Boltzmann Exploration Done Right

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Abstract

Boltzmann exploration is a classic strategy for sequential decision-making under uncertainty, and is one of the most standard tools in Reinforcement Learning (RL). Despite its widespread use, there is virtually no theoretical understanding about the limitations or the actual benefits of this exploration scheme. Does it drive exploration in a meaningful way? Is it prone to misidentifying the optimal actions or spending too much time exploring the suboptimal ones? What is the right tuning for the learning rate? In this paper, we address several of these questions for the classic setup of stochastic multi-armed bandits. One of our main results is showing that the Boltzmann exploration strategy with any monotone learning-rate sequence will induce suboptimal behavior. As a remedy, we offer a simple non-monotone schedule that guarantees near-optimal performance, albeit only when given prior access to key problem parameters that are typically not available in practical situations (like the time horizon T and the suboptimality gap Δ). More importantly, we propose a novel variant that uses different learning rates for different arms, and achieves a distribution-dependent regret bound of order $\frac{K\log^2 T}{\Delta}$ and a distribution-independent bound of order $\sqrt{KT}\log K$ without requiring such prior knowledge. To demonstrate the flexibility of our technique, we also propose a variant that guarantees the same performance bounds even if the rewards are heavy-tailed.

1. Introduction

Exponential weighting strategies are fundamental tools in a variety of areas, including Machine Learning, Optimization, Theoretical Computer Science, and Decision Theory (Arora et al., 2012). Within Reinforcement Learning (Sutton and Barto, 1998; Szepesvári, 2010), exponential weighting schemes are broadly used for balancing exploration and exploitation, and are equivalently referred to as Boltzmann, Gibbs, or softmax exploration policies (Sutton, 1990; Kaelbling et al., 1996; Sutton et al., 1999; Perkins and Precup, 2003). In the most common version of Boltzmann exploration, the probability of choosing an arm is proportional to an exponential function of the empirical mean of the reward of that arm. Despite the popularity of this policy, very little is known about its theoretical performance, even in the simplest reinforcement learning setting of *stochastic bandit problems*.

The variant of Boltzmann exploration we focus on in this paper is defined by

$$p_{t,i} \propto e^{\eta_t \widehat{\mu}_{t,i}}, \tag{1}$$

where $p_{t,i}$ is the probability of choosing arm i in round t, $\widehat{\mu}_{t,i}$ is the empirical average of the rewards obtained from arm i up until round t, and $\eta_t > 0$ is the *learning rate*. This variant is broadly used in reinforcement learning (Sutton and Barto, 1998; Szepesvári, 2010; Kaelbling et al., 1996; Vermorel and Mohri, 2005; Kuleshov and Precup, 2014; Osband et al., 2016). In the multiarmed bandit literature, exponential-weights algorithms are also widespread, but they typically use *importance-weighted* estimators for the rewards —see, e.g., (Auer et al., 1995, 2002b) (for the nonstochastic setting), (Cesa-Bianchi and Fischer, 1998) (for the stochastic setting), and (Seldin and Slivkins, 2014) (for both stochastic and nonstochastic regimes). The theoretical behavior of these algorithms is generally well understood. For example, in the stochastic bandit setting Seldin and Slivkins (2014) show a regret bound of order $\frac{K \log^2 T}{\Delta}$, where Δ is the suboptimality gap (i.e., the smallest difference between the mean reward of the optimal arm and the mean reward of any other arm).

In this paper, we aim to achieve a better theoretical understanding of the *basic* variant of the Boltzmann exploration policy that relies on the empirical mean rewards. We first show that any monotone learning-rate schedule will inevitably force the policy to either spend too much time drawing suboptimal arms or completely fail to identify the optimal arm. Then, we show that a specific non-monotone schedule of the learning rates can lead to regret bound of order $\frac{K \log T}{\Delta^2}$. However, the learning schedule has to rely on full knowledge of the gap Δ and the number of rounds T. Moreover, our negative result helps us to identify a crucial shortcoming of the Boltzmann exploration policy: it does not reason about the uncertainty of the empirical reward estimates. To alleviate this issue, we propose a variant that takes this uncertainty into account by using separate learning rates for each arm, where the learning rates account for the uncertainty of each reward estimate. We show that the resulting algorithm guarantees a distribution-dependent regret bound of order $\frac{K \log^2 T}{\Delta}$, and a distribution-independent bound of order $\sqrt{KT} \log K$.

Our algorithm and analysis is based on the so-called *Gumbel–softmax* trick that connects the exponential-weights distribution with the maximum of independent random variables from the Gumbel distribution.

2. The stochastic multi-armed bandit problem

Consider the setting of stochastic multi-armed bandits: each arm $i \in [K] \stackrel{\text{def}}{=} \{1, 2, \ldots, K\}$ yields a reward with distribution ν_i, mean μ_i, with the optimal mean reward being $\mu^* = \max_i \mu_i$. Without loss of generality, we will assume that the optimal arm is unique and has index 1. The gap of arm i is defined as $\Delta_i = \mu^* - \mu_i$. We consider a repeated game between the learner and the environment, where in each round $t = 1, 2, \ldots,$ the following steps are repeated:

1. The learner chooses an arm $I_t \in [K]$,
2. the environment draws a reward $X_{t,I_t} \sim \nu_{I_t}$ independently of the past,
3. the learner receives and observes the reward X_{t,I_t}.

The performance of the learner is measured in terms of the *pseudo-regret* defined as

$$R_T = \mu^* T - \sum_{t=1}^{T} \mathbb{E}\left[X_{t,I_t}\right] = \mu^* T - \mathbb{E}\left[\sum_{t=1}^{T} \mu_{I_t}\right] = \mathbb{E}\left[\sum_{t=1}^{T} \Delta_{I_t}\right] = \sum_{i=1}^{K} \Delta_i \mathbb{E}\left[N_{T,i}\right], \quad (2)$$

where we defined $N_{t,i} = \sum_{s=1}^{t} \mathbb{I}_{\{I_s = i\}}$, that is, the number of times that arm i has been chosen until the end of round t. We aim at constructing algorithms that guarantee that the regret grows sublinearly.

We will consider the above problem under various assumptions of the distribution of the rewards. For most of our results, we will assume that each ν_i is σ*-subgaussian* with a known parameter $\sigma > 0$, that is, that

$$\mathbb{E}\left[e^{y(X_{1,i} - \mathbb{E}[X_{1,i}])}\right] \le e^{\sigma^2 y^2/2}$$

holds for all $y \in \mathbb{R}$ and $i \in [K]$. It is easy to see that any random variable bounded in an interval of length B is B^2-subgaussian. Under this assumption, it is well known that any reasonable algorithm will suffer a regret of at least $\Omega\left(\sum_{i>1} \frac{\sigma^2 \log T}{\Delta_i}\right)$, as shown in the classic paper of Lai and Robbins (1985). There exist several algorithms guaranteeing matching upper bounds, even for finite horizons (Auer et al., 2002a; Cappé et al., 2013; Kaufmann et al., 2012). We refer to the survey of Bubeck and Cesa-Bianchi (2012) for an exhaustive treatment of the topic.

3. Boltzmann exploration done wrong

We now formally describe the heuristic form of Boltzmann exploration that is commonly used in the reinforcement learning literature (Sutton and Barto, 1998; Szepesvári, 2010; Kaelbling et al., 1996). This strategy works by maintaining the empirical estimates of each μ_i defined as

$$\widehat{\mu}_{t,i} = \frac{\sum_{s=1}^{t} X_{s,i} \mathbb{I}_{\{I_s = i\}}}{N_{t,i}} \quad (3)$$

and computing the exponential-weights distribution (1) for an appropriately tuned sequence of *learning rate* parameters $\eta_t > 0$ (which are often referred to as the *inverse temperature*). As noted on several occasions in the literature, finding the right schedule for η_t can be very difficult in practice (Kaelbling et al., 1996; Vermorel and Mohri, 2005). Below, we quantify this difficulty by showing that natural learning-rate schedules may fail to achieve near-optimal regret guarantees. More precisely, they may draw suboptimal arms too much even after having estimated all the means correctly, or commit too early to a suboptimal arm and never recover afterwards. We partially circumvent this issue by proposing an admittedly artificial learning-rate schedule that actually guarantees near-optimal performance. However, a serious limitation of this schedule is that it relies on prior knowledge of problem parameters Δ and T that are typically unknown at the beginning of the learning procedure. These observations lead us to the conclusion that the Boltzmann exploration policy as described by Equations (1) and (3) is not much more effective for regret minimization than the simplest alternative of ε-greedy exploration (Sutton and Barto, 1998; Auer et al., 2002a).

Before we present our own technical results, we mention that Singh et al. (2000) propose a learning-rate schedule η_t for Boltzmann exploration that simultaneously guarantees that all arms will be drawn infinitely often as T goes to infinity, and that the policy becomes greedy in the limit. This property is proven by choosing a learning-rate schedule adaptively to ensure that in each round t, each arm gets drawn with probability at least $\frac{1}{t}$, making it similar in spirit to ε-greedy exploration. While this strategy clearly leads to sublinear regret, it is easy to construct examples on which it suffers a regret of at least $\Omega\left(T^{1-\alpha}\right)$ for any small $\alpha > 0$. Our endeavor is more ambitious: we aim to find out whether Boltzmann exploration can actually guarantee polylogarithmic regret. In the rest of this section, we present both negative and positive results concerning the standard variant of Boltzmann exploration, and then move on to providing an efficient generalization that achieves consistency in a more universal sense.

3.1. Boltzmann exploration with monotone learning rates is suboptimal

In this section, we study the most natural variant of Boltzmann exploration that uses a monotone learning-rate schedule. It is easy to see that in order to achieve sublinear regret, the learning rate η_t needs to *increase* with t so that the suboptimal arms are drawn with less and less probability as time progresses. For the sake of clarity, we study the simplest possible setting with two arms with a gap of Δ between their means. We first show that, asymptotically, the learning rate has to increase at least at a rate $\frac{\log t}{\Delta}$ even when the mean rewards are perfectly known. In other words, this is the minimal affordable learning rate.

Proposition 1 *Let us assume that $\widehat{\mu}_{t,i} = \mu_i$ for all t and both i. If $\eta_t = o\left(\frac{\log(t\Delta^2)}{\Delta}\right)$, then the regret grows at least as fast as $R_T = \omega\left(\frac{\log T}{\Delta}\right)$.*

Proof Let us define $\eta_t^* = \frac{\log(t\Delta^2)}{\Delta}$ for all t. The probability of pulling the suboptimal arm can be asymptotically bounded as

$$\mathbb{P}\left[I_t = 2\right] = \frac{1}{1+e^{\eta_t\Delta}} \geq \frac{e^{-\eta_t\Delta}}{2} = \omega\left(\frac{e^{-\eta_t^*\Delta}}{2}\right) = \omega\left(\frac{1}{\Delta^2 t}\right).$$

Summing up for all t, we get that the regret is at least

$$R_T = \Delta\sum_{t=1}^{T}\mathbb{P}\left[I_t = 2\right] = \omega\left(\sum_{t=1}^{T}\frac{1}{\Delta^2 t}\right) = \omega\left(\frac{\log T}{\Delta}\right),$$

thus proving the statement. ■

This simple proposition thus implies an asymptotic lower bound on the schedule of learning rates η_t. In contrast, Theorem 2 below shows that all learning rate sequences that grow faster than $2\log t$ yield a linear regret, provided this schedule is adopted since the beginning of the game. This should be contrasted with Theorem 3, which exhibits a schedule achieving logarithmic regret where η_t grows faster than $2\log t$ only after the first τ rounds.

Theorem 2 *There exists a 2-armed stochastic bandit problem with rewards bounded in* $[0,1]$ *where Boltzmann exploration using any learning rate sequence* η_t *such that* $\eta_t > 2\log t$ *for all* $t \geq 1$ *has regret* $R_T = \Omega(T)$.

Proof Consider the case where arm 2 gives a reward deterministically equal to $\frac{1}{2}$ whereas the optimal arm 1 has a Bernoulli distribution of parameter $p = \frac{1}{2} + \Delta$ for some $0 < \Delta < \frac{1}{2}$. Note that the regret of any algorithm satisfies $R_T \geq \Delta(T - t_0)\mathbb{P}\left[\forall t > t_0,\ I_t = 2\right]$. Without loss of generality, assume that $\widehat{\mu}_{1,1} = 0$ and $\widehat{\mu}_{1,2} = 1/2$. Then for all t, independent of the algorithm, $\widehat{\mu}_{t,2} = 1/2$ and

$$p_{t,1} = \frac{e^{\eta_t \mathrm{Bin}(N_{t-1,1},p)}}{e^{\eta_t/2} + e^{\eta_t \mathrm{Bin}(N_{t-1,1},p)}} \quad \text{and} \quad p_{t,2} = \frac{e^{\eta_t/2}}{e^{\eta_t/2} + e^{\eta_t \mathrm{Bin}(N_{t-1,1},p)}}.$$

For $t_0 \geq 1$, Let E_{t_0} be the event that $\mathrm{Bin}(N_{t_0,1},p) = 0$, that is, up to time t_0, arm 1 gives only zero reward whenever it is sampled. Then

$$\begin{aligned}\mathbb{P}\left[\forall t > t_0\ I_t = 2\right] &\geq \mathbb{P}\left[E_{t_0}\right]\Big(1 - \mathbb{P}\left[\exists t > t_0\ I_t = 1 \mid E_{t_0}\right]\Big)\\ &\geq \left(\frac{1}{2} - \Delta\right)^{t_0}\Big(1 - \mathbb{P}\left[\exists t > t_0\ I_t = 1 \mid E_{t_0}\right]\Big).\end{aligned}$$

For $t > t_0$, let A_{t,t_0} be the event that arm 1 is sampled at time t but not at any of the times $t_0+1, t_0+2, \ldots, t-1$. Then, for any $t_0 \geq 1$,

$$\begin{aligned}\mathbb{P}\left[\exists t > t_0\ I_t = 1 \mid E_{t_0}\right] &= \mathbb{P}\left[\exists t > t_0\ A_{t,t_0} \mid E_{t_0}\right] \leq \sum_{t>t_0}\mathbb{P}\left[A_{t,t_0} \mid E_{t_0}\right]\\ &= \sum_{t>t_0}\frac{1}{1+e^{\eta_t/2}}\prod_{s=t_0+1}^{t-1}\left(1 - \frac{1}{1+e^{\eta_s/2}}\right) \leq \sum_{t>t_0} e^{-\eta_t/2}.\end{aligned}$$

Therefore

$$R_T \geq \Delta(T - t_0)\left(\frac{1}{2} - \Delta\right)^{t_0}\left(1 - \sum_{t>t_0} e^{-\eta_t/2}\right) .$$

Assume $\eta_t \geq c \log t$ for some $c > 2$ and for all $t \geq t_0$. Then

$$\sum_{t>t_0} e^{-\eta_t/2} \leq \sum_{t>t_0} t^{-\frac{c}{2}} \leq \int_{t_0}^{\infty} x^{-\frac{c}{2}}\, dx = \left(\frac{c}{2} - 1\right) t_0^{-(\frac{c}{2}-1)} \leq \frac{1}{2}$$

whenever $t_0 \geq (2a)^{\frac{1}{a}}$ where $a = \frac{c}{2} - 1$. This implies $R_T = \Omega(T)$. ∎

3.2. A learning-rate schedule with near-optimal guarantees

The above negative result is indeed heavily relying on the assumption that $\eta_t > 2 \log t$ holds since the beginning. If we instead start off from a constant learning rate which we keep for a logarithmic number of rounds, then a logarithmic regret bound can be shown. Arguably, this results in a rather simplistic exploration scheme, which can be essentially seen as an *explore-then-commit* strategy (e.g., Garivier et al. (2016)).

Theorem 3 *Assume the rewards of each arm are in* $[0, 1]$ *and let* $\tau = \frac{16eK(\log T)}{\Delta^2}$. *Then the regret of Boltzmann exploration with learning rate* $\eta_t = \mathbb{I}_{\{t<\tau\}} + \frac{\log(t\Delta^2)}{\Delta}\mathbb{I}_{\{t\geq\tau\}}$ *satisfies*

$$R_T \leq \frac{16eK(\log T)}{\Delta^2} + \frac{9K}{\Delta^2} .$$

Proof For any round t and action i,

$$\frac{e^{-\eta_t}}{K} \leq \frac{e^{\eta_t \widehat{\mu}_{t-1,i}}}{\sum_{j=1}^{K} e^{\eta_t \widehat{\mu}_{t-1,j}}} \leq e^{\eta_t\left(\widehat{\mu}_{t-1,i} - \widehat{\mu}_{t-1,1}\right)} . \tag{4}$$

Now, for any $i > 1$, we can write

$$\begin{aligned}
\mathbb{I}_{\{I_t=i\}} &= \mathbb{I}_{\left\{I_t=i,\, \widehat{\mu}_{t-1,i} - \widehat{\mu}_{t-1,1} < -\frac{\Delta_i}{2}\right\}} + \mathbb{I}_{\left\{I_t=i,\, \widehat{\mu}_{t-1,i} - \widehat{\mu}_{t-1,1} \geq -\frac{\Delta_i}{2}\right\}} \\
&\leq \mathbb{I}_{\left\{I_t=i,\, \widehat{\mu}_{t-1,i} - \widehat{\mu}_{t-1,1} < -\frac{\Delta_i}{2}\right\}} + \mathbb{I}_{\left\{\widehat{\mu}_{t-1,1} \leq \mu_1 - \frac{\Delta_i}{4}\right\}} + \mathbb{I}_{\left\{\widehat{\mu}_{t-1,i} \geq \mu_i + \frac{\Delta_i}{4}\right\}} .
\end{aligned}$$

We take expectation of the three terms above and sum over $t = \tau + 1, \ldots, T$. Because of (4), the first term is simply bounded as

$$\sum_{t=\tau+1}^{T} \mathbb{P}\left[I_t = i,\, \widehat{\mu}_{t-1,i} - \widehat{\mu}_{t-1,1} < -\frac{\Delta_i}{2}\right] \leq \sum_{t=\tau+1}^{T} e^{-\eta_t \Delta_i/2} \leq \sum_{t=\tau+1}^{T} \frac{1}{t\Delta^2} \leq \frac{\log(T+1)}{\Delta^2} .$$

We control the second and third term in the same way. For the second term we have that $\mathbb{I}_{\left\{\widehat{\mu}_{t-1,1}\le\mu_1-\frac{\Delta_i}{4}\right\}} \le \mathbb{I}_{\{N_{t-1,1}\le t_1\}} + \mathbb{I}_{\left\{\widehat{\mu}_{t-1,1}\le\mu_1-\frac{\Delta_i}{4},\, N_{t-1,1}>t_1\right\}}$ holds for any fixed t and for any $t_1 \le t-1$. Hence

$$\sum_{t=\tau+1}^{T} \mathbb{P}\left[\widehat{\mu}_{t-1,1} \le \mu_1 - \frac{\Delta_i}{4}\right] \le \sum_{t=\tau+1}^{T} \mathbb{P}\left[N_{t-1,1} \le t_1\right] + \sum_{t=\tau+1}^{T} \mathbb{P}\left[\widehat{\mu}_{t-1,1} \le \mu_1 - \frac{\Delta_i}{4},\, N_{t-1,1} > t_1\right] .$$

Now observe that, because of (4) applied to the initial τ rounds, $\mathbb{E}\left[N_{t-1,1}\right] \ge \frac{\tau}{eK}$ holds for all $t > \tau$. By setting $t_1 = \frac{1}{2}\mathbb{E}\left[N_{t-1,1}\right] \ge \frac{\tau}{2eK}$, Chernoff bounds (in multiplicative form) give $\mathbb{P}\left[N_{t-1,1} \le t_1\right] \le e^{-\frac{\tau}{8eK}}$. Standard Chernoff bounds, instead, give

$$\mathbb{P}\left[\widehat{\mu}_{t-1,1} \le \mu_1 - \frac{\Delta_i}{4},\, N_{t-1,1} > t_1\right] \le \sum_{s=t_1+1}^{t-1} e^{-\frac{s\Delta^2}{8}} \le \frac{8}{\Delta^2} e^{-\frac{t_1\Delta^2}{8}} \le \frac{8}{\Delta^2} e^{-\frac{\tau\Delta^2}{16eK}} .$$

Therefore, for the second term we can write

$$\sum_{t=\tau+1}^{T} \mathbb{P}\left[\widehat{\mu}_{t-1,1} \le \mu_1 - \frac{\Delta_i}{4}\right] \le T\left(e^{-\frac{\tau}{8eK}} + \frac{8}{\Delta^2} e^{-\frac{\tau\Delta^2}{16eK}}\right) \le 1 + \frac{8}{\Delta^2} .$$

The third term can be bounded exactly in the same way. Putting together, we have thus obtained, for all actions $i > 1$,

$$\sum_{i>1} \mathbb{E}\left[N_{T,i}\right] \le \tau + K + \frac{8K}{\Delta^2} \le \frac{16eK(\log T)}{\Delta^2} + \frac{9K}{\Delta^2} .$$

This concludes the proof. ∎

4. Boltzmann exploration done right

We now turn to give a variant of Boltzmann exploration that achieves near-optimal guarantees without prior knowledge of either Δ or T. Our approach is based on the observation that the distribution $p_{t,i} \propto \exp\left(\eta_t \widehat{\mu}_{t,i}\right)$ can be equivalently specified by the rule $I_t = \arg\max_j \left\{\eta_t \widehat{\mu}_{t,j} + Z_{t,j}\right\}$, where $Z_{t,j}$ is a standard Gumbel random variable[1] drawn independently for each arm j (see, e.g., Abernethy et al. (2014) and the references therein). As we saw in the previous section, this scheme fails to guarantee consistency in general, as it does not capture the uncertainty of the reward estimates. We now propose a variant that takes this uncertainty into account by choosing different scaling factors for each perturbation. In particular, we will use the

1. The cumulative density function of a standard Gumbel random variable is $F(x) = \exp(-e^{-x+\gamma})$ where γ is the Euler-Mascheroni constant.

simple choice $\beta_{t,i} = \sqrt{C^2/N_{t,i}}$ with some constant $C > 0$ that will be specified later. Our algorithm operates by independently drawing perturbations $Z_{t,i}$ from a standard Gumbel distribution for each arm i, then choosing action

$$I_{t+1} = \arg\max_i \left\{\widehat{\mu}_{t,i} + \beta_{t,i} Z_{t,i}\right\}. \tag{5}$$

We refer to this algorithm as *Boltzmann–Gumbel exploration*, or, in short, BGE. Unfortunately, the probabilities $p_{t,i}$ no longer have a simple closed form, nevertheless the algorithm is very straightforward to implement. Our main positive result is showing the following performance guarantee about the algorithm.

Theorem 4 *Assume that the rewards of each arm are σ^2-subgaussian and let $c > 0$ be arbitrary. Then, the regret of Boltzmann–Gumbel exploration satisfies*

$$R_T \le \sum_{i=2}^K \frac{9C^2 \log_+^2 \left(T\Delta_i/c^2\right)}{\Delta_i^2} + \sum_{i=2}^K \frac{c^2 e^\gamma + 18C^2 e^{\sigma^2/2C^2}\left(1 + e^{-\gamma}\right)}{\Delta_i} + \sum_{i=2}^K \Delta_i.$$

In particular, choosing $C = \sigma$ and $c = \sigma$ guarantees a regret bound of

$$R_T = O\left(\sum_{i=2}^K \frac{\sigma^2 \log^2(T\Delta_i^2/\sigma^2)}{\Delta_i}\right).$$

Notice that, unlike any other algorithm that we are aware of, Boltzmann–Gumbel exploration still continues to guarantee meaningful regret bounds even if the subgaussianity constant σ is underestimated—although such misspecification is penalized exponentially in the true σ^2. A downside of our bound is that it shows a suboptimal dependence on the number of rounds T: it grows asymptotically as $\sum_{i>1} \log^2(T\Delta_i^2)\big/\Delta_i$, in contrast to the standard regret bounds for the UCB algorithm of Auer et al. (2002a) that grow as $\sum_{i>1} (\log T)\big/\Delta_i$. However, our guarantee improves on the distribution-independent regret bounds of UCB that are of order $\sqrt{KT\log T}$. This is shown in the following corollary.

Corollary 5 *Assume that the rewards of each arm are σ^2-subgaussian. Then, the regret of Boltzmann–Gumbel exploration with $C = \sigma$ satisfies $R_T \le 200\sigma\sqrt{KT}\log K$.*

Notably, this bound shows optimal dependence on the number of rounds T, but is suboptimal in terms of the number of arms. To complement this upper bound, we also show that these bounds are tight in the sense that the $\log K$ factor cannot be removed.

Theorem 6 *For any K and T such that $\sqrt{K/T}\log K \le 1$, there exists a bandit problem with rewards bounded in $[0, 1]$ where the regret of Boltzmann–Gumbel exploration with $C = 1$ is at least $R_T \ge \frac{1}{2}\sqrt{KT}\log K$.*

The proofs can be found in the Appendices A.4 and A.5. Note that more sophisticated policies are known to have better distribution-free bounds. The algorithm MOSS Audibert and Bubeck (2009) achieves minimax-optimal $\sqrt{KT}$ distribution-free bounds, but distribution-dependent bounds of the form $(K/\Delta)\log(T\Delta^2)$ where $\Delta = \min_{i>1}\Delta_i$ is the suboptimality gap. A variant of UCB using action elimination and due to Auer and Ortner (2010) has regret $\sum_{i>1}\log(T\Delta_i^2)\big/\Delta_i$ corresponding to a $\sqrt{KT(\log K)}$ distribution-free bound. The same bounds are achieved by the Gaussian Thompson sampling algorithm of Agrawal and Goyal (2013), given that the rewards are subgaussian. An "optimally confident" variant of UCB recently proposed by Lattimore (2015, 2016) dominates all of these algorithms in that it achieves optimal performance guarantees in both the asymptotic and the worst-case regime.

We finally provide a simple variant of our algorithm that allows to handle heavy-tailed rewards, intended here as reward distributions that are not subgaussian. We propose to use technique due to Catoni (2012) based on the *influence function*

$$\psi(x) = \begin{cases} \log\left(1 + x + x^2/2\right), & \text{for } x \geq 0, \\ -\log\left(1 - x + x^2/2\right), & \text{for } x \leq 0. \end{cases}$$

Using this function, we define our estimates as

$$\widehat{\mu}_{t,i} = \beta_{t,i} \sum_{s=1}^{t} \mathbb{I}_{\{I_t=i\}} \psi\left(\frac{X_{t,i}}{\beta_{t,i} N_{t,i}}\right)$$

We prove the following result regarding Boltzmann–Gumbel exploration run with the above estimates.

Theorem 7 *Assume that the second moment of the rewards of each arm are bounded uniformly as $\mathbb{E}\left[X_i^2\right] \leq V$ and let $c > 0$ be arbitrary. Then, the regret of Boltzmann–Gumbel exploration satisfies*

$$R_T \leq \sum_{i=2}^{K} \frac{9C^2 \log_+^2\left(T\Delta_i/c^2\right)}{\Delta_i^2} + \sum_{i=2}^{K} \frac{c^2 e^{\gamma} + 18C^2 e^{V/2C^2}\left(1 + e^{-\gamma}\right)}{\Delta_i} + \sum_{i=2}^{K} \Delta_i.$$

Notably, this bound coincides with that of Theorem 4, except that σ^2 is replaced by V. Thus, by following the proof of Corollary 5, we can show a distribution-independent regret bound of order $\sqrt{KT}\log K$.

5. Analysis

Let us now present the proofs of our main results, Theorems 4 and 7. Our analysis builds on several ideas from Agrawal and Goyal (2013). We first provide generic tools that are independent of the reward estimator and then move on to providing specifics for both estimators.

We start with introducing some notation. We define $\widetilde{\mu}_{t,i} = \widehat{\mu}_{t,i} + \beta_{t,i} Z_{t,i}$, so that the algorithm can be simply written as $I_t = \arg\max_i \widetilde{\mu}_{t,i}$. Let $\mathcal{F}_{t-1}$ be the sigma-algebra generated by the actions taken by the learner and the realized rewards up to the beginning of round t. Let us fix thresholds x_i, y_i satisfying $\mu_i \le x_i \le y_i \le \mu_1$ and define $q_{t,i} = \mathbb{P}\left[\widetilde{\mu}_{t,1} > y_i \middle| \mathcal{F}_{t-1}\right]$. Furthermore, we define the events $E^{\widehat{\mu}}_{t,i} = \{\widehat{\mu}_{t,i} \le x_i\}$ and $E^{\widetilde{\mu}}_{t,i} = \{\widetilde{\mu}_{t,i} \le y_i\}$. With this notation at hand, we can decompose the number of draws of any suboptimal i as follows:

$$\mathbb{E}\left[N_{T,i}\right] = \sum_{t=1}^{T} \mathbb{P}\left[I_t = i, E^{\widetilde{\mu}}_{t,i}, E^{\widehat{\mu}}_{t,i}\right] + \sum_{t=1}^{T} \mathbb{P}\left[I_t = i, \overline{E^{\widetilde{\mu}}_{t,i}}, E^{\widehat{\mu}}_{t,i}\right] + \sum_{t=1}^{T} \mathbb{P}\left[I_t = i, \overline{E^{\widehat{\mu}}_{t,i}}\right]. \quad (6)$$

It remains to choose the thresholds x_i and y_i in a meaningful way: we pick $x_i = \mu_i + \frac{\Delta_i}{3}$ and $y_i = \mu_1 - \frac{\Delta_i}{3}$. The rest of the proof is devoted to bounding each term in Eq. (6). Intuitively, the individual terms capture the following events:

- The first term counts the number of times that, even though the estimated mean reward of arm i is well-concentrated and the additional perturbation $Z_{t.i}$ is not too large, arm i was drawn instead of the optimal arm 1. This happens when the optimal arm is poorly estimated or when the perturbation $Z_{t,1}$ is not large enough. Intuitively, this term measures the interaction between the perturbations $Z_{t,1}$ and the random fluctuations of the reward estimate $\widehat{\mu}_{t,1}$ around its true mean, and will be small if the perturbations are large enough and the tail of the reward estimates is light enough.

- The second term counts the number of times that the mean reward of arm i is well-estimated, but it ends up being drawn due to a large perturbation. This term can be bounded independently of the properties of the mean estimator and is small when the perturbations are not too large.

- The last term counts the number of times that the reward estimate of arm i is poorly concentrated. This term is independent of the perturbations and only depends on the properties of the reward estimator.

As we will see, the first and the last terms can be bounded in terms of the *moment generating function* of the reward estimates, which makes subgaussian reward estimators particularly easy to treat. We begin by the most standard part of our analysis: bounding the third term on the right-hand-side of (6) in terms of the moment-generating function.

Lemma 8 *Let us fix any i and define τ_k as the k'th time that arm i was drawn. We have*

$$\sum_{t=1}^{T} \mathbb{P}\left[I_t = i, \overline{E^{\widehat{\mu}}_{t,i}}\right] \le 1 + \sum_{k=1}^{T-1} \mathbb{E}\left[\exp\left(\frac{\widehat{\mu}_{\tau_k,i} - \mu_i}{\beta_{\tau_k,i}}\right)\right] \cdot e^{-\frac{\Delta_i \sqrt{k}}{3C}}.$$

Interestingly, our next key result shows that the first term can be bounded by a nearly identical expression:

Lemma 9 *Let us define τ_k as the k'th time that arm 1 was drawn. For any i, we have*

$$\sum_{t=1}^{T} \mathbb{P}\left[I_t = i, E^{\widetilde{\mu}}_{t,i}, E^{\widehat{\mu}}_{t,i}\right] \le \sum_{k=0}^{T-1} \mathbb{E}\left[\exp\left(\frac{\mu_1 - \widehat{\mu}_{\tau_k,1}}{\beta_{\tau_k,1}}\right)\right] e^{-\gamma - \frac{\Delta_i\sqrt{k}}{3C}}.$$

It remains to bound the second term in Equation (6), which we do in the following lemma where we use the notation $\log_+(\cdot) = \min\{0, \cdot\}$.

Lemma 10 *For any $i \ne 1$ and any constant $c > 0$, we have*

$$\sum_{t=1}^{T} \mathbb{P}\left[I_t = i, \overline{E^{\widetilde{\mu}}_{t,i}}, E^{\widehat{\mu}}_{t,i}\right] \le \frac{9C^2 \log_+^2\left(T\Delta_i^2/c^2\right) + c^2 e^{\gamma}}{\Delta_i^2}.$$

The proofs of these three lemmas are included in the supplementary material.

5.1. The proof of Theorem 4

For this section, we assume that the rewards are σ-subgaussian and that $\widehat{\mu}_{t,i}$ is the empirical-mean estimator. Building on the results of the previous section, observe that we are left with bounding the terms appearing in Lemmas 8 and 11. To this end, let us fix a k and an i and notice that by the subgaussianity assumption on the rewards, the empirical mean $\widetilde{\mu}_{\tau_k,i}$ is $\frac{\sigma}{\sqrt{k}}$-subgaussian (as $N_{\tau_k,i} = k$). In other words,

$$\mathbb{E}\left[e^{\alpha\left(\widehat{\mu}_{\tau_k,i} - \mu_i\right)}\right] \le e^{\alpha^2\sigma^2/2k}$$

holds for any α. In particular, using this above formula for $\alpha = 1/\beta_{\tau_k,i} = \sqrt{\frac{k}{C^2}}$, we obtain

$$\mathbb{E}\left[\exp\left(\frac{\widehat{\mu}_{\tau_k,i} - \mu_i}{\beta_{\tau_k,i}}\right)\right] \le e^{\sigma^2/2C^2}.$$

Thus, the sum appearing in Lemma 8 can be bounded as

$$\sum_{k=1}^{T-1} \mathbb{E}\left[\exp\left(\frac{\widehat{\mu}_{\tau_k,i} - \mu_i}{\beta_{\tau_k,i}}\right)\right] \cdot e^{-\frac{\Delta_i\sqrt{k}}{3C}} \le e^{\sigma^2/2C^2} \sum_{k=1}^{T-1} e^{-\frac{\Delta_i\sqrt{k}}{3C}} \le \frac{18C^2 e^{\sigma^2/2C^2}}{\Delta_i^2},$$

where the last step follows from the fact[2] that $\sum_{k=0}^{\infty} e^{c\sqrt{k}} \le \frac{2}{c^2}$ holds for all $c > 0$. The statement of Theorem 4 now follows from applying the same argument to the bound of Lemma 11, using Lemma 10, and the standard expression for the regret in Equation (2). ∎

2. This can be easily seen by bounding the sum with an integral.

5.2. The proof of Theorem 7

We now drop the subgaussian assumption on the rewards and consider reward distributions that are possibly heavy-tailed, but have bounded variance. The proof of Theorem 7 trivially follows from the arguments in the previous subsection and using Proposition 2.1 of Catoni (2012) that guarantees the bound

$$\mathbb{E}\left[\exp\left(\pm\frac{\mu_i - \widehat{\mu}_{t,i}}{\beta_{t,i}}\right)\middle| N_{t,i} = n\right] \le \exp\left(\frac{\mathbb{E}\left[X_i^2\right]}{2C^2}\right). \tag{7}$$

■

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Appendix A. Technical proofs

A.1. The proof of Lemma 8

Let τ_k denote the index of the round when arm i is drawn for the k'th time. We let $\tau_0 = 0$ and $\tau_k = T$ for $k > N_{T,i}$. Then,

$$\begin{aligned}
\sum_{t=1}^{T} \mathbb{P}\left[I_t = i, \overline{E_{t,i}^{\widehat{\mu}}}\right] &\le \mathbb{E}\left[\sum_{k=0}^{T-1} \sum_{t=\tau_k+1}^{\tau_{k+1}} \mathbb{I}_{\{I_t=i\}} \mathbb{I}_{\left\{\overline{E_{t,i}^{\widehat{\mu}}}\right\}}\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{T-1} \mathbb{I}_{\left\{\overline{E_{\tau_k,i}^{\widehat{\mu}}}\right\}} \sum_{t=\tau_k+1}^{\tau_{k+1}} \mathbb{I}_{\{I_t=i\}}\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{T-1} \mathbb{I}_{\left\{\overline{E_{\tau_k,i}^{\widehat{\mu}}}\right\}}\right] \\
&\le 1 + \sum_{k=1}^{T-1} \mathbb{P}\left[\widehat{\mu}_{\tau_k,i} \ge x_i\right] \\
&\le 1 + \sum_{k=1}^{T-1} \mathbb{P}\left[\widehat{\mu}_{\tau_k,i} - \mu_i \ge \frac{\Delta_i}{3}\right].
\end{aligned}$$

Now, using the fact that $N_{\tau_k,i} = k$, we bound the last term by exploiting the subgaussianity of the rewards through Markov's inequality:

$$\begin{aligned}
\mathbb{P}\left[\widehat{\mu}_{\tau_k,i} - \mu_i \ge \frac{\Delta_i}{3}\right] &= \mathbb{P}\left[e^{\alpha\left(\widehat{\mu}_{\tau_k,i}-\mu_i\right)} \ge e^{\alpha\frac{\Delta_i}{3}}\right] && \text{(for any } \alpha > 0\text{)} \\
&\le \mathbb{E}\left[e^{\alpha\left(\widehat{\mu}_{\tau_k,i}-\mu_i\right)}\right] \cdot e^{-\alpha\frac{\Delta_i}{3}} && \text{(Markov's inequality)} \\
&\le e^{\alpha^2\sigma^2/2k} \cdot e^{-\alpha\frac{\Delta_i}{3}} && \text{(the subgaussian property)} \\
&\le e^{\sigma^2/2C^2} \cdot e^{-\frac{\Delta_i\sqrt{k}}{3C}} && \text{(choosing } \alpha = \sqrt{k/C^2}\text{)}
\end{aligned}$$

Now, using the fact[3] that $\sum_{k=0}^{\infty} e^{c\sqrt{k}} \le \frac{2}{c^2}$ holds for all $c > 0$, the proof is concluded. ■

A.2. The proof of Lemma 9

The proof of this lemma crucially builds on Lemma 1 of Agrawal and Goyal (2013), which we state and prove below.

Lemma 11 (cf. Lemma 1 of Agrawal and Goyal (2013))

$$\mathbb{P}\left[I_t = i, E_{t,i}^{\widehat{\mu}}, E_{t,i}^{\widetilde{\mu}} \middle| \mathcal{F}_{t-1}\right] \le \frac{1 - q_{t,i}}{q_{t,i}} \cdot \mathbb{P}\left[I_t = 1, E_{t,i}^{\widehat{\mu}}, E_{t,i}^{\widetilde{\mu}} \middle| \mathcal{F}_{t-1}\right]$$

3. This can be easily seen by bounding the sum with an integral.

Proof First, note that $E^{\widehat{\mu}}_{t,i} \subseteq \mathcal{F}_{t-1}$. We only have to care about the case when $E^{\widetilde{\mu}}_{t,i}$ holds, otherwise both sides of the inequality are zero and the statement trivially holds. Thus, we only have to prove

$$\mathbb{P}\left[I_t = i \middle| \mathcal{F}_{t-1}, E^{\widetilde{\mu}}_{t,i}\right] \leq \frac{1-q_{t,i}}{q_{t,i}} \cdot \mathbb{P}\left[I_t = 1 \middle| \mathcal{F}_{t-1}, E^{\widetilde{\mu}}_{t,i}\right].$$

Now observe that $I_t = i$ under the event $E^{\widetilde{\mu}}_{t,i}$ implies $\widetilde{\mu}_{t,j} \leq y_i$ for all j (which follows from $\widetilde{\mu}_{t,j} \leq \widetilde{\mu}_{t,i} \leq y_i$). Thus, for any $i > 1$, we have

$$\begin{aligned}
\mathbb{P}\left[I_t = i \middle| \mathcal{F}_{t-1}, E^{\widetilde{\mu}}_{t,i}\right] \leq & \mathbb{P}\left[\forall j : \widetilde{\mu}_{t,j} \leq y_i \middle| \mathcal{F}_{t-1}, E^{\widetilde{\mu}}_{t,i}\right] \\
= & \mathbb{P}\left[\widetilde{\mu}_{t,1} \leq y_i \middle| \mathcal{F}_{t-1}, E^{\widetilde{\mu}}_{t,i}\right] \cdot \mathbb{P}\left[\forall j > 1 : \widetilde{\mu}_{t,j} \leq y_i \middle| \mathcal{F}_{t-1}, E^{\widetilde{\mu}}_{t,i}\right] \\
= & (1-q_{t,i}) \cdot \mathbb{P}\left[\forall j > 1 : \widetilde{\mu}_{t,j} \leq y_i \middle| \mathcal{F}_{t-1}, E^{\widetilde{\mu}}_{t,i}\right],
\end{aligned}$$

where the last equality holds because the event in question is independent of $E^{\widetilde{\mu}}_{t,i}$. Similarly,

$$\begin{aligned}
\mathbb{P}\left[I_t = 1 \middle| \mathcal{F}_{t-1}, E^{\widetilde{\mu}}_{t,i}\right] \geq & \mathbb{P}\left[\forall j > 1 : \widetilde{\mu}_{t,1} > y_i \geq \widetilde{\mu}_{t,j} \middle| \mathcal{F}_{t-1}, E^{\widetilde{\mu}}_{t,i}\right] \\
= & \mathbb{P}\left[\widetilde{\mu}_{t,1} > y_i \middle| \mathcal{F}_{t-1}, E^{\widetilde{\mu}}_{t,i}\right] \cdot \mathbb{P}\left[\forall j > 1 : \widetilde{\mu}_{t,j} \leq y_i \middle| \mathcal{F}_{t-1}, E^{\widetilde{\mu}}_{t,i}\right] \\
= & q_{t,i} \cdot \mathbb{P}\left[\forall j > 1 : \widetilde{\mu}_{t,j} \leq y_i \middle| \mathcal{F}_{t-1}, E^{\widetilde{\mu}}_{t,i}\right].
\end{aligned}$$

Combining the above two inequalities and multiplying both sides with $\mathbb{P}\left[E^{\widetilde{\mu}}_{t,i}\middle| \mathcal{F}_{t-1}\right]$ gives the result. ∎

We are now ready to prove Lemma 9.
Proof [Proof of Lemma 9] Following straightforward calculations and using Lemma 11,

$$\sum_{t=1}^{T} \mathbb{P}\left[I_t = i, E^{\widetilde{\mu}}_{t,i}, E^{\widehat{\mu}}_{t,i}\right] \leq \sum_{k=0}^{T-1} \mathbb{E}\left[\frac{1-q_{\tau_k,i}}{q_{\tau_k,i}}\right].$$

Thus, it remains to bound the summands on the right-hand side. To achieve this, we start with rewriting $q_{\tau_k,i}$ as

$$\begin{aligned}
q_{\tau_k,i} = \mathbb{P}\left[\widetilde{\mu}_{\tau_k,1} > y_i \middle| \mathcal{F}_{\tau_k-1}\right] &= \mathbb{P}\left[Z_{\tau_k,1} > \frac{\mu_1 - \widehat{\mu}_{\tau_k,1} - \frac{\Delta_i}{3}}{\beta_{\tau_k,1}} \middle| \mathcal{F}_{\tau_k-1}\right] \\
&= 1 - \exp\left(-\exp\left(-\frac{\mu_1 - \widehat{\mu}_{\tau_k,1} - \frac{\Delta_i}{3}}{\beta_{\tau_k,1}} + \gamma\right)\right),
\end{aligned}$$

so that we have

$$\frac{1-q_{\tau_k,i}}{q_{\tau_k,i}} = \frac{\exp\left(-\exp\left(-\frac{\mu_1-\widehat{\mu}_{\tau_k,1}-\frac{\Delta_i}{3}}{\beta_{\tau_k,1}}+\gamma\right)\right)}{1-\exp\left(-\exp\left(-\frac{\mu_1-\widehat{\mu}_{\tau_k,1}-\frac{\Delta_i}{3}}{\beta_{\tau_k,1}}+\gamma\right)\right)}$$
$$\leq \exp\left(\frac{\mu_1-\widehat{\mu}_{\tau_k,1}-\frac{\Delta_i}{3}}{\beta_{\tau_k,1}}-\gamma\right) = \exp\left(\frac{\mu_1-\widehat{\mu}_{t,1}}{\beta_{\tau_k,1}}\right)\cdot e^{-\gamma-\frac{\Delta_i}{3\beta_{\tau_k,1}}},$$

where we used the elementary inequality $\frac{e^{-1/x}}{1-e^{-1/x}} \leq x$ that holds for all $x \geq 0$. Taking expectations on both sides and using the definition of $\beta_{t,i}$ concludes the proof. ■

A.3. Proof of Lemma 10

Setting $L = \frac{9C^2\log^2\left(T\Delta_i^2/c^2\right)}{\Delta_i^2}$, we begin with the bound

$$\sum_{t=1}^{T}\mathbb{I}_{\left\{I_t=i,\overline{E^{\widetilde{\mu}}_{t,i}},E^{\widehat{\mu}}_{t,i}\right\}} \leq L + \sum_{t=L}^{T}\mathbb{I}_{\left\{\widetilde{\mu}_{t,i}>\mu_1-\frac{\Delta_i}{3},\widehat{\mu}_{t,i}<\mu_i+\frac{\Delta_i}{3},N_{t,i}>L\right\}}.$$

For bounding the expectation of the second term above, observe that

$$\mathbb{P}\left[\widetilde{\mu}_{t,i}>\mu_1-\frac{\Delta_i}{3},\widehat{\mu}_{t,i}<\mu_i+\frac{\Delta_i}{3},N_{t,i}>L\middle|\mathcal{F}_{t-1}\right] \leq \mathbb{P}\left[\widetilde{\mu}_{t,i}>\widehat{\mu}_{t,i}+\frac{\Delta_i}{3},N_{t,i}>L\middle|\mathcal{F}_{t-1}\right]$$
$$\leq \mathbb{P}\left[\beta_{t,i}Z_{t,i}>\frac{\Delta_i}{3},N_{t,i}>L\middle|\mathcal{F}_{t-1}\right] = \mathbb{P}\left[Z_{t,i}>\frac{\Delta_i}{3\beta_{t,i}},N_{t,i}>L\middle|\mathcal{F}_{t-1}\right].$$

By the distribution of the perturbations $Z_{t,i}$, we have

$$\mathbb{P}\left[Z_{t,i}\geq\frac{\Delta_i}{3\beta_{t,i}}\middle|\mathcal{F}_{t-1}\right] = 1-\exp\left(-\exp\left(-\frac{\Delta_i}{3\beta_{t,i}}+\gamma\right)\right)$$
$$\leq \exp\left(-\frac{\Delta_i}{3\beta_{t,i}}+\gamma\right) = \exp\left(-\frac{\Delta_i\sqrt{N_{t,i}}}{3C}+\gamma\right),$$

where we used the inequality $1-e^{-x}\leq x$ that holds for all x and the definition of $\beta_{t,i}$. Noticing that $N_{t,i}$ is measurable in $\mathcal{F}_{t-1}$, we obtain the bound

$$\mathbb{P}\left[Z_{t,i}>\frac{\Delta_i}{3\beta_{t,i}},N_{t,i}>L\middle|\mathcal{F}_{t-1}\right] \leq \exp\left(-\frac{\Delta_i\sqrt{N_{t,i}}}{3C}+\gamma\right)\cdot\mathbb{I}_{\{N_{t,i}>L\}},$$
$$\leq \exp\left(-\frac{\Delta_i\sqrt{L}}{3C}+\gamma\right)\cdot\mathbb{I}_{\{N_{t,i}>L\}} \leq \frac{c^2e^\gamma}{T\Delta_i^2},$$

where the last step follows from using the definition of L and bounding the indicator by 1. Summing up for all t and taking expectations concludes the proof. ■

A.4. The proof of Corollary 5

Following the arguments in Section 5.1, we can show that the number of suboptimal draws can be bounded as

$$\mathbb{E}\left[N_{T,i}\right] \leq 1 + \sigma^2 \frac{A + B\log^2(T\Delta_i^2/\sigma^2)}{\Delta_i^2}$$

for all arms i, with constants $A = e^{\gamma} + 18\sqrt{e}\,(1 + e^{-\gamma})$ and $B = 9$. We can obtain a distribution-independent bound by setting a threshold $\Delta > 0$ and writing the regret as

$$\begin{aligned}
R_T &\leq \sigma^2 \sum_{i:\Delta_i > \Delta} \frac{A + B\log^2(T\Delta_i^2/\sigma^2)}{\Delta_i} + \Delta T \\
&\leq \sigma^2 K \frac{A + B\log^2(T\Delta^2/\sigma^2)}{\Delta} + \Delta T \qquad \text{(since } \log^2(x^2)/x \text{ is monotone decreasing for } x \leq 1) \\
&\leq \sigma\sqrt{TK}\frac{A + B\log^2(K\log^2 K)}{\log K} + \sigma\sqrt{TK}\log K \qquad \text{(setting } \Delta = \sigma\sqrt{K/T}\log K) \\
&\leq \sigma\sqrt{TK}\frac{A + 2B\log^2(K)}{\log K} + \sigma\sqrt{TK}\log K \qquad \text{(using } 2\log\log(x) \leq \log(x)) \\
&\leq \sigma\sqrt{TK}\log K\left(2B + A/\log K\right) + \sigma\sqrt{TK}\log K \\
&\leq (2A + 2B + 1)\,\sigma\sqrt{TK}\log K,
\end{aligned}$$

where we used $\log K \geq \frac{1}{2}$ that holds for $K \geq 2$. The proof is concluded by noting that $2A + 2B + 1 \approx 187.63 < 200$. ■

A.5. The proof of Theorem 6

The simple counterexample for the proof follows the construction of Section 3 of Agrawal and Goyal (2013). Consider a problem with deterministic rewards for each arm: the optimal arm 1 always gives a reward of $\Delta = \sqrt{\frac{K}{T}}C_1$ and all the other arms give rewards of 0. Define B_{t-1} as the event that $\sum_{i=2}^{K} N_{t,i} \leq \frac{C_2\sqrt{KT}}{\Delta}$. Let us study two cases depending on the probability $\mathbb{P}\left[A_{t-1}\right]$: If $\mathbb{P}\left[A_{t-1}\right] \leq \frac{1}{2}$, we have

$$R_T \geq R_t \geq \mathbb{E}\left[\sum_i N_{t,i}\Delta \,\middle|\, \overline{A_{t-1}}\right] \cdot \frac{1}{2}. \geq \frac{1}{2}C_2\sqrt{KT}. \tag{8}$$

In what follows, we will study the other case when $\mathbb{P}\left[A_{t-1}\right] \geq \frac{1}{2}$. We will show that, under this assumption, a suboptimal arm will be drawn in round t with at least constant probability. In particular, we have

$$\begin{aligned}
\mathbb{P}\left[I_t \neq 1\right] &= \mathbb{P}\left[\exists i > 1 :\, \widetilde{\mu}_{t,1} < \widetilde{\mu}_{t,i}\right] \\
&\geq \mathbb{P}\left[\widetilde{\mu}_{t,1} < \mu_1,\, \exists i > 1 :\, \mu_1 < \widetilde{\mu}_{t,i}\right] \\
&\geq \mathbb{P}\left[\widetilde{\mu}_{t,1} < \mu_1,\, \exists i > 1 :\, \mu_1 < \widetilde{\mu}_{t,i} \middle|\, A_{t-1}\right] \mathbb{P}\left[A_{t-1}\right] \\
&\geq \mathbb{E}\left[\mathbb{P}\left[\widetilde{\mu}_{t,1} < \mu_1,\, \exists i > 1 :\, \mu_1 < \widetilde{\mu}_{t,i} \middle|\, \mathcal{F}_{t-1}, A_{t-1}\right]\right] \frac{1}{2} \\
&= \mathbb{E}\left[\mathbb{P}\left[\widetilde{\mu}_{t,1} < \mu_1 \middle|\, \mathcal{F}_{t-1}, A_{t-1}\right] \cdot \mathbb{P}\left[\exists i > 1 :\, \mu_1 < \widetilde{\mu}_{t,i} \middle|\, \mathcal{F}_{t-1}, A_{t-1}\right]\right] \frac{1}{2} \\
&= \mathbb{E}\left[\mathbb{P}\left[Z_{t,1} < 0 \middle|\, \mathcal{F}_{t-1}, A_{t-1}\right] \cdot \mathbb{P}\left[\exists i > 1 :\, \Delta < \beta_{t,i} Z_{t,i} \middle|\, \mathcal{F}_{t-1}, A_{t-1}\right]\right] \frac{1}{2}.
\end{aligned}$$

To proceed, observe that $\mathbb{P}\left[Z_{t,1} < 0\right] \geq 0.1$ and

$$\begin{aligned}
&\mathbb{P}\left[\exists i > 1 :\, \Delta < \beta_{t,i} Z_{t,i} \middle|\, \mathcal{F}_{t-1}, A_{t-1}\right] = \mathbb{P}\left[\exists i > 1 :\, \Delta\sqrt{N_{t,i}} < Z_{t,i} \middle|\, \mathcal{F}_{t-1}, A_{t-1}\right] \\
&\qquad = 1 - \prod_{i>1} \exp\left(-\exp\left(-\Delta\sqrt{N_{t,i}} + \gamma\right)\right) \\
&\qquad = 1 - \exp\left(-\sum_{i>1} \exp\left(-\Delta\sqrt{N_{t,i}} + \gamma\right)\right) \\
&\qquad = 1 - \exp\left(-\sum_{i>1} \frac{K-1}{K-1} \exp\left(-\Delta\sqrt{N_{t,i}} + \gamma\right)\right) \\
&\qquad \geq 1 - \exp\left(-\left(K-1\right)\exp\left(-\Delta\sqrt{\sum_{i>1}\frac{N_{t,i}}{K-1}} + \gamma\right)\right) \qquad \text{(by Jensen's inequality)} \\
&\qquad \geq 1 - \exp\left(-\left(K-1\right)\exp\left(-\Delta\sqrt{\frac{C_2\sqrt{KT}}{\Delta\left(K-1\right)}} + \gamma\right)\right) \\
&\qquad = 1 - \exp\left(-\left(K-1\right)\exp\left(-\Delta\sqrt{\frac{C_2 T}{C_1\left(K-1\right)}} + \gamma\right)\right) \\
&\qquad \geq 1 - \exp\left(-\exp\left(-C_1\sqrt{\frac{C_2}{C_1}} + \log(K-1) + \gamma\right)\right).
\end{aligned}$$

Setting $C_2 = C_1 = \log K$, we obtain that whenever $\mathbb{P}\left[A_{t-1}\right] \geq \frac{1}{2}$, we have

$$\begin{aligned}
\mathbb{P}\left[I_t \neq 1\right] &\geq 1 - \exp\left(-\exp\left(-\log K + \log(K-1) + \gamma\right)\right) \\
&\geq 1 - \exp\left(-\exp\left(\gamma\right)\right) \geq 0.83 > \frac{1}{2}.
\end{aligned}$$

This implies that the regret of our algorithm is at least

$$\frac{1}{2} T\Delta = \frac{1}{2}\sqrt{TK}\log K.$$

Together with the bound of Equation (8) for the complementary case, this concludes the proof. ■