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SINGLE-CROSSING RANDOM UTILITY MODELS

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SINGLE-CROSSING RANDOM UTILITY MODELS

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We propose a novel model of stochastic choice: the single-crossing random utility model (SCRUM). This is a random utility model in which the collection of preferences satisfies the single-crossing property. We offer a characterization of SCRUMs based on two easy-to-check properties: the classic Monotonicity property and a novel condition, Centrality. The identified collection of preferences and associated probabilities is unique. We show that SCRUMs nest both single-peaked and single-dipped random utility models and establish a stochastic monotone comparative result for the case of SCRUMs.

KEYWORDS: Stochastic choice, single-crossing property, single-peaked preferences, single-dipped preferences, random utility models, monotone comparative statics.

1. INTRODUCTION

IN A RANDOM UTILITY MODEL (RUM), there is a collection of preferences endowed with a probability distribution. The probability of choosing an option from the set of available alternatives is described by the sum of the probability masses associated with the preferences according to which it is the best option. This is a flexible model that can be interpreted from the perspective of an individual or a group of individuals. In the first case, the different preferences may stand for different criteria, selves, or moods of the individual, with their corresponding probabilities describing their prevalence. Accordingly, individual choice here is understood as stochastic in nature.[2] In the second case, the preferences represent individuals who differ in their tastes and the probability masses describe how prevalent these preferences are in the population. Here, the probability distribution over choices describes the frequency with which the different options are selected in the population.

Under both the individual and group interpretations, RUMs crucially allow for heterogeneity in preferences. A property that has been proven to have great practical relevance when it comes to introducing structure into the modeling of preference heterogeneity is the classic single-crossing condition (see Mirrlees (1971), Spence (1974), Milgrom and Shannon (1994)). Let the set of alternatives X be ordered by $\succ$. The single-crossing condition essentially assumes that preferences can be ordered such that whenever $x \succ y$, lower preferences in the collection rank y above x, and higher preferences rank x above y. This condition has proved critical in a number of diverse, relevant settings.[3]

This paper proposes and explores a RUM in which preference heterogeneity is modeled by the single-crossing condition, that is, the collection of preferences involved in the RUM

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[2]Agranov and Ortoleva (2016) provided recent experimental evidence for the stochastic nature of individual behavior.

[3]For example, it enables the consideration of optimal taxation and market signaling problems (Mirrlees (1971), Spence (1974)); the derivation of sharp comparative statics results (Milgrom and Shannon (1994)); the solution of the preference aggregation problem (Gans and Smart (1996)); the characterization of equilibria in incomplete information games (Athey (2001)); the examination of the value of information (Persico (2000)); a means for addressing a number of political economy issues (Persson and Tabellini (2000)); and a deeper understanding of the fundamental preference parameters such as risk, time, and altruism (see Jewitt (1987), Benoît and Ok (2007), and Cox, Friedman and Sadiraj (2008), respectively).

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satisfies the single-crossing condition. Our model, therefore, endows heterogeneity with an intuitive structure and is sufficiently flexible to be applied in a wide variety of settings. We call this model the single-crossing random utility model (SCRUM).

In our first result, we characterize SCRUMs by two simple properties, Monotonicity and Centrality, thus providing testable foundations for the model. Monotonicity is a classic property known to be satisfied by all RUMs; it simply states that the probability of choosing an option from a set should not increase as more alternatives are added. Centrality is a new property that exploits the structure that the single-crossing condition brings to the model. Consider three ordered alternatives $x \succ y \succ z$. Centrality imposes that, if the central alternative y is minimally attractive, that is, if it is chosen with strictly positive probability in $\{x, y, z\}$, then the probability of choosing one of the extreme alternatives does not depend on the presence of the other. The intuition is that all of the reasons for choosing an extreme alternative in the triplet are inherited by the central alternative when the former is absent. Note that the Centrality property uses only triplets and binary sets, and hence is computationally easy to check. Theorem 1 shows that these properties characterize SCRUMs. Furthermore, the proof of Theorem 1 is constructive; the collection of preferences and their associated weights are obtained from the revealed stochastic choices. In addition and in contrast to the multiplicity of representations obtained in the case of RUMs, we show in Proposition 1 that identification in SCRUM is unique.

Section 4 is devoted to the study of some important subclasses of SCRUMs. In particular, we investigate the case in which the RUM is composed of preferences that are single-peaked (SPRUM). This is an economically relevant property with many applications, and known to be independent of single-crossing. Notably, in Corollary 1 we characterize SPRUMs by using the property of Monotonicity together with a stronger version of Centrality, which we call Strong Centrality. This immediately shows that, in terms of random choice data, SPRUM is no more than a special case of SCRUM.[4]

The concept of standard deterministic monotone comparative statics has had significant impact in the literature. Milgrom and Shannon (1994) were the first to formally introduce the single-crossing condition, with the purpose of examining the way in which optimal choices vary under different preferences. They established that the optimal alternative of a higher-ordered preference in the collection of single-crossing preferences is higher than the optimal alternative of a lower-ranked preference. In Section 5, we establish the stochastic analogue of the monotone comparative statics result using SCRUMs. This result may then be instrumental in settings where the aim is to compare the stochastic behavior of two individuals or populations. We identify a partial order on SCRUMs such that the choices of a high SCRUM first-order stochastically dominate the choices generated by a low SCRUM.

We close this section by situating our work within the relevant literature. RUMs have a long tradition in economics. In early work, Block and Marschak (1960) provided a thorough theoretical treatment but left their characterization as an open question. Subsequent contributions by Falmagne (1978), Barberà and Pattanaik (1986), and McFadden and Richter (1990) solved the challenge posed by Block and Marschak by offering a full characterization of the model. However, the nature of the characterizations is algorithmic, and hence the properties are difficult to interpret and operationalize. Notably, Gul and Pesendorfer (2006) revisited the issue, characterizing the case in which alternatives are lotteries and the collection of utilities consist of expected utility functions. By exploiting the structure of expected utility, they were able to provide a more intuitive characterization using properties that are analogous to the standard properties in the deterministic

[4]In addition, we show that RUMs composed of single-dipped preferences are also special cases of SCRUMs.

study of decision under risk. Relatedly, Lu and Saito (2016) provided easy-to-interpret foundations for the case where alternatives are consumption streams and utilities are discounted utility functions. We contribute to the study of RUMs by endowing them with a flexible structure that makes them applicable to a number of diverse settings, and, as in Gul and Pesendorfer and Lu and Saito, the special structure makes the model tractable and testable.

There are a number of recent papers studying variations of the Luce (1959) model, probably the most popular probabilistic choice model. Recently, Fudenberg and Strzalecki (2015) characterized a dynamic version of the Luce model, while Gul, Natenzon, and Pesendorfer (2014) extended it to the consideration of stochastic-attribute-based choice. Fudenberg, Iijima, and Strzalecki (2015) relaxed Luce's IIA axiom, the key axiom characterizing the Luce model, in order to consider nonlinear perturbations of utility. In Section 6, we discuss how Luce-type models are substantially different from SCRUMs.

2. BASIC DEFINITIONS

Let $(X, \succ)$ be a finite, strictly linearly ordered set of alternatives. A stochastic choice function is a mapping $\rho : X \times 2^X \setminus \emptyset \to [0, 1]$ such that, for every menu $A \in 2^X \setminus \emptyset$, the following properties hold: (i) $\rho(x, A) > 0$ implies that $x \in A$ and (ii) $\sum_{x\in A} \rho(x, A) = 1$.[5] We interpret $\rho(x, A)$ as the probability of choosing alternative x from menu A.

In a random utility model (RUM), an individual randomly entertains preferences over the alternatives, that we assume to be strict linear orders. That is, denoting by $\mathcal{P}$ the collection of all strict linear orders on X, a RUM μ is a probability distribution on $\mathcal{P}$. At the moment of choice, preference $P \in \mathcal{P}$ is realized with probability $\mu(P)$, and, from the menu of available alternatives A, the individual chooses the alternative that is best under P. We denote this by $b_P(A)$, that is, $b_P(A)Px$ for every $x \in A \setminus \{b_P(A)\}$. The RUM stochastic choice function is therefore $\rho_\mu(x, A) = \sum_{P\in\mathcal{P}:x=b_P(A)} \mu(P)$.

Given a RUM μ, we denote its support by $\mathcal{P}_\mu$, that is, $\mathcal{P}_\mu = \{P \in \mathcal{P} : \mu(P) > 0\}$. We say that the RUM μ is single-crossing (SCRUM) if the support of μ can be ordered as $\mathcal{P}_\mu = \{P_1, \dots, P_T\}$ to satisfy the single-crossing condition. Namely, for every $x \succ y$ and every $s > t$, whenever $xP_t y$, then $xP_s y$.[6] That is, P_s is more aligned with $\succ$ than P_t. Put differently, the single-crossing condition states that the ranking of any pair of alternatives reverses at most once in the ordered collection of preferences, with low preferences opting for the low alternative, and high preferences opting for the high alternative. A SCRUM stochastic choice function is defined accordingly.

Settings fitting these assumptions abound and cover all the main preference parameters of interest. The following are examples of SCRUM stochastic choice functions, illustrating this point in three archetypical settings: risk-taking behavior, inter-temporal decision making, and the political economy of taxation.[7]

EXAMPLE 1: An investor has a monetary endowment m and has to decide the amount $x \in \{0, 1, \dots, m\}$ to keep in his endowment and the amount $m - x$ to invest in a risky asset

[5]The results of this paper hold when considering domains that include all menus with size 2 and 3.

[6]Note that, given the order $\succ$ on X, there is at most one order of the support compatible with single-crossing. In most applications, the order of alternatives is well-known by the analyst and thus, we have taken it as given, thereby easing the exposition. Our model can be easily extended to consider an endogenous unknown order over the alternatives, which can be recovered from the choice data. Details available upon request.

[7]More details showing the relationship of these models with SCRUM are available upon request.

that yields $k > 0$ with probability p and 0 otherwise.[8] Let the investor behave stochastically, entertaining a probability distribution over a collection of preferences represented by CRRA or CARA utilities ordered by the risk-aversion coefficient. The choices of the investor are a SCRUM stochastic choice function on the amount kept in the endowment.

EXAMPLE 2: Given the gross interest rate $1 + r$, a consumer has to decide the amount of her income $x \in \{0, 1, \dots, m\}$ saved for future consumption. That is, consumption is $m - x$ today and $x(1 + r)$ in the future.[9] Let the individual randomly entertain a set of exponential discounting preferences with, for example, a log curvature over monetary payoffs. The choices are a SCRUM stochastic choice function on the amount saved.

EXAMPLE 3: A voter has to choose from a finite set of possible income tax rates. Persson and Tabellini (2000, Example 1) showed that, whenever preferences over consumption and leisure are quasi-linear, we can represent preferences over taxation as a single-crossing collection of preferences on the productivity of individuals. It follows then that if society is heterogeneous with respect to productivity levels, the distribution of choices within the society can be understood as a SCRUM stochastic choice function.

3. A CHARACTERIZATION OF SINGLE-CROSSING RANDOM UTILITY MODELS

In this section, we introduce two basic properties of stochastic choice functions that characterize SCRUMs. The first property, Monotonicity, is a classic condition in the study of stochastic choice already discussed by Block and Marschak (1960) in their analysis of general RUMs. It states that the probability of selecting an option does not increase when more alternatives are added to the menu.

Monotonicity (MON). If $B \subseteq A$, then $\rho(x, A) \leq \rho(x, B)$.

The second property, Centrality, uses the structure brought to RUMs by the single-crossing condition. It states that in a triplet, when the intermediate alternative is minimally attractive to the decision-maker in the sense that it is chosen with strictly positive probability, the two extreme alternatives become mutually irrelevant. That is, the choice probability of one extreme alternative is independent of the presence of the other. Intuitively, given the ordered structure of the alternatives, the arguments for choosing an extreme alternative from the triplet are inherited by the central alternative when the extreme alternative is absent. Centrality imposes this only when the central alternative is not completely unattractive to the decision-maker.

Centrality (CEN). If $x \succ y \succ z$ and $\rho(y, \{x, y, z\}) > 0$, then $\rho(x, \{x, y\}) = \rho(x, \{x, y, z\})$ and $\rho(z, \{y, z\}) = \rho(z, \{x, y, z\})$.

Theorem 1 shows that these properties are necessary and sufficient for SCRUMs. Therefore, given the simplicity of the properties, Theorem 1 shows that SCRUMs are easily testable.

THEOREM 1: *A stochastic choice function ρ satisfies MON and CEN if and only if ρ is a SCRUM stochastic choice function.*

The proof of Theorem 1 explicitly constructs a SCRUM that explains the revealed choice data satisfying the two properties. The construction is intuitive and easy to implement in practice, as it is based exclusively on the revealed stochastic choices in the binary

[8] For an influential study along these lines, see, for example, Gneezy and Potters (1997).
[9] For a recent study of a similar model, see Andreoni and Sprenger (2012).

sets. First, define P_1 as follows: for every pair $x \succ y$, let yP_1x whenever $\rho(y, \{x, y\}) > 0$, and xP_1y otherwise. Construct P_2 from P_1 by reversing the ranking of exactly the pair(s) $x \succ y$ with the lowest nonzero binary choice probability $\rho(y, \{x, y\})$. Once P_t is constructed, identify again the pair(s) $x \succ y$, not considered previously, with the lowest binary choice probabilities $\rho(y, \{x, y\})$, and construct P_{t+1} from P_t by reversing the order of such pair(s). Proceed in this ordered way until all the binary comparisons are exhausted.[10] Notice, then, that by construction we have a single-crossing collection of preferences. With respect to the probability masses, define $\mu(P_1)$ as the lowest nonzero binary choice probability. Given $\mu(P_1), \dots, \mu(P_t)$ and the lowest choice probability in step $t+1$, $\rho(y, \{x, y\})$, define $\mu(P_{t+1}) = \rho(y, \{x, y\}) - \sum_{s=1}^{t} \mu(P_s)$. Proceed in this way until the last preference and assign to it the remaining mass $\mu(P_T) = 1 - \sum_{s=1}^{T-1} \mu(P_s)$. It is therefore the case that all the binary probabilities are intuitively explained by the constructed SCRUM, and the proof shows how MON and CEN extend this result to larger menus of alternatives.

Notably, another important aspect of the characterization result is that the representation is unique. That is, the SCRUM described is the unique SCRUM that generates the stochastic choice function. This is in sharp contrast to unrestricted RUM stochastic choice functions, which are well-known to admit multiple RUM representations (see Fishburn (1998)). This is stated formally in Proposition 1. Moreover, the proof of the proposition shows that choices over pairs are sufficient for identifying SCRUM.

PROPOSITION 1: *Let μ and μ' be SCRUMs such that $\rho_\mu = \rho_{\mu'}$. Then, $\mu = \mu'$.*

4. SINGLE-PEAKED AND SINGLE-DIPPED RANDOM UTILITY MODELS

We discuss here how the class of SCRUM stochastic choice functions covers a broad class of behaviors and encompasses other important heterogeneous preference structures. We start with the analysis of the well-known single-peaked preferences. Formally, we say that the RUM μ is single-peaked (SPRUM) if every preference P in $\mathcal{P}_\mu$ satisfies the condition that whenever $y \succ x \succ b_P(X)$ or $b_P(X) \succ x \succ y$, then xPy. That is, every preference has a unique peak and alternatives that are closer to the peak are more preferred. Notice that single-peakedness is related to the (strict) convexity of preferences or the (strict) quasi-concavity of utility functions representing these preferences. That is, if all the alternatives in X are ordered on a line, single-peakedness is fulfilled by the restriction to X of any strictly convex preference defined on that line.

Single-peakedness and single-crossing are distinct properties. The former establishes a condition for each of the preferences in a collection, while the latter establishes a condition that links all the preferences of the collection. As the literature has clearly shown, these properties turn out to be independent and consequently, we can find RUMs that are single-crossing but not single-peaked and vice versa. The following examples illustrate this.

EXAMPLE 4: Let $X = \{x, y, z\}$ with $x \succ y \succ z$ and μ_1 be such that $\mathcal{P}_{\mu_1} = \{P_1, P_2\}$, with xP_1zP_1y and xP_2yP_2z. μ_1 is clearly single-crossing, since x is always better than y and z, and y is better than z only for P_2. However, μ_1 is not single-peaked, since for P_1, x is maximal and y is below z.

[10]Notice that if all the binary probabilities considered in this exercise are strictly positive, then $x \succ y$ if and only if yP_1x. Similarly, if all probabilities are different from 1, then $x \succ y$ if and only if xP_Ty. The proof also helps to explain how P_1 is always a strict linear order and how only those alternatives that are consecutive at a given step can be reversed.

EXAMPLE 5: Let $X = \{w, x, y, z\}$ with $w \succ x \succ y \succ z$ and μ_2 such that $\mathcal{P}_{\mu_2} = \{P_1, P_2\}$, with $yP_1xP_1wP_1z$ and $xP_2yP_2zP_2w$. μ_2 is clearly single-peaked. It is not single-crossing, however, since P_1 should precede P_2 to accommodate the order of alternatives x and y, but alternatives w and z force P_2 to precede P_1.

Despite the fact that single-peakedness and single-crossing are independent properties, we show that the class of SPRUM stochastic choice functions is but a subset of the class of SCRUM stochastic choice functions. We show this by providing a characterization of SPRUMs that uses a strengthening of CEN.

Strong Centrality (SCEN). If $x \succ y \succ z$, then $\rho(x, \{x, y\}) = \rho(x, \{x, y, z\})$ and $\rho(z, \{y, z\}) = \rho(z, \{x, y, z\})$.

COROLLARY 1: *A stochastic choice function ρ satisfies MON and SCEN if and only if ρ is a SPRUM stochastic choice function.*

Corollary 1 implies that any SPRUM stochastic choice function is also a SCRUM stochastic choice function and, consequently, SPRUM has no empirical content beyond that of SCRUM. In other words, any SPRUM that fails to be single-crossing generates a stochastic choice function that can be replicated by a SCRUM. Indeed, the proof of Corollary 1 shows that, for this task, we can choose (uniquely) a RUM that is both single-crossing and single-peaked. To illustrate, consider the RUM μ_2 in Example 5, and recall that it is single-peaked, but not single-crossing. We now construct an alternative RUM that is both single-crossing and single-peaked, and that generates the same stochastic choices as μ_2. The exercise is trivial if $\mu_2(P_1) = \mu_2(P_2) = \frac{1}{2}$ because, in this case, we can just swap alternatives w and z in both preferences P_1 and P_2 to obtain $yP'xP'zP'w$ and $xP''yP''wP''z$. It is immediate to see that we can reproduce the stochastic choices by simply assigning probability $\frac{1}{2}$ to each preference in the single-crossing and single-peaked collection of preferences $\{P', P''\}$. If one of the preferences in the support of μ_2, say P_1, has a larger probability than the other, the stochastic choices can be generated by the single-crossing and single-peaked collection of preferences $\{P', P_1, P''\}$ with probabilities $\mu_2(P_2)$, $\mu_2(P_1) - \mu_2(P_2)$, and $\mu_2(P_2)$, respectively.

We now show that the class inversely related to SPRUM, the single-dipped RUM, represents another interesting subclass of SCRUM stochastic choice functions. Formally, denote by $w_P(X)$ the worst alternative in X according to preference P. Then, we say that the RUM μ is single-dipped (SDRUM) if every preference P in $\mathcal{P}_\mu$ satisfies: whenever $y \succ x \succ w_P(X)$ or $w_P(X) \succ x \succ y$, we have that yPx. It is immediate to see that we can characterize the class of SDRUM stochastic choice functions using MON and the following strengthening of CEN.[11]

Extremality (EXT). If $x \succ y \succ z$, then $\rho(y, \{x, y, z\}) = 0$.

COROLLARY 2: *A stochastic choice function ρ satisfies MON and EXT if and only if ρ is a SDRUM stochastic choice function.*

5. STOCHASTIC MONOTONE COMPARATIVE STATICS

The single-crossing condition has been instrumental in establishing the so-called monotone comparative statics results. These are results that formalize the relationship between

[11]Extremality is an ordinal analogue of the Extremeness axiom from Gul and Pesendorfer (2006).

preference parameters and optimal choices. Let us briefly describe the classic result. Suppose that $\{P_1, \dots, P_T\}$ is an ordered collection of preferences satisfying the single-crossing condition and consider $s > t$. Then, for every menu A, either $b_{P_s}(A) \succ b_{P_t}(A)$ or $b_{P_s}(A) = b_{P_t}(A)$.

In this section, we address the question of monotone comparative statics results from the stochastic perspective of SCRUMs. To do this, we need to introduce an order on SCRUMs and another one on stochastic choices. For the former, as in the proof of Proposition 1, we denote by $\tilde{\mu}^{-1}$ the inverse of the c.d.f. function of a SCRUM μ. That is, $\tilde{\mu}^{-1}(\theta)$ is the preference P_t for which the cumulated mass of the preferences P_s with $s < t$ is strictly smaller than θ, while the cumulated mass of preferences P_s with $s \leq t$ is weakly larger than θ. We can then say that μ is higher than ν if, for every $\theta \in (0, 1]$, whenever $x \succ y$ and $x\tilde{\nu}^{-1}(\theta)y$, then $x\tilde{\mu}^{-1}(\theta)y$. In other words, the cumulative preference of μ is always more aligned with $\succ$ than that of ν. We rank the stochastic choices by the standard first-order stochastic dominance criterion. That is, if we enumerate the alternatives in a menu A as $a_{|A|} \succ \cdots \succ a_2 \succ a_1$, we say that ρ_μ first-order stochastically dominates ρ_ν for the menu A if, for every $i \in \{1, 2, \dots, |A|\}$, $\sum_{j=i}^{|A|} \rho_\mu(a_j, A) \geq \sum_{j=i}^{|A|} \rho_\nu(a_j, A)$. In other words, ρ_μ assigns larger probabilities of choice to higher alternatives than ρ_ν does.

PROPOSITION 2: *SCRUM μ is higher than SCRUM ν if and only if ρ_μ first-order stochastically dominates ρ_ν for every menu A.*

Notice, first, that Proposition 2 encompasses the classic result whenever both μ and ν are degenerate, that is, whenever they assign mass 1 to a unique preference. Suppose now that both SCRUMs have the same support, that is, $\mathcal{P}_\mu = \mathcal{P}_\nu$. In this case, the idea of a higher SCRUM is equivalent to stochastic dominance on the cdfs over the common single-crossing support. That is, our result states that the c.d.f. of μ first-order stochastically dominates the c.d.f. of ν if and only if, for every menu A, the choices of the former first-order stochastically dominate the choices of the latter. This has important practical implications, as the following example illustrates.

EXAMPLE 6: Let $X = \{x, y, z\}$ with $x \succ y \succ z$. Consider two SCRUMs μ and ν with the same ordered support, $\mathcal{P}_\mu = \mathcal{P}_\nu = \{P_1, \dots, P_4\}$, where zP_1yP_1x, zP_2xP_2y, xP_3zP_3y, and xP_4yP_4z, and masses $\mu = (0.06, 0.24, 0.23, 0.47)$ and $\nu = (0.17, 0.42, 0.2, 0.21)$. Clearly, μ is higher than ν, and hence Proposition 2 implies that ρ_μ first-order stochastically dominates ρ_ν.

These SCRUMs follow from considering two individuals in the following risk setting. Alternatives x, y, and z are equiprobable lotteries with payoffs $(30, 30)$, $(20, 40)$, and $(15, 50)$, respectively. The individuals entertain two different CRRA parameters, 0.7 for μ and 0.4 for ν, subject to a logistic perturbation.[12] Thus, in this context of decision-making under risk, and for the case of a binary comparison of lotteries, our stochastic monotone comparative result can be read as follows: the more risk-averse agent μ chooses safer alternatives more often than does the less risk-averse agent ν.

Suppose, in contrast, that the two individuals entertain the same CRRA levels of 0.7 and 0.4, but the perturbation is now on the utility values rather than on the space of

[12] In both cases, the scale of the logistic distribution is 0.25.

CRRA parameters.[13] This case gives rise to two RUMs with support on all possible preferences $\mathcal{P} = \{P_1, P_2, P, P', P_3, P_4\}$, where $yPzPx$ and $yP'xP'z$. Specifically, they are $\mu_1 = (0.11, 0.18, 0.11, 0.17, 0.22, 0.21)$ and $\mu_2 = (0.17, 0.35, 0.10, 0.06, 0.25, 0.08)$. Obviously, these RUMs fail to be SCRUMs and Proposition 2 cannot be applied. However, the standard approach in the literature is to consider the first individual as more risk-averse than the second, given the order on the CRRA parameters. It is then that problems arise. Consider alternatives $x \succ y$, for instance. The allegedly more risk-averse agent chooses the safer alternative x with a 0.61 probability, less often than does the allegedly less risk-averse agent, who does so with probability 0.67. For general results on this problem, see Apesteguia and Ballester (2016).

Proposition 2 also establishes comparative statics for the more general case in which the SCRUMs do not share the same support. Notice that, in this case, $\mathcal{P}_\mu \cup \mathcal{P}_\nu$ may fail to be single-crossing, thus making it difficult to compare the cdfs of the two individuals. We can, however, invert these cdfs and check whether, for every $\theta \in (0, 1]$, the associated preference for μ is higher than that of ν. When this is the case, Proposition 2 guarantees first-order stochastic dominance in terms of choices.

6. FINAL REMARKS

In this paper, we have proposed and studied a new stochastic choice model that can be used in a wide variety of settings, specifically those in which the single-crossing property applies. We have also shown that the model is easily testable in practice. This is in contrast to the traditional characterization of RUMs which involves the complex higher-order Block–Marschak inequalities. By exploiting the structure that the single-crossing condition brings to RUMs, our approach is able to do with such complicated properties.[14]

We close this paper by commenting on the relationship between SCRUMs and two classical properties and, thereby, on the relationship between SCRUMs and other stochastic choice models. Let us start with Luce's well-known Independence of Irrelevant Alternatives axiom (Luce-IIA). This property essentially requires that the choice ratio between two alternatives is independent of other available alternatives, that is, $\frac{\rho(x,A)}{\rho(y,A)} = \frac{\rho(x,B)}{\rho(y,B)}$, whenever $\rho(y, A) > 0$ and $\rho(y, B) > 0$. It can be seen that Luce-IIA is in direct conflict with CEN by simply considering three ordered alternatives $x \succ y \succ z$. Notice that, when eliminating x from $\{x, y, z\}$, $\rho(x, \{x, y, z\})$ must be completely inherited by y according to CEN, whenever y is chosen with strictly positive probability, but, by Luce-IIA, must be distributed proportionally between y and z. Contrary to CEN, Luce-IIA ignores the structure of heterogeneity. As mentioned in the Introduction, in recent years Luce's model has been extended in many directions by considering relaxations of Luce-IIA, and therefore, all these models differ in structure from SCRUMs.

We conclude with some remarks on stochastic transitivity, another cornerstone concept for understanding stochastic choice models. It is well-known that RUMs fail to satisfy even the weakest version, namely, $\rho(x, \{x, y\}) > \frac{1}{2}$, and that $\rho(y, \{y, z\}) > \frac{1}{2}$ implies that $\rho(x, \{x, z\}) \geq \frac{1}{2}$. For purposes of illustration, consider the Condorcet cycle defined by three equiprobable preferences P_1, P_2, and P_3 over three alternatives x, y, and z, with

[13] The most standard approach considers i.i.d. extreme type I additive perturbations for all available alternatives, giving rise to the well-known logit model. We use this distribution in our example, with a scale parameter of 0.25.

[14] Future research should study the generalization to partial orders of the orders imposed on X and on $\mathcal{P}_\mu$, and their consequences on the tractability of the higher-order Block–Marschak inequalities.

xP_1yP_1z, yP_2zP_2x, and zP_3xP_3y. It is clear that the RUM probabilities are $\rho(x, \{x, y\}) = \rho(y, \{y, z\}) = \frac{2}{3} > \frac{1}{2}$ but $\rho(x, \{x, z\}) = \frac{1}{3}$. However, notice that this example is not single-crossing. Indeed, the single-crossing condition implies that Condorcet cycles of this kind cannot occur.[15] This implies, in turn, that the Manzini and Mariotti (2014) model is not a special case of our model, because theirs is compatible with this type of Condorcet cycles. To see that SCRUMs are not a special case of their model, simply notice that their i-Asymmetry property is in direct conflict with Centrality.

PROOFS

PROOF OF THEOREM 1: We divide the proof into a series of claims. Claims 1 to 6 prove the sufficiency of the axioms. Claim 7 proves their necessity.

CLAIM 1: *Let ρ satisfy MON and CEN and $x \succ y \succ z$. Then, $\rho(z, \{y, z\}) \leq \rho(z, \{x, z\}) \leq \rho(y, \{x, y\})$ whenever $\rho(y, \{x, y, z\}) > 0$ and $\rho(z, \{y, z\}) \geq \rho(z, \{x, z\}) \geq \rho(y, \{x, y\})$ whenever $\rho(y, \{x, y, z\}) = 0$.*

PROOF: Suppose, first, that $\rho(y, \{x, y, z\}) > 0$. CEN implies that $\rho(z, \{y, z\}) = \rho(z, \{x, y, z\})$, while MON implies that $\rho(z, \{x, y, z\}) \leq \rho(z, \{x, z\})$, leading to $\rho(z, \{y, z\}) \leq \rho(z, \{x, z\})$. Also, MON guarantees that $\rho(z, \{x, z\}) = 1 - \rho(x, \{x, z\}) \leq 1 - \rho(x, \{x, y, z\})$, while CEN implies that $\rho(x, \{x, y, z\}) = \rho(x, \{x, y\})$, guaranteeing that $\rho(z, \{x, z\}) \leq 1 - \rho(x, \{x, y\}) = \rho(y, \{x, y\})$. Next, let $\rho(y, \{x, y, z\}) = 0$. It must then be that $\rho(x, \{x, y, z\}) + \rho(z, \{x, y, z\}) = 1 = \rho(x, \{x, z\}) + \rho(z, \{x, z\})$ and MON implies both that $\rho(x, \{x, y, z\}) = \rho(x, \{x, z\})$ and that $\rho(z, \{x, y, z\}) = \rho(z, \{x, z\})$. The use of MON guarantees that $\rho(z, \{y, z\}) \geq \rho(z, \{x, y, z\}) = \rho(z, \{x, z\})$ and $\rho(z, \{x, z\}) = 1 - \rho(x, \{x, z\}) = 1 - \rho(x, \{x, y, z\}) \geq 1 - \rho(x, \{x, y\}) = \rho(y, \{x, y\})$. *Q.E.D.*

CLAIM 2: *Let ρ satisfy MON and CEN, and $A = \{a_1, a_2, \dots, a_{|A|}\}$ be such that $a_{|A|} \succ \cdots \succ a_2 \succ a_1$ with $\rho(a_i, \{a_{i-1}, a_i, a_{i+1}\}) > 0$ for every $i \in \{2, \dots, |A| - 1\}$. Then, $\rho(a_1, A) = \rho(a_1, \{a_1, a_2\})$, $\rho(a_i, A) = \rho(a_i, \{a_{i-1}, a_i, a_{i+1}\})$, $2 \leq i \leq |A| - 1$, and $\rho(a_{|A|}, A) = \rho(a_{|A|}, \{a_{|A|-1}, a_{|A|}\})$.*

PROOF: The result is trivial if $|A| < 3$. Otherwise, repeated use of CEN guarantees that $1 = \rho(a_1, \{a_1, a_2, a_3\}) + \rho(a_2, \{a_1, a_2, a_3\}) + \rho(a_3, \{a_1, a_2, a_3\}) = \rho(a_1, \{a_1, a_2\}) + \rho(a_2, \{a_1, a_2, a_3\}) + \rho(a_3, \{a_2, a_3\}) = \rho(a_1, \{a_1, a_2\}) + \rho(a_2, \{a_1, a_2, a_3\}) + (1 - \rho(a_2, \{a_2, a_3\})) = \rho(a_1, \{a_1, a_2\}) + \rho(a_2, \{a_1, a_2, a_3\}) + (1 - \rho(a_2, \{a_2, a_3, a_4\})) = \rho(a_1, \{a_1, a_2\}) + \rho(a_2, \{a_1, a_2, a_3\}) + \rho(a_3, \{a_2, a_3, a_4\}) + \rho(a_4, \{a_2, a_3, a_4\}) = \cdots = \rho(a_1, \{a_1, a_2\}) + \sum_{i=2}^{|A|-1} \rho(a_i, \{a_{i-1}, a_i, a_{i+1}\}) + \rho(a_{|A|}, \{a_{|A|-2}, a_{|A|-1}, a_{|A|}\}) = \rho(a_1, \{a_1, a_2\}) + \sum_{i=2}^{|A|-1} \rho(a_i, \{a_{i-1}, a_i, a_{i+1}\}) + \rho(a_{|A|}, \{a_{|A|-1}, a_{|A|}\})$. By MON, it must be the case that $\rho(a_1, \{a_1, a_2\}) + \sum_{i=2}^{|A|-1} \rho(a_i, \{a_{i-1}, a_i, a_{i+1}\}) + \rho(a_{|A|}, \{a_{|A|-1}, a_{|A|}\}) \geq \sum_{i=1}^{|A|} \rho(a_i, A) = 1$ and hence the result follows. *Q.E.D.*

CLAIM 3: *Let ρ satisfy MON and CEN. Given a menu A, denote the set $\{x \in A : \rho(x, A) > 0\}$ by $\bar{A}$ and its elements by $\bar{a}_{|\bar{A}|} \succ \cdots \succ \bar{a}_2 \succ \bar{a}_1$. Then:*

(1) *For every A' such that $\bar{A} \subseteq A' \subseteq A$ and for every $x \in A'$, $\rho(x, A) = \rho(x, A')$.*
(2) *For every $x \in A \setminus \bar{A}$ such that $\bar{a}_1 \succ x$, $\rho(x, \{x, \bar{a}_1\}) = 0$.*

[15] Details available upon request.

(3) *For every* $x \in A \setminus \bar{A}$ *such that* $x \succ \bar{a}_{|\bar{A}|}$, $\rho(x, \{\bar{a}_{|\bar{A}|}, x\}) = 0$.
(4) *For every* $x \in A \setminus \bar{A}$ *such that* $\bar{a}_{i+1} \succ x \succ \bar{a}_i$, $\rho(x, \{\bar{a}_i, x, \bar{a}_{i+1}\}) = 0$.

PROOF: For the first part, simply notice that $\sum_{x\in A'} \rho(x, A) = 1 = \sum_{x\in A'} \rho(x, A')$ and hence, the result follows immediately from MON. For the second and third parts, notice that MON implies that every alternative $\bar{a}_i \in \bar{A}$ is chosen with positive probability in every triplet of $\bar{A} \cup \{x\}$ and then menu $\bar{A} \cup \{x\}$ is bound by the conditions described in Claim 2. Hence $\rho(x, \bar{A} \cup \{x\})$ is equal to $\rho(x, \{x, \bar{a}_1\})$ in the second part, and to $\rho(x, \{\bar{a}_{|\bar{A}|}, x\})$ in the third part. By the first part, these values are equal to zero. For the last part, suppose, by way of contradiction, that there exists $x \notin \bar{A}$ such that $\bar{a}_{i+1} \succ x \succ \bar{a}_i$ and $\rho(x, \{\bar{a}_i, x, \bar{a}_{i+1}\}) > 0$. This again guarantees that menu $\bar{A} \cup \{x\}$ is bound by the conditions described in Claim 2, thereby implying that $\rho(x, \bar{A} \cup \{x\}) = \rho(x, \{\bar{a}_i, x, \bar{a}_{i+1}\}) > 0$. This contradicts the first part, and the claim is proved. *Q.E.D.*

CLAIM 4: *Let* ρ *satisfy MON and CEN. Define, for every* $\theta \in (0, 1]$, *the binary relation* $xP_\theta y \Leftrightarrow [y \succ x$ *and* $\rho(x, \{x, y\}) \geq \theta]$ *or* $[x \succ y$ *and* $\rho(y, \{x, y\}) < \theta]$. *Then,* P_θ *is a strict linear order.*

PROOF: Completeness and asymmetry of P_θ are immediate. To prove transitivity, let $xP_\theta y$ and $yP_\theta z$. We prove that $xP_\theta z$ by using Claim 1 accordingly:
(1) $z \succ y \succ x$. Notice that $xP_\theta y$ and $yP_\theta z$ imply, respectively, $\theta \leq \rho(x, \{x, y\})$ and $\theta \leq \rho(y, \{y, z\})$ and hence $\theta \leq \min\{\rho(x, \{x, y\}), \rho(y, \{y, z\})\} \leq \rho(x, \{x, z\})$, thus implying that $xP_\theta z$.
(2) $y \succ z \succ x$. Then, it is the case that $\rho(z, \{y, z\}) < \theta \leq \rho(x, \{x, y\})$ and it must be the case that $\rho(z, \{y, z\}) \leq \rho(x, \{x, y\}) \leq \rho(x, \{x, z\})$, thus implying that $\theta \leq \rho(x, \{x, z\})$ or $xP_\theta z$.
(3) $z \succ x \succ y$. Then, $\rho(y, \{x, y\}) < \theta \leq \rho(y, \{y, z\}) \leq \rho(x, \{x, z\})$ and $xP_\theta z$.
(4) $x \succ z \succ y$. Then, $\rho(z, \{x, z\}) \leq \rho(y, \{x, y\}) < \theta \leq \rho(y, \{y, z\})$ and, given that $x \succ z$, we again obtain that $xP_\theta z$.
(5) $y \succ x \succ z$. Then, $\rho(z, \{x, z\}) \leq \rho(z, \{y, z\}) < \theta \leq \rho(x, \{x, y\})$ and $xP_\theta z$.
(6) $x \succ y \succ z$. Then, $\rho(z, \{x, z\}) \leq \max\{\rho(z, \{y, z\}), \rho(y, \{x, y\})\} < \theta$ and $xP_\theta z$. *Q.E.D.*

CLAIM 5: *Let* ρ *satisfy MON and CEN. Consider the RUM* μ *that assigns, to any* $P \in \mathcal{P}$, *the value* $\mu(P) = \mathcal{L}\{\theta : P = P_\theta\}$, *where* P_θ *is defined as in Claim 4 and* $\mathcal{L}$ *is the Lebesgue measure. Then,* $\rho = \rho_\mu$.

PROOF: We prove that, for every $x \in \bar{A}$, $\rho(x, A) = \rho_\mu(x, A)$ and thus obtain that $\sum_{x\in\bar{A}} \rho(x, A) = 1 = \sum_{x\in\bar{A}} \rho_\mu(x, A)$ and, since ρ_μ is a stochastic choice function, it must assign mass zero to any alternative $x \notin \bar{A}$, thus concluding the proof of the claim. Consider then $x \in \bar{A}$. By the first part of Claim 3, we know that $\rho(x, A) = \rho(x, \bar{A})$. We now consider three cases, depending on the position of x in menu $\bar{A}$, borrowing the notation of Claim 4.
(1) Let $x = \bar{a}_1$. Since menu $\bar{A}$ is bound by the conditions described in Claim 2, it is the case that $\rho(\bar{a}_1, \bar{A}) = \rho(\bar{a}_1, \{\bar{a}_1, \bar{a}_2\})$. Hence, given the construction of μ, it is sufficient to show that $\bar{a}_1 = b_{P_\theta}(A)$ if and only if $\theta \leq \rho(\bar{a}_1, \{\bar{a}_1, \bar{a}_2\})$. For every $\theta \leq \rho(\bar{a}_1, \{\bar{a}_1, \bar{a}_2\})$, it is clearly the case that $\bar{a}_1 P_\theta \bar{a}_2$, and for any $\bar{a}_j$ with $j > 2$, the first part of Claim 1 also guarantees that $\rho(\bar{a}_1, \{\bar{a}_1, \bar{a}_j\}) \geq \rho(\bar{a}_1, \{\bar{a}_1, \bar{a}_2\})$, and hence, $\bar{a}_1 P_\theta \bar{a}_j$. Let $y \in A \setminus \bar{A}$. If $\bar{a}_1 \succ y$, then

the second part of Claim 3 guarantees that $\rho(y, \{y, \bar{a}_1\}) = 0$, thereby implying that $\bar{a}_1 P_\theta y$ for every θ. If $\bar{a}_2 \succ y \succ \bar{a}_1$, the fourth part of Claim 3 guarantees that $\rho(y, \{\bar{a}_1, y, \bar{a}_2\}) = 0$, and Claim 1 implies that $\rho(\bar{a}_1, \{\bar{a}_1, y\}) \geq \rho(\bar{a}_1, \{\bar{a}_1, \bar{a}_2\})$, and thus, $\bar{a}_1 P_\theta y$ for every $\theta \leq \rho(\bar{a}_1, \{\bar{a}_1, \bar{a}_2\})$. Finally, if $y \succ \bar{a}_2 \succ \bar{a}_1$, MON implies that $\rho(\bar{a}_2, \{\bar{a}_1, \bar{a}_2, y\}) > 0$ and Claim 1 again guarantees that $\rho(\bar{a}_1, \{\bar{a}_1, y\}) \geq \rho(\bar{a}_1, \{\bar{a}_1, \bar{a}_2\})$, leading to $\bar{a}_1 P_\theta y$ for every $\theta \leq \rho(\bar{a}_1, \{\bar{a}_1, \bar{a}_2\})$. Thus, $\bar{a}_1 = b_{P_\theta}(A)$ whenever $\theta \leq \rho(\bar{a}_1, \{\bar{a}_1, \bar{a}_2\})$. Clearly, if $\theta > \rho(\bar{a}_1, \{\bar{a}_1, \bar{a}_2\})$, $\bar{a}_2 P_\theta \bar{a}_1$ and hence, $\bar{a}_1 \neq b_{P_\theta}(A)$. This concludes the proof of this case.

(2) Let $x = \bar{a}_{|\bar{A}|}$. It is the case that $\rho(\bar{a}_{|\bar{A}|}, \bar{A}) = \rho(\bar{a}_{|\bar{A}|}, \{\bar{a}_{|\bar{A}|-1}, \bar{a}_{|\bar{A}|}\}) = 1 - \rho(\bar{a}_{|\bar{A}|-1}, \{\bar{a}_{|\bar{A}|-1}, \bar{a}_{|\bar{A}|}\})$. Given the construction of μ, it is sufficient to show that $\bar{a}_{|\bar{A}|} = b_{P_\theta}(A)$ if and only if $\theta > \rho(\bar{a}_{|\bar{A}|-1}, \{\bar{a}_{|\bar{A}|-1}, \bar{a}_{|\bar{A}|}\})$. For every $\theta \leq \rho(\bar{a}_{|\bar{A}|-1}, \{\bar{a}_{|\bar{A}|-1}, \bar{a}_{|\bar{A}|}\})$, we have $\bar{a}_{|\bar{A}|-1} P_\theta \bar{a}_{|\bar{A}|}$, and thus $\bar{a}_{|\bar{A}|} \neq b_{P_\theta}(A)$. For every $\theta > \rho(\bar{a}_{|\bar{A}|-1}, \{\bar{a}_{|\bar{A}|-1}, \bar{a}_{|\bar{A}|}\})$, it is clearly the case that $\bar{a}_{|\bar{A}|} P_\theta \bar{a}_{|\bar{A}|-1}$, and also, for any $\bar{a}_j$ with $j < |\bar{A}| - 1$, Claim 1 guarantees that $\rho(\bar{a}_j, \{\bar{a}_j, \bar{a}_{|\bar{A}|}\}) \leq \rho(\bar{a}_{|\bar{A}|-1}, \{\bar{a}_{|\bar{A}|-1}, \bar{a}_{|\bar{A}|}\})$, and hence it is also the case that $\bar{a}_{|\bar{A}|} P_\theta \bar{a}_j$. Now, let $y \in A \setminus \bar{A}$. If $\bar{a}_{|\bar{A}|} \succ \bar{a}_{|\bar{A}|-1} \succ y$, MON implies that $\rho(\bar{a}_{|\bar{A}|-1}, \{y, \bar{a}_{|\bar{A}|-1}, \bar{a}_{|\bar{A}|}\}) > 0$ and Claim 1 guarantees that $\rho(y, \{y, \bar{a}_{|\bar{A}|}\}) \leq \rho(\bar{a}_{|\bar{A}|-1}, \{\bar{a}_{|\bar{A}|-1}, \bar{a}_{|\bar{A}|}\}) < \theta$, leading to $\bar{a}_{|\bar{A}|} P_\theta y$. If $\bar{a}_{|\bar{A}|} \succ y \succ \bar{a}_{|\bar{A}|-1}$, the fourth part of Claim 3 guarantees that $\rho(y, \{\bar{a}_{|\bar{A}|-1}, y, \bar{a}_{|\bar{A}|}\}) = 0$ and Claim 1 implies that $\rho(y, \{y, \bar{a}_{|\bar{A}|}\}) \leq \rho(\bar{a}_{|\bar{A}|-1}, \{\bar{a}_{|\bar{A}|-1}, \bar{a}_{|\bar{A}|}\}) < \theta$, leading, again, to $\bar{a}_{|\bar{A}|} P_\theta y$. Finally, if $y \succ \bar{a}_{|\bar{A}|}$, the third part of Claim 3 implies that $\rho(y, \{y, \bar{a}_{|\bar{A}|}\}) = 0$, which leads to $\bar{a}_{|\bar{A}|} P_\theta y$ for every θ. Therefore, $\bar{a}_{|\bar{A}|} = b_{P_\theta}(A)$ whenever $\theta > \rho(\bar{a}_{|\bar{A}|-1}, \{\bar{a}_{|\bar{A}|-1}, \bar{a}_{|\bar{A}|}\})$ and the proof of the case is complete.

(3) Let $x = \bar{a}_i$, with $1 < i < |\bar{A}|$. The use of Claim 2 and CEN guarantees that $\rho(\bar{a}_i, \bar{A}) = \rho(\bar{a}_i, \{\bar{a}_{i-1}, \bar{a}_i, \bar{a}_{i+1}\}) = 1 - \rho(\bar{a}_{i-1}, \{\bar{a}_{i-1}, \bar{a}_i, \bar{a}_{i+1}\}) - \rho(\bar{a}_{i+1}, \{\bar{a}_{i-1}, \bar{a}_i, \bar{a}_{i+1}\}) = 1 - \rho(\bar{a}_{i-1}, \{\bar{a}_{i-1}, \bar{a}_i\}) - \rho(\bar{a}_{i+1}, \{\bar{a}_i, \bar{a}_{i+1}\}) = \rho(\bar{a}_i, \{\bar{a}_i, \bar{a}_{i+1}\}) - \rho(\bar{a}_{i-1}, \{\bar{a}_{i-1}, \bar{a}_i\})$. Given the construction of μ, it is sufficient to prove that $\bar{a}_i = b_{P_\theta}(A)$ holds if and only if $\theta \in (\rho(\bar{a}_{i-1}, \{\bar{a}_{i-1}, \bar{a}_i\}), \rho(\bar{a}_i, \{\bar{a}_i, \bar{a}_{i+1}\})]$. Now, for every $\theta \leq \rho(\bar{a}_{i-1}, \{\bar{a}_{i-1}, \bar{a}_i\})$ and $\theta > \rho(\bar{a}_i, \{\bar{a}_i, \bar{a}_{i+1}\})$, it is the case that $\bar{a}_{i-1} P_\theta \bar{a}_i$ and $\bar{a}_{i+1} P_\theta \bar{a}_i$, respectively, and hence, $\bar{a}_i \neq b_{P_\theta}(A)$. For every $\theta \in (\rho(\bar{a}_{i-1}, \{\bar{a}_{i-1}, \bar{a}_i\}), \rho(\bar{a}_i, \{\bar{a}_i, \bar{a}_{i+1}\})]$, we know that both $\bar{a}_i P_\theta \bar{a}_{i-1}$ and $\bar{a}_i P_\theta \bar{a}_{i+1}$ hold, and Claim 1 again guarantees that $\bar{a}_i P_\theta \bar{a}_j$ whenever $j \leq i - 1$ or $j \geq i + 1$. Then, consider $y \in A \setminus \bar{A}$. If $\bar{a}_{i-1} \succ y$ or $\bar{a}_i \succ y \succ \bar{a}_{i-1}$, we can apply Claim 1 over the triplet $\{y, \bar{a}_{i-1}, \bar{a}_i\}$ to get $\rho(y, \{y, \bar{a}_i\}) \leq \rho(\bar{a}_{i-1}, \{\bar{a}_{i-1}, \bar{a}_i\}) < \theta$ and hence, $\bar{a}_i P_\theta y$. If $\bar{a}_{i+1} \succ y \succ \bar{a}_i$ or $y \succ \bar{a}_{i+1}$, we can apply Claim 1 over the triplet $\{\bar{a}_i, y, \bar{a}_{i+1}\}$ to get $\rho(\bar{a}_i, \{\bar{a}_i, y\}) \geq \rho(\bar{a}_i, \{\bar{a}_i, \bar{a}_{i+1}\}) \geq \theta$ and hence, $\bar{a}_i P_\theta y$. Therefore, $\bar{a}_i = b_{P_\theta}(A)$ whenever $\theta \in (\rho(\bar{a}_{i-1}, \{\bar{a}_{i-1}, \bar{a}_i\}), \rho(\bar{a}_i, \{\bar{a}_i, \bar{a}_{i+1}\})]$, and the proof of this case is complete. *Q.E.D.*

CLAIM 6: *Let ρ satisfy MON and CEN. Then the RUM μ defined in Claim 5 is single-crossing.*

PROOF: Relabel the preferences in $\mathcal{P}_\mu$ as $P_1, P_2, \ldots, P_T$ by using the induced order $s > t \Leftrightarrow \inf\{\theta : P_s = P_\theta\} > \inf\{\theta : P_t = P_\theta\}$. Then, let $x \succ y$ and $s > t$ be such that $xP_t y$. Since $xP_t y$, it must be that $\rho(y, \{x, y\}) \leq \inf\{\theta : P_t = P_\theta\} < \inf\{\theta : P_s = P_\theta\}$ and hence, $xP_s y$, thus proving the single-crossing property. *Q.E.D.*

CLAIM 7: *Let ρ_μ be a SCRUM stochastic choice function. Then ρ_μ satisfies MON and CEN.*

PROOF: It is well-known that any RUM stochastic choice function satisfies MON. For CEN, consider a triplet $x \succ y \succ z$ with $\rho_\mu(y, \{x, y, z\}) > 0$. Given the order of the single-crossing collection of preferences in $\mathcal{P}_\mu$, let $t_y = \min\{t : y = b_{P_t}(\{x, y, z\})\}$, which must exist because $\rho_\mu(y, \{x, y, z\}) > 0$. Since $x \succ y$ and $yP_{t_y}x$, single-crossing guarantees that yP_tx for every $t < t_y$ and, by the definition of t_y, it must then be that zP_tyP_tx for every $t < t_y$. Single-crossing also guarantees that yP_tz for every $t \geq t_y$. Hence, it must be the case that $\rho_\mu(z, \{x, y, z\}) = \sum_{t<t_y} \mu(P_t) = \rho_\mu(z, \{y, z\})$. Analogously, let $t^y = \max\{t : y = b_{P_t}(\{x, y, z\})\}$. It must be the case that xP_tyP_tz for every $t > t^y$ and yP_tx for every $t \leq t^y$, which implies that $\rho_\mu(x, \{x, y, z\}) = \sum_{t>t^y} \mu(P_t) = \rho_\mu(x, \{x, y\})$. *Q.E.D.*

PROOF OF PROPOSITION 1: Given a SCRUM μ with ordered support $P_1, \dots, P_T$, denote by $\tilde{\mu}$ the cumulative distribution function (c.d.f.) of μ, that is, $\tilde{\mu}(P_t) = \sum_{s\leq t} \mu(P_s)$. Denote by $\tilde{\mu}^{-1}$ the inverse of the c.d.f., that is, $\tilde{\mu}^{-1}(\theta) = \{P_t : \tilde{\mu}(P_{t-1}) < \theta \leq \tilde{\mu}(P_t)\}$.[16] Suppose, by way of contradiction, that $\rho_\mu = \rho_{\mu'}$ but $\mu \neq \mu'$. Then, there must exist θ^* such that $\tilde{\mu}^{-1}(\theta^*) \neq \tilde{\mu}'^{-1}(\theta^*)$ for some $\theta^* \in (0, 1]$. In other words, there exist $x \succ y$ for which, w.l.o.g., $x\tilde{\mu}^{-1}(\theta^*)y$ and $y\tilde{\mu}'^{-1}(\theta^*)x$. The single-crossing condition implies that $\rho_\mu(y, \{x, y\}) < \theta^*$ and $\rho_{\mu'}(y, \{x, y\}) \geq \theta^*$, which is a contradiction. *Q.E.D.*

PROOF OF COROLLARY 1: Let ρ satisfy MON and SCEN. Notice that we can reproduce the first part of Claim 1 in the proof of Theorem 1 to show that $\rho(z, \{y, z\}) \leq \rho(z, \{x, z\}) \leq \rho(y, \{x, y\})$ holds whenever $x \succ y \succ z$. We now prove that the strict linear order P_θ, with $\theta \in (0, 1]$, constructed in Claim 4 of the proof of Theorem 1, satisfies single-peakedness. If $b_{P_\theta}(X) \succ x \succ y$, it must be that $\rho(y, \{y, x\}) \leq \rho(x, \{x, b_{P_\theta}(X)\}) < \theta$ and hence $xP_\theta y$. If $y \succ x \succ b_P P_\theta(X)$, it must be that $\theta \leq \rho(b_{P_\theta}(X), \{x, b_{P_\theta}(X)\}) \leq \rho(x, \{x, y\})$ and hence $xP_\theta y$. Then, the SCRUM constructed in the proof of Theorem 1 is also a SPRUM, and sufficiency follows. To show that a SPRUM stochastic choice function ρ_μ satisfies SCEN, consider three alternatives such that $x \succ y \succ z$. Single-peakedness guarantees that $\rho_\mu(x, \{x, y, z\}) = \sum_{P:xPy,xPz} \mu(P) = \sum_{P:xPy} \mu(P) = \rho_\mu(x, \{x, y\})$ and $\rho_\mu(z, \{x, y, z\}) = \sum_{P:zPy,zPx} \mu(P) = \sum_{P:zPy} \mu(P) = \rho_\mu(z, \{z, y\})$, and necessity follows. *Q.E.D.*

PROOF OF COROLLARY 2: Let ρ satisfy MON and EXT. We can reproduce the second part of Claim 1 in the proof of Theorem 1 to show that $\rho(z, \{y, z\}) \geq \rho(z, \{x, z\}) \geq \rho(y, \{x, y\})$ holds whenever $x \succ y \succ z$. We now prove the single-dippedness of every preference P_θ defined in the proof of Theorem 1. If $w_{P_\theta}(X) \succ x \succ y$, then it must be that $\rho(y, \{y, x\}) \geq \rho(x, \{x, w_{P_\theta}(X)\}) \geq \theta$ and ultimately $yP_\theta x$. If $y \succ x \succ w_{P_\theta}(X)$, it must be that $\theta > \rho(w_{P_\theta}(X), \{x, w_{P_\theta}(X)\}) \geq \rho(x, \{x, y\})$ and hence $yP_\theta x$. Necessity is immediate, and thus omitted. *Q.E.D.*

PROOF OF PROPOSITION 2: For the "only if" part, suppose that μ is higher than ν and assume, by way of contradiction, that ρ_μ does not first-order stochastically dominate ρ_ν. That is, there exists i such that $\sum_{j=i}^{|A|} \rho_\nu(a_j, A) > \sum_{j=i}^{|A|} \rho_\mu(a_j, A)$. Then it must be $1 < i$ and we have $\sum_{j=1}^{i-1} \rho_\mu(a_j, A) > \sum_{j=1}^{i-1} \rho_\nu(a_j, A)$. Let x and y be the best alternatives in A for preferences $\tilde{\mu}^{-1}(\sum_{j=1}^{i-1} \rho_\mu(a_j, A))$ and $\tilde{\nu}^{-1}(\sum_{j=1}^{i-1} \rho_\mu(a_j, A))$, respectively. Clearly, $a_i \succ x$ and $y \succ a_{i-1}$, implying $y \succ x$. However, $x\tilde{\mu}^{-1}(\sum_{j=1}^{i-1} \rho_\mu(a_j, A))y$ but $y\tilde{\nu}^{-1}(\sum_{j=1}^{i-1} \rho_\mu(a_j, A))x$, a contradiction with the fact that μ is higher than ν.

[16]Notice that, in Claims 4 to 6 of the sufficiency part of the proof of Theorem 1, we discuss how to construct a SCRUM from the choice data. We started by defining P_θ in Claim 4, which can be thought of as $\tilde{\mu}^{-1}$, and from this constructed the associated RUM.

For the "if" part, suppose that ρ_μ first-order stochastically dominates ρ_ν and assume, by way of contradiction, that μ is not higher than ν. That is, there exist $x \succ y$ and $\theta^* \in (0, 1]$ such that $x\tilde{\nu}^{-1}(\theta^*)y$ but $y\tilde{\mu}^{-1}(\theta^*)x$. Clearly, by the definition of P_θ, $\rho_\mu(y, \{x, y\}) \geq \theta^*$ and $\rho_\nu(y, \{x, y\}) < \theta^*$, which implies $\rho_\nu(x, \{x, y\}) > \rho_\mu(x, \{x, y\})$ and contradicts the fact that ρ_μ first-order stochastically dominates ρ_ν in menu $\{x, y\}$. *Q.E.D.*

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