Existence of sparsely supported correlated equilibria*

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Abstract

We show that every N-player $K_1 \times \cdots \times K_N$ game possesses a correlated equilibrium with at least $\prod_{i=1}^N K_i - 1 - \sum_{i=1}^N K_i(K_i - 1)$ zero entries. In particular, the largest N-player $K \times \cdots \times K$ games with unique fully supported correlated equilibrium are two-player games.
Keywords Correlated equilibrium; support; finite games.

1 The result

Consider an N-player $K_1 \times \cdots \times K_N$ normal form game $\gamma = (N, S, \{\gamma^i\}_{i=1}^N)$, where, for each player $i = 1, \ldots, N$, S_i is a set of pure strategies with

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$K_i = \#S_i \geq 2$ and $\gamma^i : S \to I\!R$ is a payoff function defined on the set of pure strategy profiles $S = \times_{i=1}^N S_i$. Denote by $\Delta(S)$ the set of probability distributions on S and by $S_{-i} = \times_{j \neq i} S_j$ the set of pure strategy profiles of the players other than i. Given $s^i \in S_i$ and $s^{-i} \in S_{-i}$, we sometimes write $s = (s^{-i}, s^i)$ for a generic element of S.

A probability distribution $p = (p(s))_{s \in S} \in \Delta(S)$ is a *correlated equilibrium* of the game γ if it satisfies, for all $i = 1, \ldots, N$, and all $s^i, t^i \in S_i$ with $s^i \neq t^i$,

$$\sum_{s^{-i} \in S_{-i}} p(s^{-i}, s^i) \left(\gamma^i(s^{-i}, s^i) - \gamma^i(s^{-i}, t^i) \right) \geq 0.$$

We refer to these $\sum_{i=1}^N K_i(K_i - 1)$ inequalities as the *incentive constraints*. The notion of correlated equilibrium was introduced by Aumann [2] as a rich generalization of Nash equilibrium. The set $\mathcal{C} \subset \Delta(S)$ of correlated equilibria is a nonempty convex polytope defined by the incentive constraints, as well as the nonnegativity constraints, $p(s) \geq 0$, $s \in S$, and the constraint $\sum_{s \in S} p(s) = 1$, the latter two guaranteeing $p \in \Delta(S)$. Recall that a correlated equilibrium that is a product measure is also a Nash equilibrium.

The purpose of this note is to point out that games with many players have sparsely supported correlated equilibria. More precisely, the main result is the following.

Theorem 1 *Any N-player $K_1 \times \cdots \times K_N$ game possesses a correlated equilibrium with at least $\prod_{i=1}^N K_i - 1 - \sum_{i=1}^N K_i(K_i - 1)$ zero entries.*

To better understand the result, consider an N-player $2 \times \cdots \times 2$ game. There are 2^N pure strategy profiles, yet the theorem implies the existence of a correlated equilibrium with at least $2^N - 1 - 2N$ zeros in its support. Thus, there always exists a correlated equilibrium concentrated on an *exponentially small fraction* of pure strategy profiles.

The result also has implications for the (non)existence of games with unique fully supported correlated equilibria.

Corollary 1 *For $N \geq 3$, there exist no $K \times \cdots \times K$ games with unique fully supported correlated equilibrium.*

To see this, notice that, for any $N \geq 3$ and $K \geq 2$, we have

$$K^{N-1} - N(K-1) \geq 1.$$

Hence $K^N - NK^2 + NK = K(K^{N-1} - N(K-1)) \geq 2$ since $K \geq 2$. Therefore,

$$\prod_{i=1}^{N} K_i - 1 - \sum_{i=1}^{N} K_i(K_i - 1) = K^N - NK^2 + NK - 1 \geq 1,$$

which means that, for $N \geq 3$, there always exists a correlated equilibrium with at least one zero entry and which cannot be fully supported.

This is to be contrasted with Nitzan [4], who shows that the set of two-player $K \times K$ games possessing a unique fully supported correlated equilibrium has positive measure for any K. It complements Nitzan's results by showing that not only do such games have zero measure as soon as $N \geq 3$, but that such games simply cannot exist in these "remaining" cases.

Finally, recall that if an N-player game has a unique correlated equilibrium, then it must also be a Nash equilibrium and hence a product measure. For simplicity, consider again $2 \times \cdots \times 2$ games. The correlated equilibrium has at least $2^N - 1 - 2N$ zero entries and thus at most $2N + 1$ atoms, and, since it is a product measure, there are at most $\log_2(2N+1)$ non-degenerate marginal distributions. This implies the following fact suggesting that large games with a unique correlated equilibrium must be quite "degenerate" in some sense.

Corollary 2 *Consider an N-player $2 \times \cdots \times 2$ game with a unique correlated equilibrium. At this equilibrium there are at most $\log_2(2N + 1)$ players who use a non-degenerate mixed strategy, all others play a pure strategy.*

2 The proof

To simplify notation, set $d = \prod_{i=1}^{N} K_i - 1$, $m = \sum_{i=1}^{N} K_i(K_i - 1)$, and assume, without loss of generality, $d > m$. We identify a probability distribution $p \in \Delta(S)$ over the set of pure strategy profiles with a $(d+1)$-vector p of non-negative components p_j, satisfying $\sum_{j=1}^{d+1} p_j = 1$. Each incentive constraint takes the form of a linear inequality, which can be written as $C_k p \geq 0$, $k = 1, \ldots, m$.

Fix $q^0 \in \mathcal{C}$, which exists since $\mathcal{C}$ is nonempty, (e.g., Aumann [2], Hart and Schmeidler [3]), and consider the affine subspace,

$$H_0 = \{p \in I\!R^{d+1} : \sum_{j=1}^{d+1} p_j = 1 \text{ and } C_k p = C_k q^0, \quad k = 1, \ldots, m\}.$$

By the dimension theorem, (e.g., Artin [1]), H_0 has dimension at least $d - m$, and any point in H_0 satisfies all the incentive constraints defining $\mathcal{C}$. Moreover, since $H_0 \subset I\!R^{d+1}$ is defined by $m+1$ equalities, there exist $d-m$ entries whose values can be set arbitrarily, yet the system of $m+1$ equations defining H_0 has a solution with these restricted values in the $d-m$ entries. In particular, there exists $\overline{q}^0 \in H_0$ in which $d-m$ entries are equal to, say, -1, and, without loss of generality, we can assume $\overline{q}^0$ to be of the form

$$\overline{q}^0 = (-1, \ldots, -1; p_{d-m+1}, \ldots, p_{d-m_1}; p_{d-m_1+1}, \ldots, p_{d+1}), \qquad m_1 \leq m,$$

where the first $d-m$ entries are -1's, the next $m-m_1$ entries, $p_{d-m-1}, .., p_{d-m_1}$, are nonpositive, and the remaining m_1+1 entries, $p_{d-m_1+1}, .., p_{d+1}$, are positive (m_1+1 is thus the number of positive entries of $\overline{q}^0$).

Consider now the line segment $L \subset H_0$ between q^0 and $\overline{q}^0$. It intersects the union of hyperplanes, $\cup_{j=1}^{d-m_1}\{p : p_j = 0\}$, at least once and at most $d - m_1$ times. Take the first intersection encountered when moving from q^0 ($\in \mathcal{C}$) towards $\overline{q}^0$ along L and denote the point of intersection by $q^1 \in \mathcal{C}$. Let $N_1 \subset \{1, \ldots, d - m_1\}$ be the set of nonnegativity constraints holding with equality at q^1. Set $\#N_1 = n_1$ and notice that, since $d - m > 0$, we have $n_1 > 0$. If $d - m - n_1 \leq 0$, then we are done, since we have found a point $q^1 \in \mathcal{C}$ with at least $n_1 \geq d - m$ zero entries. If, however, $d - m - n_1 > 0$, then repeating the procedure (at most $d - m$ times) will eventually lead to a point in $\mathcal{C}$ with the desired property. More specifically, starting with $\ell = 1$, consider the following.

PROCEDURE: Suppose $q^\ell \in \mathcal{C}$ is given, together with the corresponding set $N_\ell \subset \{1, \ldots, d - m_\ell\}$, and numbers n_ℓ and m_ℓ, and suppose $d - m - n_\ell > 0$. Consider the affine subspace

$$H_\ell = \{p \in I\!R^{d+1} : p_j = 0, j \in N_\ell, \ \sum_{j=1}^{d+1} p_j = 1, \ \ C_k p = C_k q^\ell, k = 1, \ldots, m\}.$$

By the dimension theorem, H_ℓ has dimension at least $d - m - n_\ell$, and any point in H_ℓ satisfies: the nonnegativity constraints in N_ℓ with equality, the constraint $\sum_{j=1}^{d+1} p_j = 1$, and all the incentive constraints defining $\mathcal{C}$. Therefore there exists $\overline{q}^\ell \in H_\ell$ in which $d - m - n_\ell$ entries are equal to, say, -1, and, again without loss, the positive entries coincide with the last $m_{\ell+1}+1$ entries, for some $0 \leq m_{\ell+1} \leq m$; (it is always possible to relabel the coordinates and the matrix C defining the incentive constraints at each iteration ℓ).

Next, as before, consider the line segment from q^ℓ ($\in \mathcal{C}$) to $\overline{q}^\ell$. Again, the segment is entirely contained in H_ℓ and it eventually leads to a (first) intersection, say at $q^{\ell+1} \in \mathcal{C}$, with one or more of the hyperplanes defining $\cup_{j=1, j \notin N_\ell}^{d-m_{\ell+1}} \{p : p_j = 0\}$. Once again, the point $q^{\ell+1}$ implies a corresponding set $N_{\ell+1} \subset \{1, \ldots, d - m_{\ell+1}\}$ of nonnegativity constraints holding with equality at $q^{\ell+1}$, as well as numbers $n_{\ell+1} = \#N_{\ell+1}$ ($> n_\ell$) and $m_{\ell+1}$ ($\leq m$), where $m_{\ell+1} + 1$ is the number of positive entries of $\overline{q}^\ell$.

If $d - m - n_{\ell+1} \leq 0$, then, again, we are done; otherwise, repeat the above procedure with $\ell = \ell + 1$. Notice that while $d - m - n_\ell > 0$, repeating the procedure always yields a new point $\overline{q}^\ell$ with at least one entry equal to -1, which in turn yields a point $q^{\ell+1} \in \mathcal{C}$ with at least one additional entry equal to zero, thus $n_{\ell+1} > n_\ell$. Therefore, there exists $\overline{\ell} \leq d-m$, such that repeating the procedure $\overline{\ell}$ times, eventually yields an affine space $H_{\overline{\ell}}$ of dimension greater or equal to zero, and a point $q^{\overline{\ell}}$ satisfying at least $d-m$ nonnegativity constraints with equality, the constraint $\sum_{j=1}^{d+1} p_j = 1$, as well as all the incentive constraints defining $\mathcal{C}$. In other words, it eventually yields a point $p = q^{\overline{\ell}} \in \mathcal{C}$ with at least $d-m$ zero entries, which completes the proof.

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