcal{U}^*$, utility U^p must correspond to some type, that we denote t_p, i.e., $\gamma^*(t_p) = U^p$. We need to consider the distribution g^* such that the cumulative probability at type t_p is equal to p for every $p \in (0,1)$, and it can be seen that the pair (γ^*, g^*) is an ORUM-rationalization of the data. ■

Proof of Theorem 2: Since the necessity of the axioms is straightforward, we will now prove sufficiency. Consider any menu j. We construct a sequence of open intervals of types, $\{I_j^0, I_j^1, \dots, I_j^n, \dots\}$, and a sequence of real functions defined over them, $\{G_j^0, G_j^1, \dots, G_j^n, \dots\}$, satisfying the following four properties:

(1) For every n, $I_j^n \subseteq I_j^{n+1}$.
(2) For every n, G_j^{n+1} extends G_j^n.
(3) For every n, G_j^n takes values in $(0,1)$, is continuous, and strictly increasing. Moreover, if I_j^n is bounded from above (respectively, from below), the function G_j^n must be strictly bounded from above by some value $k < 1$ (respectively, strictly bounded from below by some value $k > 0$).
(4) For every n and every four types t_1, t_2, t_1', t_2' in I_j^n, if $t_1 + t_2 = t_1' + t_2'$ then $\log \frac{G_j^n(t_1)}{1-G_j^n(t_1)} + \log \frac{G_j^n(t_2)}{1-G_j^n(t_2)} = \log \frac{G_j^n(t_1')}{1-G_j^n(t_1')} + \log \frac{G_j^n(t_2')}{1-G_j^n(t_2')}$.

The first interval of types, I_j^0, corresponds to the set of types that have a maximizer in $A_j \setminus \{\underline{x}_j, \overline{x}_j\}$.[20] The first function, G_j^0, corresponds to the function that choice data F_j induces over these types, i.e., for every $t \in I_j^0$, $G_j^0(t) = F_j(x_j^{\gamma^*(t)})$. The function G_j^0 is well-defined given the assumptions made on F_j. It is obviously strictly increasing and takes values in $(0,1)$. Moreover, if the interval I_j^0 is bounded from above (respectively, from below), there is an interval of types selecting $\overline{x}_j$ (respectively, $\underline{x}_j$) and hence, $\lim_{x \to \overline{x}_j} F_j(x) < 1$ (respectively, $\lim_{x \to \underline{x}_j} F_j(x) > 0$), and the boundedness conditions hold for G_j^0. That is, property (3) is satisfied. We now show that G_j^0 must satisfy property (4). To see this, notice that we can apply CLA with menu $j' = j$, using the collection of non-corner alternatives $x_j^{\gamma^*(t_1)}, x_j^{\gamma^*(t_2)}, x_j^{\gamma^*(t_1')}, x_j^{\gamma^*(t_2')}$, which brings property (4) over G_j^0.

The remaining intervals and functions are now defined recursively. Given collections $\{I_j^0, I_j^1, \dots, I_j^n\}$ and $\{G_j^0, G_j^1, \dots, G_j^n\}$ which satisfy all the properties, we define interval I_j^{n+1} and function G_j^{n+1} in such a way as to guarantee that collections $\{I_j^0, I_j^1, \dots, I_j^{n+1}\}$ and $\{G_j^0, G_j^1, \dots, G_j^{n+1}\}$ also satisfy the properties. The definition of the new interval

[20] Notice that this set of types depends on the assumed type-utility map γ^*. Since γ^* is fixed and to simplify the exposition, we will avoid some references to γ^* in the arguments that follow.

of types, I_j^{n+1}, depends on the parity of n. If n is an even (respectively, an odd) integer, we define interval I_j^{n+1} as follows: (i) if I_j^n is not bounded from above (respectively, from below), define $I_j^{n+1} = I_j^n$ and (ii) if I_j^n is bounded from above (respectively, from below), define I_j^{n+1} as the union of the previous interval I_j^n, the least upper bound (respectively, the greatest lower bound) z_j^n of interval I_j^n, and the types t for which there exists $t' \in I_j^n$ with $t = 2z_j^n - t'$.[21]

We now consider the definition of function G_j^{n+1}. For every $t \in I_j^n$, define $G_j^{n+1}(t) = G_j^n(t)$. For the limit type z_j^n, define $G_j^{n+1}(z_j^n) = \lim_{s \to z_j^n} G_j^n(s)$, where the right-hand or left-hand bound must be considered, depending on the parity. Finally, for any other type t belonging to I_j^{n+1}, we know that there exists a unique value $t' \in I_j^n$ such that $t = 2z_j^n - t'$, so we can define $G_j^{n+1}(t)$ as the unique real value satisfying the equation:

$$\log \frac{G_j^{n+1}(t)}{1 - G_j^{n+1}(t)} = 2 \log \frac{G_j^{n+1}(z_j^n)}{1 - G_j^{n+1}(z_j^n)} - \log \frac{G_j^n(t')}{1 - G_j^n(t')}.$$

It is then evident that the function G_j^{n+1} is well defined on I_j^{n+1} and it is straightforward to see that $I_j^n \subseteq I_j^{n+1}$ and, hence, property (1) holds. Similarly, note that the construction guarantees that the function G_j^{n+1} extends G_j^n, and therefore property (2) is satisfied.

We now discuss property (3). Notice that, by the continuity of G_j^n and the fact that all values belong to $(0, 1)$, it is guaranteed that the limit value at z_j^n is well defined when needed. The continuity of the function G_j^{n+1} is then a direct consequence of this limit definition at z_j^n. To appreciate the strictly increasing nature of the new function, consider two types $t_1 < t_2$. If both types belong to I_j^n, we know that $G_j^{n+1}(t_1) < G_j^{n+1}(t_2)$ must hold because G_j^{n+1} extends the strictly increasing function G_j^n. If $t_1 \in I_j^n$ but t_2 does not, it must be the case that n is even and there exists $t'_2 \in I_j^n$ such that $t_2 = 2z_j^n - t'_2$. Since $\log \frac{G_j^n(z_j^n)}{1 - G_j^n(z_j^n)} > \log \frac{G_j^n(t'_2)}{1 - G_j^n(t'_2)}$, it is $\log \frac{G_j^{n+1}(t_1)}{1 - G_j^{n+1}(t_1)} = \log \frac{G_j^n(t_1)}{1 - G_j^n(t_1)} < \log \frac{G_j^n(z_j^n)}{1 - G_j^n(z_j^n)} < 2 \log \frac{G_j^n(z_j^n)}{1 - G_j^n(z_j^n)} - \log \frac{G_j^n(t'_2)}{1 - G_j^n(t'_,)} = \log \frac{G_j^{n+1}(t_2)}{1 - G_j^{n+1}(t_2)}$, as desired. If t_1 is not in I_j^n

[21]Intuitively we are extending the original right-bounded (respectively, left-bounded) interval I_j^n beyond its boundary and adding the boundary point. This step is not needed when there are no corner choices because then the initial interval I_j^0 equals the set of all types, $\mathbb{R}$. When choices are observed in only one of the corner alternatives, or, equivalently, I_j^0 is bounded on one side, the logic requires a unique duplication, which already forms the entire real line. If choices are observed in both corner alternatives, or, equivalently, the initial interval is bounded on both sides, we need to duplicate the initial bounded interval an infinite number of times, as the proof indicates.

but t_2 is, an analogous argument applies in which n is odd and z_j^n is the lower bound of I_j^n. If neither is in I_j^n, they must both be above or below z_j^n, depending on the parity. There must exist $t_1', t_2' \in I_j^n$ such that $t_1 = 2z_j^n - t_1'$ and $t_2 = 2z_j^n - t_2'$. It clearly must be that $t_1' > t_2'$ and we know that $G_j^n(t_1') > G_j^n(t_2')$. The definition of $G_j^{n+1}(t_1)$ and $G_j^{n+1}(t_2)$ guarantees that the former is strictly smaller than the latter. Hence, we have shown that G^{n+1} is strictly increasing and, to complete property (3), we need to show that this function takes values in $(0,1)$ and is bounded as required. We show the case of n being even, the other case being analogous. If I_j^n is not bounded from above, the new function replicates the original one and the property holds. If I_j^n is bounded from above, we know that the value $G^{n+1}(z_j^n)$ must be strictly lower than 1 by virtue of the boundedness condition. For every $t \in I_j^{n+1}$ with $t > z_j^n$, the construction guarantees that G_j^{n+1} takes values in $(0,1)$. To show boundedness, notice that nothing changes at the lower end of the interval and, since G_j^{n+1} extends G_j^n, the property is satisfied. For the upper end of the interval, suppose that I_j^{n+1} is bounded from above, in which case it must be that I_j^n is bounded from below (say, with largest lower bound k). It then follows that $\log \frac{G_j^{n+1}(t)}{1-G_j^{n+1}(t)} < 2 \log \frac{G_j^{n+1}(z_j^n)}{1-G_j^{n+1}(z_j^n)} - \log \frac{G_j^n(k)}{1-G_j^n(k)}$, and hence $G_j^{n+1}(t)$ must be strictly lower than 1. This completes the proof that G_j^{n+1} satisfies property (3).

To see that property (4) holds, consider any four types t_1, t_2, t_1', t_2' in I_j^{n+1} such that $t_1 + t_2 = t_1' + t_2'$ and assume, without loss of generality, that $t_1 < t_1' \leq t_2' < t_2$.[22] Again, we show the case of n even, the other case being analogous. We start by noticing that property (4) holds over the closure of I_j^n, denoted by $\overline{I}_j^n$, thanks to the recursive assumption on G_j^n, the fact that G_j^{n+1} extends G_j^n, and the limit construction at z_j^n. Hence, we only need to consider cases where not all four types belong to $\overline{I}_j^n$:

- Case 1: None of the four types belongs to $\overline{I}_j^n$. There must exist $s_1, s_2, s_1', s_2' \in I_j^n$ such that $t_1 = 2z_j^n - s_1$, $t_1' = 2z_j^n - s_1'$, $t_2 = 2z_j^n - s_2$ and $t_2' = 2z_j^n - s_2'$. Clearly, it must be that $s_1 + s_2 = s_1' + s_2'$ and hence, we know that $\log \frac{G_j^n(s_1)}{1-G_j^n(s_1)} + \log \frac{G_j^n(s_2)}{1-G_j^n(s_2)} = \log \frac{G_j^n(s_1')}{1-G_j^n(s_1')} + \log \frac{G_j^n(s_2')}{1-G_j^n(s_2')}$, which is equivalent to $\log \frac{G_j^n(s_1)}{1-G_j^n(s_1)} + \log \frac{G_j^n(s_2)}{1-G_j^n(s_2)} + 4G_j^{n+1}(z_j^n) = \log \frac{G_j^n(s_1')}{1-G_j^n(s_1')} + \log \frac{G_j^n(s_2')}{1-G_j^n(s_2')} + 4G_j^{n+1}(z_j^n)$, which implies $\log \frac{G_j^n(t_1)}{1-G_j^n(t_1)} + \log \frac{G_j^n(t_2)}{1-G_j^n(t_2)} = \log \frac{G_j^n(t_1')}{1-G_j^n(t_1')} + \log \frac{G_j^n(t_2')}{1-G_j^n(t_2')}$, as desired.
- Case 2: $t_1 \in \overline{I}_j^n$. There must exist $s_2, s_1', s_2' \in I_j^n$ such that $t_1' = 2z_j^n - s_1'$, $t_2 = 2z_j^n - s_2$ and $t_2' = 2z_j^n - s_2'$. It must be that $t_1 + s_1' + s_2' = s_2 + 2z_j^n$.

[22]Notice that if the types were equal across the two pairs, the property would be trivially satisfied.

Define $\hat{t} = s_2 + z_j^n - t_1$, which belongs to I_j^n. Given that $t_1 + \hat{t} = s_2 + z_j^n$, property (4) holds over these four types. Now, notice that it must also be that $s_1' + s_2' = \hat{t} + z_j^n$ and hence property (4) holds over these four types. We can combine the two expressions to verify that property (4) holds over t_1, t_2, t_1' and t_2', as desired.

- Case 3: $t_1, t_1' \in \overline{I}_j^n$. There must exist $s_2, s_2' \in I_j^n$ such that $t_2 = 2z_j^n - s_2$, and $t_2' = 2z_j^n - s_2'$. It must be that $t_1 + s_2' = t_1' + s_2$ and hence, we know that $\log \frac{G_j^n(t_1)}{1-G_j^n(t_1)} + \log \frac{G_j^n(s_2')}{1-G_j^n(s_2')} = \log \frac{G_j^n(t_1')}{1-G_j^n(t_1')} + \log \frac{G_j^n(s_2)}{1-G_j^n(s_2)}$, which implies $\log \frac{G_j^n(t_1)}{1-G_j^n(t_1)} + \log \frac{G_j^n(s_2')}{1-G_j^n(s_2')} + 2G_j^{n+1}(z_j^n) = \log \frac{G_j^n(t_1')}{1-G_j^n(t_1')} + \log \frac{G_j^n(s_2)}{1-G_j^n(s_2)} + 2G_j^{n+1}(z_j^n)$, which implies $\log \frac{G_j^n(t_1)}{1-G_j^n(t_1)} + \log \frac{G_j^n(t_2)}{1-G_j^n(t_2)} = \log \frac{G_j^n(t_1')}{1-G_j^n(t_1')} + \log \frac{G_j^n(t_2')}{1-G_j^n(t_2')}$, as desired.
- Case 4: $t_1, t_1', t_2' \in \overline{I}_j^n$. There must exist $s_2 \in I_j^n$ such that $t_2 = 2z_j^n - s_2$. It must be that $t_1 + 2z_j^n = t_1' + t_2' + s_2$. Define $\hat{t} = t_1 + z_j^n - t_1'$, which belongs to I_j^n. Given that $t_1' + \hat{t} = t_1 + z_j^n$, property (4) holds over these four types. Now, notice that it must also be that $\hat{t} + z_j^n = t_2' + s_2$ and hence property (4) holds over these four types. We can combine the two expressions to verify that property (4) holds over t_1, t_2, t_1' and t_2', as desired.

This completes the proof that the collections $\{I_j^0, I_j^1, \ldots, I_j^{n+1}\}$ and $\{G_j^0, G_j^1, \ldots, G_j^{n+1}\}$ satisfy all the properties. The limit interval of the sequence $\{I_j^0, I_j^1, \ldots, I_j^n, \ldots\}$ is the entire set of reals. The limit function of the sequence $\{G_j^0, G_j^1, \ldots, G_j^n, \ldots\}$, which we denote by G_j, must be a continuous, strictly increasing CDF over the reals. Moreover, it extends G_j^0 and must also satisfy property (4) above.

Consider the median type of distribution G_j, i.e., the type τ_j such that $G_j(\tau_j) = .5$. Define the function H_j over the reals as follows:

$$H_j(w) = G_j(\tau_j + w).$$

We claim that H_j is a continuous, strictly increasing CDF over the reals, and is symmetric with respect to the origin. We need to show symmetry. For this, consider $t_1 = \tau_j - w$, $t_2 = \tau_j + w$ and $t_1' = t_2' = \tau_j$. Then, since $t_1 + t_2 = t_1' + t_2'$, we know that $\log \frac{G_j(t_1)}{1-G_j(t_1)} + \log \frac{G_j(t_2)}{1-G_j(t_2)} = \log \frac{G_j(t_1')}{1-G_j(t_1')} + \log \frac{G_j(t_2')}{1-G_j(t_2')} = 0 + 0 = 0$. Hence, it must be that $\log \frac{G_j(t_1)}{1-G_j(t_1)} = \log \frac{1-G_j(t_2)}{G_j(t_2)}$ and $G_j(t_1) = 1 - G_j(t_2)$ follows. As a result, $H_j(-w) = G_j(t_1) = 1 - G_j(t_2) = 1 - H_j(w)$, and the symmetry of H_j has been proved.

Consider now the following function defined over the positive reals:

$$O_j(w) = \frac{1 - H_j(w)}{H_j(w)}.$$

Since H_j is a continuous, strictly increasing CDF over the reals with $H_j(0) = .5$, $1 - O_j(w)$ must be a continuous, strictly increasing CDF over the positive reals with no strictly positive mass at zero. Moreover, given that G_j satisfies property (4) above, the definition of H_j and O_j guarantees that $O_j(w)O_j(z) = O_j(w + z)$ must hold for every pair of positive real values w and z. One can then reproduce the standard argument dating back to Cauchy (1821), which is described in Galambos and Kotz (1978; Theorem 1.3.1), to guarantee that O_j must be an exponential distribution, with no strictly positive mass at the origin.[23] That is, there exists $\sigma_j \in \mathbb{R}_{++}$ such that

$$1 - O_j(w) = 1 - \frac{1 - H_j(w)}{H_j(w)} = 1 - e^{-w/\sigma_j},$$

and hence, for every $w \geq 0$, it is true that $H_j(w) = \frac{1}{1+e^{-w/\sigma_j}}$. Moreover, given the symmetry of H_j with respect to the origin, for every $w < 0$, it must also be true that $H_j(w) = 1 - H_j(-w) = 1 - \frac{1}{1+e^{w/\sigma_j}} = \frac{1}{1+e^{-w/\sigma_j}}$. That is, H_j is a logistic distribution with location parameter equal to zero and scale parameter σ_j, and G_j is ordered logistic with location parameter τ_j and scale parameter σ_j. Since G_j extends G_j^0, all choices in menu j are explained by this distribution.

Consider now two menus j and j'. By our richness assumption, there exists a sequence of menus $j^0 = j, j^1, \ldots, j^k, \ldots, j^K = j'$ such that, for every $k \in \{0, \ldots, K-1\}$, $I^0_{j^k} \cap I^0_{j^{k+1}} \neq \emptyset$. Consider $t \in I^0_{j^k} \cap I^0_{j^{k+1}}$ and take $t_1 = t_2 = t'_1 = t'_2 = t$. Using the ordered-logit structure of G_{j^k} and $G_{j^{k+1}}$, it follows that they must both have a common location parameter τ and a common scale parameter σ. The recursive application of this argument shows that G_j and $G_{j'}$ must have the same common parameters τ and σ, which concludes the proof. ■

Proof of Theorem 3: We first claim that a choice function c defined over J menus has the intersection property if and only if it can be deterministically rationalized by a single-peaked utility function. For the necessity part, suppose that all choices are generated by a single-peaked utility function, with peak in $z \in \mathbb{R}$. Notice that for every menu j, I^c_j must include the peak alternative z: either because z belongs to A_j and is

[23] The property is satisfied by exponential distributions with and without strictly positive mass at zero. Since we know that O has no strictly positive mass at zero, it must be one of the latter.

thus chosen, or because z does not belong to A_j and the chosen alternative is the closest corner to z in A_j. Thus, the intersection property trivially holds. For the sufficiency part, consider the following two cases. First, there exists one menu A_j such that $c(A_j) \neq \{\underline{x}_j, \overline{x}_j\}$. Denote this chosen element by z and construct the Euclidean utility with peak at z, $-(x-z)^2$. Euclidean utilities are trivially single-peaked utilities. We show that this utility rationalizes all choices. It trivially rationalizes menu j. For any other j', it can be that $y \in A_{j'}$ or $y \notin A_{j'}$. In the former case, the intersection property requires that $c(A_{j'}) = y$, and rationalization holds. In the latter case, the intersection property requires that $c(A_{j'})$ is the corner alternative closest to y, and rationalization holds. Second, suppose that corner alternatives are selected in all menus. By the intersection property, it must be the case that whenever $c(A_j) = \underline{x}_j$ and $c(A_{j'}) = \overline{x}_{j'}$, it is $\underline{x}_j \geq \overline{x}_{j'}$. Hence, the largest right-corner choice, denoted y_1, must be lower than the smallest left-corner choice, denoted y_2. One can set the peak of the Euclidean utility to be any alternative in $[y_1, y_2]$ and this utility rationalizes all choices. The result then follows from Theorem 1. ■

Proof of Theorem 4: Consider the type-utility map E. First, notice that for every menu A_j, the non-empty interval of types $(-\infty, \underline{x}_j]$ (respectively $[\overline{x}_j, \infty)$) is mapped into utilities that select alternative $\underline{x}_j$ (respectively, alternative $\overline{x}_j$). For Corner extremeness to be satisfied, corners should have strictly positive mass. This is guaranteed by assumption (i). Second, notice that for any type t and menu A_j, if t has non-corner maximizer, it must be the case that $x_j^{E(t)}$ coincides with the peak of $E(t)$. Then, assumption (ii) guarantees that CLA holds. The application of Theorem 2 concludes the proof. ■

Proof of Theorem 5: The necessity of WARP for deterministic rationalizability with strictly convex and strictly monotone utilities is well-known in the consumption literature. Similarly, sufficiency requires to use the classical argument developed by Rose (1958) for the two-dimensional consumption setting and conclude that a choice function satisfying WARP must also satisfy the Strong Axiom of Revealed Preference and, as a result, this allows us to construct a utility function that rationalizes all choices. In particular, this utility function can be selected to be strictly quasi-concave and strictly monotone in both goods (see, e.g., Matzkin and Richter (1991)). The application of Theorem 1 concludes the proof. ■

Proof of Theorem 6: Consider the type-utility map CD. First, notice that for every menu A_j, the non-empty interval of types $(-\infty, 0]$ (respectively $[1, \infty)$) is mapped into utilities that select alternative $\underline{x}_j$ (respectively, alternative $\overline{x}_j$). Again, assumption (i) guarantees that CE holds. Second, notice that for any type t and menu A_j, if t has non-corner maximizer, it must be the case that $t \in (0, 1)$, and the maximizer $x_j^{CD(t)}$ must be such that the fraction of income spent in the first good is equal to t. Then, assumption (ii) guarantees that CLA holds. The application of Theorem 2 concludes the proof. ■

Proof of Theorem 7: Theorem 1 in Kubler, Selden and Wei (2014) shows that a deterministic choice function, over an arbitrary set of linear budget sets is rationalizable by a utility in $\mathcal{U}$ if and only if it satisfies SAREU. We can then use our Theorem 1 and the result follows. ■

Proof of Theorem 8: Consider the type-utility map EU_{CRRA}. First, notice that for every menu A_j, the non-empty interval of types $(-\infty, 0]$ is mapped into utilities that select alternative $\underline{x}_j$ and that no type is mapped into a utility that selects alternative $\overline{x}_j$. Hence, assumption (i) guarantees that CE holds. Second, non-corner maximizers are associated to types $t > 0$. Moreover, it is immediate to see that the first-order condition of a CRRA expected utility maximization over a linear budget set can be expressed as $t = -\kappa(x_j) > 0$. Thus, assumption (ii) guarantees that CLA holds. The application of Theorem 2 concludes the proof. ■

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