Concentration of the spectral norm of Erdős-Rényi random graphs *

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Abstract

We present results on the concentration properties of the spectral norm $\|A_p\|$ of the adjacency matrix A_p of an Erdős-Rényi random graph $G(n,p)$. We prove sharp sub-Gaussian moment inequalities for $\|A_p\|$ for all $p \in [c \log^3 n/n, 1]$ that improve the general bounds of Alon, Krivelevich, and Vu [1] for small values of p. We also consider the Erdős-Rényi random graph *process* and prove that $\|A_p\|$ is *uniformly* concentrated.

1 Introduction

An Erdős-Rényi random graph $G(n,p)$, named after the authors of the pioneering work [9], is a graph defined on the vertex set $[n] = \{1,\ldots,n\}$ in which any two vertices $i,j \in [n]$, $i \neq j$, are connected by an edge independently, with probability p. Such a random graph is represented by its adjacency matrix A_p. A_p is a symmetric

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matrix whose entries are

$$A_{i,j}^{(p)} = \begin{cases} 0 & \text{if } i = j \\ \mathbb{1}_{U_{i,j}<p} & \text{if } 1 \le i < j \le n \\ \mathbb{1}_{U_{i,j}<p} & \text{if } 1 \le j < i \le n \,, \end{cases} \tag{1.1}$$

where $(U_{i,j})_{1\le i<j\le n}$ are independent random variables, uniformly distributed on $[0,1]$ and $\mathbb{1}$ stands for the indicator function. We call the family of random matrices $(A_p)_{p\in[0,1]}$ the *Erdős-Rényi random graph process*.

Spectral properties of adjacency matrices of random graphs have received considerable attention, see Füredi and Komlós [11], Krivelevich and Sudakov [13], Vu [18], Erdős, Knowles, Yau, and Yin [10], Benaych-Georges, Bordenave, and Knowles [3, 4], Jung and Lee [12], Tran, Vu, and Wang [16], among many other papers.

In this paper we are primarily concerned with concentration properties of the spectral norm $\|A_p\|$ of the adjacency matrix. It follows from a general concentration inequality of Alon, Krivelevich, and Vu [1] for the largest eigenvalue of symmetric random matrices with bounded independent entries that for all $n \ge 1$, $p \in [0,1]$, and $t > 0$,

$$\mathbb{P}\left\{ \left| \|A_p\| - \mathbb{E}\|A_p\| \right| > t \right\} \le 2e^{-t^2/32} \,. \tag{1.2}$$

In particular, $\mathrm{Var}(\|A_p\|) \le C$ for a universal constant C. (One may take $C = 16$, see [7, Example 3.14].) In this paper we strengthen (1.2) in two different ways. First we show that, for small values of p, $\|A_p\|$ is significantly more concentrated than what this bound suggests. Indeed, we prove that there exists a universal constant C such that

$$\mathrm{Var}(\|A_p\|) \le Cp$$

for all n and $p \ge C\log^3 n/n$. We also prove sub-Gaussian inequalities for moments of $\|A_p\|$ of higher order (up to order approximately np). The precise statement is given in Theorem 1 in Section 2.1 below.

The other results of this paper concern *uniform* concentration of the spectral norm. In particular, we prove that there exists a universal constant C such that

$$\mathbb{E} \sup_{p\ge C\log n/n} \left| \|A_p\| - \mathbb{E}\|A_p\| \right| \le C$$

(see Theorem 2 below). We leave open the question whether the restriction to the range $p \in [C\log n/n, 1]$ is necessary for uniform concentration. For the entire range $p \in [0,1]$, we are able to prove the slightly weaker inequality

$$\mathbb{E} \sup_{p\in[0,1]} \left| \|A_p\| - \mathbb{E}\|A_p\| \right| \le C\sqrt{\log\log n}$$

for a constant C (Theorem 3).

We also prove

$$\mathbb{P}\left\{\sup_{p\geq C\log n/n}\left|\|A_p\|-\mathbb{E}\|A_p\|\right|>t\right\}\leq e^{-t^2/C} ,$$

a uniform version of the sub-Gaussian inequality (1.2).

Note that it follows from the Perron-Frobenius theorem that the spectral norm of A_p equals the largest eigenvalue of A_p, that is, $\|A_p\|=\lambda_p$. We use both interchangeably throughout the paper, depending on the particular interpretation that is convenient.

The proof of both inequalities crucially hinges on the so-called *delocalization* property of the eigenvector corresponding to the largest eigenvalue (see Erdős, Knowles, Yau, and Yin [10], Mitra [15]), that is, the fact that the normalized eigenvector corresponding to the largest eigenvalue is close, in a certain sense, to the vector $(1/\sqrt{n},\dots,1/\sqrt{n})$. We provide delocalization bounds for the top eigenvector of A_p tailored to our needs (Lemma 1) and a uniform delocalization inequality (Lemma 4).

The rest of the paper is organized as follows. In Section 2 we formalize and discuss the results of the paper, including the moment inequalities for $\|A_p\|$ and the uniform concentration results. The proofs are presented in Section 3.

2 Results

2.1 Moment inequalities for the spectral norm

The first result of the paper shows that typical deviations of $\|A_p\|$ from its expected value are of the order of $\sqrt{p}$. This is in accordance with the asymptotic normality theorem of Füredi and Komlós [11]. However, while the result of [11] holds for fixed p as $n\to\infty$, the theorem below is non-asymptotic. In particular, it holds for $p=o(1)$ as long as np is at least of the order of $\log^3 n$. Note that the non-asymptotic concentration inequality of [1] only implies that typical deviations are $O(1)$.

Theorem 1. *There exist constants $c,C,C',\kappa>0$ such that for all n and $p\geq\kappa\log^3(n)/n$*

$$\mathrm{Var}(\|A_p\|)\leq Cp .$$

Moreover, for every $k\in\left(2,\frac{c\left(\frac{\log(np)}{\log n}\right)^2 p(n-1)-\log(8(n-1))}{\log(\frac{1}{p})+\log(11^5/4)}\right]$,

$$\mathbb{E}\left[\left(\|A_p\|-\mathbb{E}\|A_p\|\right)_+\right]^{1/k}\leq(Ckp)^{\frac{1}{2}}$$

and

$$\mathbb{E}\left[\left(\|A_p\| - \mathbb{E}\|A_p\|\right)_-\right]^{1/k} \le (C'kp)^{\frac{1}{2}} .$$

It is natural to ask whether the condition $p \ge \kappa \log^3(n)/n$ is necessary. The fact that the inequality $\mathrm{Var}(\|A_p\|) \le Cp$ cannot hold for all values of p is easily seen by taking $p = c/n^2$ for a positive constant c. In this case, the probability that the graph $G(n,p)$ is empty is bounded away from zero. In that case $\|A_p\| = 0$. On the other hand, with a probability bounded away from zero, the graph $G(n,p)$ contains a single edge, in which case $\|A_p\| = 1$. Thus, for $p = c/n^2$, $\mathrm{Var}(\|A_p\|) = \Omega(1)$, showing that the bound of [1] is sharp in this range. Understanding the concentration properties of $\|A_p\|$ in the range $n^{-2} \ll p \ll \log^3(n)/n$ is an intriguing open question.

The proof of Theorem 1 is presented in Section 3.1. The proof reveals that for the values of the constants one may take $\kappa = 2 \times 835^2$, $C = 966306$, $C' = 1339945$, and $c = 1/9408$. However, these values have not been optimized. In the rest of this discussion we assume these numerical values.

Using the moment bound with $k = t^2/(2Cp)$, Markov's inequality implies that for all $0 < t \le 2\sqrt{Ccp}\sqrt{n-1}\log(np)/(\log n \log(1/p))$,

$$\mathbb{P}\left\{\|A_p\| > \mathbb{E}\|A_p\| + t\right\} \le 2^{-t^2/(2Cp)} .$$

The proof is based on general moment inequalities of Boucheron, Bousquet, Lugosi, and Massart [6] (see also [7, Theorems 15.5 and 15.7]) that state that if $Z = f(X_1,\dots,X_n)$ is a real random variable that is a function of the independent random variables $X_1,\dots,X_n$, then for all $k \ge 2$,

$$\mathbb{E}\left[(Z - \mathbb{E}Z)_+^k\right]^{1/k} \le \sqrt{3k}\left(\mathbb{E}\left[{V_+}^{k/2}\right]\right)^{1/k} , \qquad (2.1)$$

and

$$\mathbb{E}\left[(Z - \mathbb{E}Z)_-^k\right]^{1/k} \le \sqrt{4.16k}\left(\left(\mathbb{E}\left[{V_+}^{k/2}\right]\right)^{1/k} \vee \sqrt{k}\left(\mathbb{E}\left[M^k\right]\right)^{1/k}\right) , \qquad (2.2)$$

where the random variable V_+ is defined as

$$V_+ = \mathbb{E}' \sum_{i=1}^{n} (Z - Z_i')_+^2 .$$

Here $Z_i' = f(X_1,\dots,X_{i-1},X_i',X_{i+1},\dots,X_n)$ with $X_1',\dots,X_n'$ being independent copies of $X_1,\dots,X_n$ and $\mathbb{E}'$ denotes expectation with respect to $X_1',\dots,X_n'$. Moreover,

$$M = \max_i (Z - Z_i')_+ .$$

Recall also that, by the Efron-Stein inequality (e.g., (see also [7, Theorem 3.1])

$$\mathrm{Var}(Z) \leq \mathbb{E}V_+ .$$

The proof of Theorem 1 is based on (2.1), applied for the random variable $Z = \|A_p\|$. In order to bound moments of the random variable V_+, we make use of the fact that the eigenvector of A_p corresponding to the largest eigenvalue is nearly uniform. An elegant way of proving such results appears in Mitra [15]. We follow Mitra's approach though we need to modify his arguments in order to achieve stronger probabilistic guarantees for weak ℓ_∞ delocalization bounds. In Lemma 1 we provide the bound we need for the proof of Theorem 1.

2.2 Uniform concentration for the Erdős-Rényi random graph process

Next we state our inequalities for the uniform concentration of the spectral norm $\|A_p\|$—or, equivalently, for the largest eigenvalue λ_p of the adjacency matrix A_p defined by (1.1). Our first result shows that

Theorem 2. *There exists a constant C such that, for all n,*

$$\mathbb{E} \sup_{p \in \left[\frac{64\log n}{n},1\right]} \left|\lambda_p - \mathbb{E}\lambda_p\right| \leq C .$$

Moreover, for all $t \geq 2C$,

$$\mathbb{P}\left\{ \sup_{p \in \left[\frac{64\log n}{n},1\right]} \left|\lambda_p - \mathbb{E}\lambda_p\right| \geq t \right\} \leq \exp(-t^2/128) .$$

For the numerical constant, our proof provides the (surely suboptimal) value $C = 5 \times 10^8$. Once again, our proof is based on the fact that the normalized eigenvector corresponding to the largest eigenvalue of A_p stays close to the vector $(1/\sqrt{n}, \ldots, 1/\sqrt{n})$. In Lemma 4 we prove an ℓ_2 bound that holds uniformly over intervals of the form $[q, 2q]$ when $q \in [4\log n/n, 1/2]$. It is because of the restriction of the range of q in the uniform delocalization lemma that we need to impose $p \geq 64\log n/n$ in Theorem 2. We do not know whether the uniform concentration bound holds over the entire interval $p \in [0,1]$. However, we are able to prove the following, only slightly weaker bound.

Theorem 3. *There exists a constant C' such that, for all n,*

$$\mathbb{E} \sup_{p \in [0,1]} \left|\lambda_p - \mathbb{E}\lambda_p\right| \leq C' \sqrt{\log\log n} .$$

The proof of Theorem 3 uses direct approximation arguments to handle the interval $p \in [0, 64\log n/n]$. In particular, we show that

$$\mathbb{E} \sup_{p\in[0,64\log n/n]} |\lambda_p - \mathbb{E}\lambda_p| \le 5\sqrt{16 + 2\log\log n} ,$$

which, combined with Theorem 2 implies Theorem 3.

3 Proofs

3.1 Proof of Theorem 1

Let v_p denote an eigenvector corresponding to the largest eigenvalue of A_p such that $\|v_p\| = 1$. Recall that $\kappa = 2 \times 835^2$ and $c = 1/9408$. One of the key elements of the proof is the following variant of a delocalization inequality of Mitra [15].

Lemma 1. *Let $n \ge 7$ and $p \ge \kappa \log^3(n)/n$. Let v_p denote an eigenvector corresponding to the largest eigenvalue λ_p of A_p with $\|v_p\|_2 = 1$. Then, with probability at least*

$$1 - 4(n-1)\exp\left(-2c\left(\frac{\log(np)}{\log n}\right)^2 (n-1)p\right) ,$$

$$\left\|v_p\right\|_\infty \le \frac{11}{\sqrt{n}} .$$

The lemma is proved in Section 3.3 below. Based on this lemma, we may prove Theorem 1:

Proof of Theorem 1. We apply (2.1) for the random variable $Z = \|A_p\|$, as a function of the $\binom{n}{2}$ independent Bernoulli random variables $A_{i,j} = A^{(p)}_{i,j}$, $1 \le i < j \le n$. Let E_1 denote the event $\|v_p\|_\infty \le 11/\sqrt{n}$. By Lemma 1,

$$\mathbb{P}\{E_1\} \ge 1 - 4(n-1)\exp\left(-\frac{1}{4704}\left(\frac{\log(np)}{\log n}\right)^2 (n-1)p\right) .$$

For $1 \le i < j \le n$, denote by $\lambda'_{i,j}$ the largest eigenvalue of the adjacency matrix obtained by replacing $A_{i,j}$ (and $A_{j,i}$) by an independent copy $A'_{i,j}$ and keeping all other entries unchanged. If the components of the eigenvector v_p (corresponding to the eigenvalue λ_p) are $(v_1, \dots, v_n)$, then

$$V_+ = \mathbb{E}' \sum_{i<j}^{n} (\lambda_p - \lambda'_{i,j})_+^2 \le 4 \sum_{i<j}^{n} \mathbb{E}'\left[v_i^2 v_j^2 (A_{i,j} - A'_{i,j})^2\right] = 4 \sum_{i<j}^{n} v_i^2 v_j^2 (p + (1-2p)A_{i,j})_+ .$$

Since $(A_{i,j} - A'_{i,j})^2 \le 1$ and $\sum_i v_i^2 = 1$, we always have $V_+ \le 4$. On the event E_1, we have a better control:

$$V_+ \mathbb{1}_{E_1} \le \frac{4 \cdot 11^4}{n^2} \left(\binom{n}{2} p + (1-2p) \sum_{i<j} A_{i,j} \right) .$$

Let E_2 denote the event that $\sum_{i<j}^{n} A_{i,j} \le pn(n-1)$. By Bernstein's inequality, $\mathbb{P}\{E_2\} \ge 1 - \exp(-\frac{3pn(n-1)}{8})$. Then

$$V_+ \mathbb{1}_{E_1 \cap E_2} \le 11^5 p .$$

Thus,

$$\begin{aligned}
\mathbb{E}\left[(V_+)^{\frac{k}{2}}\right] &= \mathbb{E}\left[(V_+)^{\frac{k}{2}} \mathbb{1}_{E_1 \cap E_2}\right] + \mathbb{E}\left[(V_+)^{\frac{k}{2}} \left(\mathbb{1}_{\overline{E}_1} + \mathbb{1}_{\overline{E}_2}\right)\right] \\
&\le \left(11^5 p\right)^{k/2} + 4^{k/2} \left(\mathbb{P}\{\overline{E}_1\} + \mathbb{P}\{\overline{E}_2\}\right) \\
&\le 2\left(11^5 p\right)^{k/2} ,
\end{aligned}$$

whenever $\mathbb{P}\{\overline{E}_1\} + \mathbb{P}\{\overline{E}_2\} \le \left(11^5 p/4\right)^{k/2}$. This holds whenever

$$8(n-1)\exp\left(-\frac{1}{4704}\left(\frac{\log(np)}{\log n}\right)^2 (n-1)p\right) \le \left(11^5 p/4\right)^{k/2} ,$$

guaranteed by assumption on k. The proof of the bound for the upper tail follows from (2.1). The bound for the variance follows from the Efron-Stein inequality.

For the bound for the lower tail we use (2.2). Note that

$$\max_{i<j} (\lambda_p - \lambda'_{i,j})_+ \mathbb{1}_{E_1} \le 2(v_i v_j (A_{i,j} - A'_{i,j}))_+ \mathbb{1}_{E_1} \le \frac{72}{n} ,$$

and therefore

$$\mathbb{E} \max_{i<j} (v_i v_j (A_{i,j} - A'_{i,j}))_+^k \mathbb{1}_{E_1} \le \left(\frac{72}{n}\right)^k .$$

Moreover,

$$\mathbb{E} \max_{i<j} (v_i v_j (A_{i,j} - A'_{i,j}))_+^k \mathbb{1}_{\overline{E}_1} \le \mathbb{P}\left\{\overline{E}_1\right\} \le 4(n-1)\exp\left(-\frac{1}{4704}\left(\frac{\log(np)}{\log(n)}\right)^2 (n-1)p\right) .$$

We require

$$\left(\frac{72}{n}\right)^k \ge 4(n-1)\exp\left(-\frac{1}{4704}\left(\frac{\log(np)}{\log(n)}\right)^2 (n-1)p\right)$$

which holds whenever

$$k \le \frac{\frac{1}{4704}\left(\frac{\log(np)}{\log(n)}\right)^2 (n-1)p - \log(4(n-1))}{\log(\frac{n}{72})} .$$

Under this condition

$$\left(\mathbb{E}\max_{i<j}(v_i v_j (A_{i,j} - A'_{i,j}))_+^k\right)^{\frac{1}{k}} \le \frac{144}{n} .$$

Under our conditions for k, we have $k(144/n)^2 \le 2\cdot 11^5 p$ and therefore (2.2) implies the last inequality of Theorem 1.

3.2 Proof of Theorem 2

We begin by noting that, if $p \le q$, then A_q is element-wise greater than or equal to A_p and therefore $\|A_p\| \le \|A_q\|$ whenever $p \le q$. (see Corollary 1.5 in [5]).

We start with a lemma for the expected spectral norm for a sparse Erdős-Rényi graph. Since the largest eigenvalue of the adjacency matrix is always bounded by the maximum degree of the graph, $\mathbb{E}\|A_{\frac{1}{n}}\|$ is at most of the order $\log n$. The next lemma improves this naive bound to $O(\sqrt{\log n})$. With more work, it is possible to improve the rate to $\sqrt{\frac{\log n}{\log\log n}}$ (see the asymptotic result in [13]). However, this slightly weaker version is sufficient for our purposes.

Lemma 2. *For all n,*

$$\mathbb{E}\|A_{\frac{1}{n}}\| \le 173\sqrt{\log n} .$$

Proof. First write

$$\mathbb{E}\|A_{\frac{1}{n}}\| \le \mathbb{E}\|A_{\frac{1}{n}} - \mathbb{E}A_{\frac{1}{n}}\| + \|\mathbb{E}A_{\frac{1}{n}}\| \le \mathbb{E}\|A_{\frac{1}{n}} - \mathbb{E}A_{\frac{1}{n}}\| + 1 .$$

Denote $B = A_{\frac{1}{n}} - \mathbb{E}A_{\frac{1}{n}}$ and let B' be an independent copy of B. Denoting by $\mathbb{E}'$ the expectation operator with respect to B', note that $\mathbb{E}'B' = 0$ and therefore, by Jensen's inequality,

$$\mathbb{E}\|B\| = \mathbb{E}\|B - \mathbb{E}'B'\| \le \mathbb{E}\|B - B'\| .$$

The matrix $B - B'$ is zero mean, its non-diagonal entries have a symmetric distribution with variance $(2/n)(1-1/n)$ and all entries have absolute value bounded by 2. Now, applying Corollary 3.6 of Bandeira and van Handel [2] with $\alpha = 3$,

$$\mathbb{E}\|B - B'\| \le e^{\frac{2}{3}}(2\sqrt{2} + 84\sqrt{\log n}) \le 6 + 164\sqrt{\log n} .$$

Thus,

$$\mathbb{E}\|A_{\frac{1}{n}}\| \le 7 + 164\sqrt{\log n} \le 173\sqrt{\log n} .$$

∎

The next lemma and the uniform delocalization inequality of Lemma 4 (presented in Section 3.3) are the crucial building blocks of the proof of Theorem 2.

Lemma 3. *For all n and $q \in [\log n/n, \frac{1}{2}]$,*

$$\mathbb{P}\left\{ \sup_{p\in[q,2q]} \|A_p - \mathbb{E}A_p\| > 420\sqrt{nq} \right\} \le e^{-nq/64} .$$

Proof. By (1.2), for each fixed p and for all $t > 0$, we have

$$\mathbb{P}\Big\{ \|A_p - \mathbb{E}A_p\| - \mathbb{E}\|A_p - \mathbb{E}A_p\| > t \Big\} \le e^{-t^2/32} .$$

On the other hand, using the same symmetrization trick as in Lemma 2, Corollary 3.6 of Bandeira, van Handel [2] implies that for any $p \ge \log n/n$,

$$\mathbb{E}\|A_p - \mathbb{E}A_p\| \le e^{\frac{2}{3}}(2\sqrt{2np} + 84\sqrt{\log n}) \le 170\sqrt{np} .$$

These two results imply

$$\mathbb{P}\Big\{ \|A_p - \mathbb{E}A_p\| > 172\sqrt{np} \Big\} \le e^{-np/8} .$$

Let now $q \ge \log n/n$ and for $i = 0, 1, \dots, \lceil nq \rceil$, define $p_i = q + i/n$. Then

$$\begin{aligned}
\sup_{p\in[p_i,p_{i+1}]} \Big(\|A_p - \mathbb{E}A_p\| - \|A_{p_i} - \mathbb{E}A_{p_i}\| \Big) &\le \sup_{p\in[p_i,p_{i+1}]} \Big(\|A_p - A_{p_i}\| + \|\mathbb{E}A_p - \mathbb{E}A_{p_i}\| \Big) \\
&= \sup_{p\in[p_i,p_{i+1}]} \Big(\|A_p - A_{p_i}\| + \|\mathbb{E}A_{p-p_i}\| \Big) \\
&= \|A_{p_{i+1}} - A_{p_i}\| + \|\mathbb{E}A_{1/n}\| \\
&\le \|A_{p_{i+1}} - A_{p_i}\| + 1 \\
&= \mathbb{E}\|A_{1/n}\| + \Big(\|A_{p_{i+1}} - A_{p_i}\| - \mathbb{E}\|A_{p_{i+1}} - A_{p_i}\| \Big) + 1 \\
&\le 1 + 173\sqrt{\log n} + \sqrt{nq} \\
&\le 176\sqrt{nq}
\end{aligned}$$

with probability at least $1 - e^{-nq/32}$, where we used Lemma 2 and (1.2). Thus, by the union bound, with probability at least $1 - nqe^{-nq/32} - nqe^{-np/8} \ge 1 - e^{-nq/64}$,

$$\begin{aligned}
\sup_{p\in[q,2q]} \|A_p - \mathbb{E}A_p\| &\le \max_{i\in\{0,\dots,\lceil nq\rceil\}} \|A_{p_i} - \mathbb{E}A_{p_i}\| + 176\sqrt{nq} \\
&\le 172\sqrt{2nq} + 176\sqrt{nq} \\
&\le 420\sqrt{nq}.
\end{aligned}$$

as desired. ■

Proof of Theorem 2. Denote by $\overline{1} \in \mathbb{R}^n$ the vector whose components are all equal to 1. Let $B_2^n = \{x \in \mathbb{R}^n : \|x\|_2 \le 1\}$ be the unit Euclidean ball. Define the event E_1 that $v_p \in \frac{\overline{1}}{\sqrt{n}} + \frac{2896}{\sqrt{np}} B_2^n$ for all $p \in [64 \log n/n, 1]$. By Lemma 4 (see Section 3.3 below), for $n \ge 7$,

$$\mathbb{P}\{E_1\} \ge 1 - 4\sum_{j=0}^{\infty} \exp\left(-2^j \log n\right) \ge 1 - 4\sum_{j=0}^{\infty} \left(\frac{1}{n}\right)^{2^j} \ge 1 - \frac{4}{n}\sum_{j=0}^{\infty} \left(\frac{1}{7}\right)^{j} = 1 - \frac{32}{7n} .$$

Now define the event E_2 that for all $p \in \left[\frac{64 \log n}{n}, 1\right]$, $\|A_p - \mathbb{E}A_p\| \le 420\sqrt{2np}$. Similarly to the calculation above, by Lemma 3, $\mathbb{P}\{E_2\} \ge 1 - \frac{32}{7n}$.

Denoting by $S^{n-1} = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$ the Euclidean unit sphere in $\mathbb{R}^n$, define

$$\overline{\lambda}_p = \sup_{x \in S^{n-1}} x^T A_p x \mathbb{1}_{E_1 \cap E_2} \quad \text{and} \quad \overline{A}_p = A_p \mathbb{1}_{E_2} .$$

Then we may write the decomposition

$$\overline{\lambda}_p = \sup_{x \in \frac{\overline{1}}{\sqrt{n}} + \frac{2896}{\sqrt{np}} B_2^n} x^T \overline{A}_p x = \frac{\overline{1}}{\sqrt{n}} \overline{A}_p \frac{\overline{1}}{\sqrt{n}} + 2 \sup_{z \in \frac{2896}{\sqrt{np}} B_2^n} z^T \overline{A}_p \left(\frac{\overline{1}}{\sqrt{n}} + \frac{z}{2}\right) .$$

Then

$$\begin{aligned}
&\sup_{p \in \left[\frac{64 \log n}{n}, 1\right]} |\overline{\lambda}_p - \mathbb{E}\overline{\lambda}_p| \\
&\le 2 \sup_{p \in \left[\frac{64 \log n}{n}, 1\right]} \left| \sup_{z \in \frac{2896}{\sqrt{np}} B_2^n} (z^T \overline{A}_p (\frac{\overline{1}}{\sqrt{n}} + \frac{z}{2})) - \mathbb{E} \sup_{z \in \frac{2896}{\sqrt{np}} B_2^n} (z^T \overline{A}_p (\frac{\overline{1}}{\sqrt{n}} + \frac{z}{2})) \right| \\
&\quad + \sup_{p \in \left[\frac{64 \log n}{n}, 1\right]} \left| \frac{\overline{1}}{\sqrt{n}}^T \overline{A}_p \frac{\overline{1}}{\sqrt{n}} - \mathbb{E} \frac{\overline{1}}{\sqrt{n}}^T \overline{A}_p \frac{\overline{1}}{\sqrt{n}} \right| .
\end{aligned} \tag{3.1}$$

For the second term on the right-hand side of (3.1), since $A_p - \overline{A}_p = A_p \mathbb{1}_{\overline{E}_2}$ we have

$$\begin{aligned}
&\mathbb{E} \sup_{p \in \left[\frac{64 \log n}{n}, 1\right]} \left| \frac{\overline{1}}{\sqrt{n}}^T \overline{A}_p \frac{\overline{1}}{\sqrt{n}} - \mathbb{E} \frac{\overline{1}}{\sqrt{n}}^T \overline{A}_p \frac{\overline{1}}{\sqrt{n}} \right| \\
&\quad \le \mathbb{E} \sup_{p \in \left[\frac{64 \log n}{n}, 1\right]} \left| \frac{\overline{1}}{\sqrt{n}}^T A_p \frac{\overline{1}}{\sqrt{n}} - \mathbb{E} \frac{\overline{1}}{\sqrt{n}}^T A_p \frac{\overline{1}}{\sqrt{n}} \right| + 2nP(\overline{E}_2) .
\end{aligned}$$

Note that $\frac{\overline{1}}{\sqrt{n}}^T A_p \frac{\overline{1}}{\sqrt{n}} = (2/n) \sum_{i<j} \mathbb{1}_{U_{i,j} < p}$. Thus, the first term on the right-hand side is just the maximum deviation between the cumulative distribution function of a uniform random variable and its empirical counterpart based on $\binom{n}{2}$ random samples. This may be bounded by the classical Dvoretzky-Kiefer-Wolfowitz theorem [8]. Indeed, by Massart's version [14], we have

$$\begin{aligned}
\mathbb{E} \sup_{p\in\left[\frac{64\log n}{n},1\right]} \left| \frac{\overline{1}}{\sqrt{n}}^T A_p \frac{\overline{1}}{\sqrt{n}} - \mathbb{E}\frac{\overline{1}}{\sqrt{n}}^T A_p \frac{\overline{1}}{\sqrt{n}} \right| &\le \mathbb{E} \sup_{p\in[0,1]} \left| \frac{\overline{1}}{\sqrt{n}}^T A_p \frac{\overline{1}}{\sqrt{n}} - \mathbb{E}\frac{\overline{1}}{\sqrt{n}}^T A_p \frac{\overline{1}}{\sqrt{n}} \right| \\
&\le 4 \int_{t=0}^{\infty} \exp(-2t^2)dt = \sqrt{2\pi} .
\end{aligned}$$

Thus, the second term on the right-hand side of (3.1) is bounded by the absolute constant $\sqrt{2\pi} + \frac{64}{7} \le 12$ since $P(\overline{E}_2) \le \frac{32}{7n}$.

In order to bound the first term on the right-hand side of (3.1), we write

$$\begin{aligned}
&\sup_{p\in\left[\frac{64\log n}{n},1\right]} \left| \sup_{z\in\frac{2896}{\sqrt{np}}B_2^n} z^T\overline{A}_p\left(\frac{\overline{1}}{\sqrt{n}}+\frac{z}{2}\right) - \mathbb{E} \sup_{z\in\frac{2896}{\sqrt{np}}B_2^n} z^T\overline{A}_p\left(\frac{\overline{1}}{\sqrt{n}}+\frac{z}{2}\right)\right| \\
&\le \sup_{p\in\left[\frac{64\log n}{n},1\right]} \sup_{z\in\frac{2896}{\sqrt{np}}B_2^n} \left| z^T\overline{A}_p\left(\frac{\overline{1}}{\sqrt{n}}+\frac{z}{2}\right) - \mathbb{E}z^T\overline{A}_p\left(\frac{\overline{1}}{\sqrt{n}}+\frac{z}{2}\right)\right| \\
&\le \sup_{p\in\left[\frac{64\log n}{n},1\right]} \frac{2896}{\sqrt{np}} \sup_{z\in\frac{2896}{\sqrt{np}}B_2^n} \left\| \frac{\overline{1}}{\sqrt{n}}+\frac{z}{2} \right\| \cdot \|\overline{A}_p - \mathbb{E}\overline{A}_p\| \\
&\le \sup_{p\in\left[\frac{64\log n}{n},1\right]} 2896\times 594\left(1+\frac{1448}{\sqrt{np}}\right) \\
&\le 2896\times 594\left(1+\frac{1448}{\sqrt{64\log(7)}}\right) \le 4.5\times 10^8 .
\end{aligned}$$

Finally, note that with probability at least $1-\frac{64}{7n}$ for all $p \in \left[\frac{64\log n}{n},1\right]$ we have $\overline{\lambda}_p = \lambda_p$. Moreover, for all p,

$$\mathbb{E}\lambda_p - \mathbb{E}\lambda'_p \le \mathbb{E} \sup_{x\in S^{n-1}} (x^T A_p x(1-\mathbb{1}_{E_1\cap E_2})) \le nP(\overline{E}_1 \cup \overline{E}_2) \le \frac{64}{7} .$$

Thus,

$$\mathbb{E} \sup_{p\in\left[\frac{64\log n}{n},1\right]} |\lambda_p - \mathbb{E}\lambda_p| \le \frac{128}{7} + \mathbb{E} \sup_{p\in\left[\frac{64\log n}{n},1\right]} |\overline{\lambda}_p - \mathbb{E}\overline{\lambda}_p| \le 5\times 10^8 ,$$

proving the first inequality of the theorem.

To prove the second inequality, we follow the argument of Example 3.14 in [7]. Denote $Z = \sup_{p\in\left[\frac{64\log n}{n},1\right]}|\lambda_p - \mathbb{E}\lambda_p|$ and $Z'_{i,j} = \sup_{p\in\left[\frac{64\log n}{n},1\right]}|\lambda'_p - \mathbb{E}\lambda_p|$ where λ'_p is the largest eigenvalue of the adjacency matrix A'_p of the random graph that is obtained from A_p by replacing $U_{i,j}$ by an independent copy. Denoting the first eigenvector of A_p by v_p and the first eigenvector of A'_p by v'_p and the (random) maximizer $\sup_{p\in\left[\frac{64\log n}{n},1\right]}|v_p^T A_p v_p - \mathbb{E}\lambda_p|$ by p^*, we have

$$\begin{aligned}
(Z - Z'_{i,j})_+ &\le \left(\sup_{p\in\left[\frac{64\log n}{n},1\right]}|v_p^T A_p v_p - \mathbb{E}\lambda_p| - \sup_{p\in\left[\frac{64\log n}{n},1\right]}|v_p'^T A'_p v'_p - \mathbb{E}\lambda_p|\right)\mathbb{1}_{Z\ge Z'_{i,j}} \\
&\le \left|v_{p^*}^T A_{p^*} v_{p^*} - \mathbb{E}\lambda_{p^*} - v_{p^*}'^T A'_{p^*} v'_{p^*} - \mathbb{E}\lambda_{p^*}\right|\mathbb{1}_{Z\ge Z'_{i,j}} \\
&\le \left|v_{p^*}^T (A_{p^*} - A'_{p^*}) v_{p^*}\right|\mathbb{1}_{Z\ge Z'_{i,j}} \\
&\le 4|v_{p^*}^i v_{p^*}^j| \,.
\end{aligned}$$

This implies $\sum_{1\le i\le j\le n}(Z - Z'_{i,j})_+^2 \le 16$. Thus, for any $t \ge 0$,

$$\mathbb{P}\left\{\sup_{p\in\left[\frac{64\log n}{n},1\right]}|v_p^T A_p v_p - \mathbb{E}\lambda_p| - \mathbb{E}\sup_{p\in\left[\frac{64\log n}{n},1\right]}|v_p^T A_p v_p - \mathbb{E}\lambda_p| \ge t\right\} \le \exp(-t^2/32)\,.$$

Using the bound $\mathbb{E}\sup_{p\in\left[\frac{64\log n}{n},1\right]}(v_p^T A_p v_p - \mathbb{E}\lambda_p) \le 5\times 10^8$, we have for $t' = t + 5\times 10^8$

$$\mathbb{P}\left\{\sup_{p\in\left[\frac{64\log n}{n},1\right]}|v_p^T A_p v_p - \mathbb{E}\lambda_p| \ge t'\right\} \le \exp(-(t' - 5\times 10^8)^2/32)\,.$$

For $t' \ge 10^9$ the claim follows.

3.3 Delocalization bounds

In this section we prove the "delocalization" inequalities that state that the eigenvector v_p corresponding to the largest eigenvalue of A_p is close to the "uniform" vector $n^{-1/2}\overline{1}$. The following lemma is crucial in the proof of Theorem 2. This proof is based on an argument of Mitra [15]. However, we need to modify it to get uniformity and also significantly better concentration guarantees.

Lemma 4. *Let $n \ge 7$ and $q \in [\frac{4\log n}{n}, \frac{1}{2}]$. Then, with probability $1 - 4\exp(-nq/64)$,*

$$\sup_{p\in[q,2q]}\left\|v_p - \frac{\overline{1}}{\sqrt{n}}\right\|_2 \le \frac{2896}{\sqrt{nq}}\,.$$

Proof.

First note that there exists a unique vector $v_p^\perp$ with $(v_p^\perp, v_p) = 0$ and $\|v_p^\perp\|_2 = 1$ such that

$$\overline{1}/\sqrt{n} = \alpha v_p + \beta v_p^\perp \tag{3.2}$$

for some $\alpha, \beta \in \mathbb{R}$. By Lemma 3, with probability at least $1 - \exp(-nq/64)$,

$$\sup_{p\in[q,2q]} \|A_p - \mathbb{E}A_p\| \le 420\sqrt{nq} \ .$$

Notice that $\mathbb{E}A_p = pn\frac{\overline{1}}{\sqrt{n}}\frac{\overline{1}^T}{\sqrt{n}} - pI_n$, where I_n is an identity $n\times n$ matrix. Since the graph with adjacency matrix A_q is connected with probability at least $1-(n-1)\exp(-nq/2)$ (see, e.g., [17, Section 5.3.3]), by monotonicity of the property of connectedness, the same holds simultaneously for all graphs A_p for $p \in [q, 2q]$. Also, by the Perron-Frobenius theorem, if the graph is connected, the components of v_p are all nonnegative for all $p \in [q, 2q]$. Using that $\alpha = \left(\frac{\overline{1}}{\sqrt{n}}, v_p\right)$,

$$\begin{aligned}
(A_p - \mathbb{E}A_p)v_p &= \lambda_p v_p - pn\frac{\overline{1}}{\sqrt{n}}\frac{\overline{1}^T}{\sqrt{n}} v_p + pv_p \\
&= \lambda_p v_p - pn\alpha\frac{\overline{1}}{\sqrt{n}} + pv_p \\
&= \lambda_p v_p - pn\alpha(\alpha v_p + \beta v_p^\perp) + pv_p \\
&= (\lambda_p + p - pn\alpha^2)v_p - pn\alpha\beta v_p^\perp .
\end{aligned}$$

This leads to

$$(\lambda_p + p - pn\alpha^2)^2 \le 420^2 nq \ . \tag{3.3}$$

Since $\alpha \in [0,1]$, this implies that, with probability at least $1 - \exp(-nq/64) - (n-1)\exp(-nq/2)$, simultaneously for all $p \in [q, 2q]$

$$\lambda_p \le p(n-1) + 420\sqrt{nq} \ . \tag{3.4}$$

We may get a lower bound for λ_p by noting that

$$\lambda_p \ge \frac{1}{n}\overline{1}^T A_p \overline{1} = \frac{2}{n}\sum_{i<j}^{n} \mathbb{1}_{U_{ij}<p} \ .$$

Applying Massart's version of the Dvoretzky-Kiefer-Wolfowitz theorem [14], we have, for all $t \ge 0$,

$$\mathbb{P}\left\{\sup_{p\in[0,1]} \left|\frac{2}{n}\sum_{i<j}^{n} \mathbb{1}_{U_{ij}<p} - (n-1)p\right| \ge (n-1)t\right\} \le 2\exp\left(-n(n-1)t^2\right) \ .$$

Choosing $t = \frac{\sqrt{nq}}{n-1}$, we have, with probability at least $1 - 2\exp(-nq/2)$,

$$\lambda_p \geq p(n-1) - \sqrt{nq}\ . \tag{3.5}$$

This lower bound, together with (3.3) gives

$$\alpha \geq \alpha^2 \geq \frac{\lambda_p + p}{pn} - \frac{420\sqrt{nq}}{pn} \geq 1 - \frac{421}{\sqrt{nq}} \tag{3.6}$$

with probability at least $1 - \exp(-nq/64) - (n-1)\exp(-nq/2) - 2\exp(-nq/2) \geq 1 - 4(n-1)\exp(-nq/64)$. For the rest of the proof, we denote this event by E.

Next, write

$$\left\|\frac{\overline{1}}{\sqrt{n}} - v_p\right\|_2 \leq \left\|\frac{A_p}{\lambda_p}\frac{\overline{1}}{\sqrt{n}} - v_p\right\|_2 + \left\|\frac{A_p}{\lambda_p}\frac{\overline{1}}{\sqrt{n}} - \frac{\overline{1}}{\sqrt{n}}\right\|_2 . \tag{3.7}$$

We analyze both terms on the right-hand side. Observe that $\mathbb{E}A_p\frac{\overline{1}}{\sqrt{n}} = \frac{(n-1)p\overline{1}}{\sqrt{n}}$. The second term on the right-hand side of (3.7) may be bounded on the event E, for all $p \in [q, 2q]$, as

$$\begin{aligned}
\left\|\frac{A_p}{\lambda_p}\frac{\overline{1}}{\sqrt{n}} - \frac{\overline{1}}{\sqrt{n}}\right\|_2 &\leq \frac{1}{\lambda_p}\left\|A_p\frac{\overline{1}}{\sqrt{n}} - \frac{(n-1)p\overline{1}}{\sqrt{n}}\right\|_2 + \frac{1}{\lambda_p}\left\|\frac{((n-1)p - \lambda_p)\overline{1}}{\sqrt{n}}\right\|_2 \\
&= \frac{1}{\lambda_p}\left\|A_p\frac{\overline{1}}{\sqrt{n}} - \mathbb{E}A_p\frac{\overline{1}}{\sqrt{n}}\right\|_2 + \frac{|(n-1)p - \lambda_p|}{\lambda_p} \\
&\leq \frac{\left\|A_p - \mathbb{E}A_p\right\| + |(n-1)p - \lambda_p|}{\lambda_p} \\
&\leq \frac{420\sqrt{nq} + 420\sqrt{nq}}{p(n-1) - \sqrt{nq}} \\
&\leq \frac{1640}{\sqrt{nq}}\ .
\end{aligned}$$

Thus, on the event E, for all $p \in [q, 2q]$,

$$\left\|\frac{\overline{1}}{\sqrt{n}} - v_p\right\|_2 \leq \left\|\frac{A_p}{\lambda_p}\frac{\overline{1}}{\sqrt{n}} - v_p\right\|_2 + \frac{1640}{\sqrt{nq}}\ .$$

For each p, we may write $v_p^\perp = \sum_{i=2}^{n} \gamma_i v_p^i$, where v_p^i is the i-th orthonormal eigenvector of A_p. Then

$$\frac{A_p}{\lambda_p}\frac{\overline{1}}{\sqrt{n}} = \alpha v_p + \beta \sum_{i=2}^{n} \frac{\gamma_i \lambda_i v_p^i}{\lambda_p}\ ,$$

where λ_i is i-th eigenvalue of A_p. By the Perron-Frobenius theorem, we have $|\lambda_i| \le \lambda_p$ for all $i = 2, \ldots, n$. Moreover, from Füredi and Komlós [11, Lemmas 1 and 2] , for all $t \in \mathbb{R}$ we have that $|\lambda_i| \le \|A_p - t\frac{\overline{1}}{\sqrt{n}}\frac{\overline{1}^T}{\sqrt{n}}\|$ for $i \ge 2$. Choosing $t = np$ we obtain $|\lambda_i| \le \|A_p - \mathbb{E}A_p\| + p\|I_n\| \le 420\sqrt{nq} + p \le 422\sqrt{nq}$. Thus, using (3.6), on the event E,

$$\left\|\frac{A_p}{\lambda_p}\frac{\overline{1}}{\sqrt{n}} - v_p\right\|_2 \le 1 - \alpha + \beta \max_{i\ge 2}\frac{|\lambda_i|}{\lambda_p} + \frac{1640}{\sqrt{nq}} \le \frac{2061}{\sqrt{nq}} + \frac{422\sqrt{nq}}{(n-1)p - \sqrt{nq}} \le \frac{2896}{\sqrt{nq}} ,$$

as desired. ■

We close this section by proving the "weak" delocalization bound of Lemma 1.

Proof of Lemma 1. We use the notation introduced in the proof of Lemma 4. Here we fix $p \ge \kappa \log^3 n/n$. Fix $\ell \in \mathbb{N}$ and write

$$\|v_p\|_\infty \le \left\|\left(\frac{A_p}{\lambda_p}\right)^\ell \frac{\overline{1}}{\sqrt{n}} - v_p\right\|_\infty + \left\|\left(\frac{A_p}{\lambda_p}\right)^\ell \frac{\overline{1}}{\sqrt{n}}\right\|_\infty . \tag{3.8}$$

We bound both terms on the right-hand side. We start with the second term and rewrite it as

$$\left\|\left(\frac{A_p}{\lambda_p}\right)^\ell \frac{\overline{1}}{\sqrt{n}}\right\|_\infty = \frac{1}{\sqrt{n}}\left|\frac{(n-1)p}{\lambda_p}\right|^\ell \left\|\left(\frac{A_p}{(n-1)p}\right)^\ell \overline{1}\right\|_\infty .$$

Denote by $D_i = \sum_{j=1}^n A_{i,j}$ the degree of vertex i. By standard tail bounds for the binomial distribution we have, for a fixed i and $0 \le \Delta \le 1$,

$$\mathbb{P}\{D_i < p(n-1) - p(n-1)\Delta\} \le \exp\left(\frac{-\Delta^2 p(n-1)}{2}\right)$$

and

$$\mathbb{P}\{D_i > p(n-1) + p(n-1)\Delta\} \le \exp\left(-\frac{3\Delta^2 p(n-1)}{8}\right) .$$

Using the union bound, we have

$$\mathbb{P}\left\{\max_i |D_i - p(n-1)| > p(n-1)\Delta\right\} \le 2(n-1)\exp\left(-\frac{3\Delta^2 p(n-1)}{8}\right) .$$

We denote the event

$$\max_i |D_i - p(n-1)| \le p(n-1)\Delta$$

by E_1. Observe that when E_1 holds we have $D_i \le p(n-1)(1+\Delta)$ and $D_i \ge p(n-1)(1-\Delta)$ for all i.

Assume that $u \in \mathbb{R}^n$ is such that

$$\|u - \overline{1}\|_\infty \leq 2t\Delta \tag{3.9}$$

for some $t \leq \ell$. In what follows we choose $\ell = \left\lfloor \frac{21 \log n}{\log(np)} \right\rfloor$ and $\Delta = \frac{\log(np)}{42 \log n}$. Observe that $\ell\Delta \leq \frac{1}{2}$. Since $t\Delta^2 \leq \ell\Delta^2 \leq \frac{1}{2}\Delta$, we have $\Delta + 2t\Delta^2 \leq 2\Delta$. Thus, on the event E_1, using the last inequality together with (3.9),

$$\left(\frac{A_p}{(n-1)p} u\right)_i \leq \frac{p(n-1)(1+\Delta)(1+2t\Delta)}{(n-1)p} = 1 + \Delta + 2t\Delta + 2t\Delta^2 \leq 1 + 2(t+1)\Delta \,. \tag{3.10}$$

Now consider the term $\left|\frac{(n-1)p}{\lambda_p}\right|^\ell$. Using (3.5) we have, with probability at least $1 - 2\exp(-np/2)$ (denote the corresponding event by E_2),

$$\left|\frac{(n-1)p}{\lambda_p}\right|^\ell \leq \left(1 - \frac{1}{\sqrt{p(n-1)}}\right)^{-\ell} .$$

Since $\ell \leq \sqrt{p(n-1)}$, we obtain $\left|\frac{(n-1)p}{\lambda_p}\right|^\ell \leq e$. Thus, applying (3.10) ℓ times for vectors satisfying (3.9), on the event $E_1 \cap E_2$, we have, for all i,

$$\left(\left(\frac{A_p}{\lambda_p}\right)^\ell \overline{1}\right)_i = \left|\frac{(n-1)p}{\lambda_p}\right|^\ell \left(\left(\frac{A_p}{(n-1)p}\right)^\ell \overline{1}\right)_i \leq e(1 + 2\ell\Delta) \leq 2e \,.$$

We may similarly derive a lower bound since, for any vector satisfying (3.9),

$$\left(\frac{A_p}{(n-1)p} u\right)_i \geq \frac{p(n-1)(1-\Delta)(1-2t\Delta)}{(n-1)p} = 1 - \Delta - 2t\Delta + 2t\Delta^2 \geq 1 - 2(t+1)\Delta \,. \tag{3.11}$$

Analogously, applying (3.11) ℓ times, on the event $E_1 \cap E_2$, we have

$$\left(\left(\frac{A_p}{\lambda_p}\right)^\ell \overline{1}\right)_i = \left|\frac{(n-1)p}{\lambda_p}\right|^\ell \left(\left(\frac{A_p}{(n-1)p}\right)^\ell \overline{1}\right)_i \geq \left|\frac{(n-1)p}{\lambda_p}\right|^\ell (1 - 2\ell\Delta) \geq 0 \,.$$

Hence, on the event $E_1 \cap E_2$,

$$\left\|\left(\frac{A_p}{\lambda_p}\right)^\ell \frac{\overline{1}}{\sqrt{n}}\right\|_\infty \leq \frac{2e}{\sqrt{n}} \,. \tag{3.12}$$

Next we bound the first term on the right-hand side of (3.8). Recall that for the decomposition $\overline{1}/\sqrt{n} = \alpha v_p + \beta v_p^\perp$ from (3.6) we have $\alpha \geq 1 - \frac{421}{\sqrt{np}}$ on an event E_3 of probability at least $1 - 4(n-1)\exp(-np/64)$. As before, we may write $v_p^\perp = \sum_{i=2}^n \gamma_i v_p^i$, where v_p^i is the i-th orthonormal eigenvector of A_p. Using $\overline{1}/\sqrt{n} = \alpha v_p + \beta v_p^{\perp}$, we have

$$\left(\frac{A_p}{\lambda_p}\right)^{\ell} \frac{\overline{1}}{\sqrt{n}} = \alpha v_p + \beta \sum_{i=2}^{n} \gamma_i v_p^i \left(\frac{\lambda_i}{\lambda_p}\right)^{\ell},$$

where λ_i is i-th eigenvalue of A_p. Using Füredi and Komlós [11, Lemmas 1 and 2] once again, for all $t \in \mathbb{R}$ we have that $|\lambda_i| \le \left\| A_p - t\frac{\overline{1}}{\sqrt{n}}\frac{\overline{1}^T}{\sqrt{n}} \right\|$ for $i \ge 2$. Choosing $t = np$ we obtain $|\lambda_i| \le \|A_p - \mathbb{E}A_p\| + p\|I_n\| \le 420\sqrt{np} + p \le 422\sqrt{np}$ on an event E_4 of probability at least $1 - 4(n-1)\exp(-np/64)$. Thus, on E_4 we have $\frac{|\lambda_i|}{\lambda_p} \le \frac{835}{\sqrt{np}}$ for $i \ge 2$, and therefore

$$\left\| \left(\frac{A_p}{\lambda_p}\right)^{\ell} \frac{\overline{1}}{\sqrt{n}} - v_p \right\|_{\infty} \le (1-\alpha)\|v_p\|_{\infty} + \beta \max_{i \ge 2} \left(\frac{|\lambda_i|}{\lambda_p}\right)^{\ell}. \tag{3.13}$$

Define $\kappa_1 = \frac{\log(835)}{\log(2\times 835^2)}$. Observe that $\kappa_1 < \frac{1}{2}$. Using $np \ge 2 \times 835^2 = \kappa$,

$$\begin{aligned}
\beta \max_{i\ge 2}\left(\frac{|\lambda_i|}{\lambda_p}\right)^{\ell} &\le \beta\left(\frac{835}{\sqrt{np}}\right)^{\ell} \\
&\le \left(\frac{835}{(np)^{\kappa_1}}\right)^{\ell} \exp\left((\frac{1}{2} - \kappa_1)\log(\frac{1}{np})\frac{21\log n}{\log(np)}\right) \\
&\le \exp\left(-21(\frac{1}{2} - \kappa_1)\log n\right) \le \frac{1}{\sqrt{n}},
\end{aligned}$$

where we used $\left(\frac{835}{(np)^{\kappa_1}}\right)^{\ell} \le 1$ and the inequality $21(\frac{1}{2} - \kappa_1) > \frac{1}{2}$. Finally, on the event $E_1 \cap E_2 \cap E_3 \cap E_4$ we have, using the decomposition (3.8) combined with (3.12) and (3.13), that

$$\|v_p\|_{\infty} \le \frac{1}{\alpha}\left(\frac{1+2e}{\sqrt{n}}\right) \le \frac{1}{1 - \frac{421}{\sqrt{np}}}\left(\frac{1+2e}{\sqrt{n}}\right) \le \frac{11}{\sqrt{n}}.$$

3.4 Proof of Theorem 3

It suffices to prove that

$$\mathbb{E} \sup_{p \in [0, \frac{64\log n}{n}]} |\lambda_p - \mathbb{E}\lambda_p| \le 5\sqrt{16 + 2\log\log n}.$$

Observe that

$$\mathbb{E}\sup_{p\in[0,1]} |\lambda_p - \mathbb{E}\lambda_p| \le \mathbb{E} \sup_{p\in[0,\frac{64\log n}{n}]} |\lambda_p - \mathbb{E}\lambda_p| + \mathbb{E} \sup_{p\in[\frac{64\log n}{n},1]} |\lambda_p - \mathbb{E}\lambda_p|$$

Let $p_0, p_1, \ldots, p_M$ be such that $0 = p_0 \le p_1 \le \cdots \le p_M = \frac{64\log n}{n}$ and $\mathbb{E}(\lambda_{p_j} - \lambda_{p_{j-1}}) = \varepsilon$ for some $\varepsilon > 0$ to be specified later. Such a choice is possible since λ_p is nondecreasing in p. We have

$$\varepsilon M = \mathbb{E}\lambda_M \le \mathbb{E}\|A_{p_M} - \mathbb{E}A_{p_M}\| + \|\mathbb{E}A_{p_M}\| \le 170\sqrt{np_M} + np_M \le 1424\log n\,. \qquad (3.14)$$

Denote for $p \in [0, p_M]$ the value $\pi_+[p] = \min\{q \in \{p_0, p_1, \ldots, p_M\} |\, q \ge p\}$ and $\pi_-[p] = \max\{q \in \{p_0, p_1, \ldots, p_M\} |\, p \ge q\}$. We have

$$\begin{aligned}
\mathbb{E} \sup_{p\in[0,\frac{64\log n}{n}]} |\lambda_p - \mathbb{E}\lambda_p| &= \mathbb{E} \sup_{p\in[0,\frac{64\log n}{n}]} \max(\lambda_p - \mathbb{E}\lambda_p, \mathbb{E}\lambda_p - \lambda_p) \\
&\le \mathbb{E} \sup_{p\in[0,\frac{64\log n}{n}]} \max(\lambda_{\pi_+[p]} - \mathbb{E}\lambda_{\pi_+[p]} + \varepsilon, \mathbb{E}\lambda_{\pi_-[p]} - \lambda_{\pi_-[p]} + \varepsilon) \\
&= \varepsilon + \mathbb{E} \sup_{p\in[0,\frac{64\log n}{n}]} \max(\lambda_{\pi_+[p]} - \mathbb{E}\lambda_{\pi_+[p]}, \mathbb{E}\lambda_{\pi_-[p]} - \lambda_{\pi_-[p]}) \\
&\le \varepsilon + \mathbb{E} \sup_{q\in\{p_0,\ldots,p_M\}} |\lambda_q - \mathbb{E}\lambda_q|\,.
\end{aligned}$$

Since for each p_i, the random variable $|\lambda_q - \mathbb{E}\lambda_q|$ has sub-Gaussian tails by (1.2), for their maximum we obtain the bound

$$\mathbb{E} \sup_{q\in\{p_0,\ldots,p_M\}} |\lambda_q - \mathbb{E}\lambda_q| \le 4\sqrt{2\log 2M}\,.$$

Finally, using (3.14)

$$\mathbb{E} \sup_{p\in[0,\frac{64\log n}{n}]} |\lambda_p - \mathbb{E}\lambda_p| \le \inf_{\varepsilon>0}(\varepsilon + 4\sqrt{2\log(2848\log n/\varepsilon)}) \le 5\sqrt{2\log(2848\log n)}\,,$$

as desired.

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