COMPETITIVE BARGAINING EQUILIBRIUM

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Abstract. In a simple exchange economy we propose a bargaining procedure that leads to a Walrasian outcome as the agents become increasingly patient. The competitive outcome therefore obtains even if agents have market power and are not price-takers. Moreover, where in other bargaining protocols the final outcome depends on bargaining power or relative impatience, the outcome here is determinate and depends only on preferences and endowments. Our bargaining procedure involves bargaining over prices and maximum quantity constraints, and it guarantees convergence to a Walrasian outcome for *any* standard exchange economy. In contrast, without quantity constraints we show that equilibrium is generically inefficient.

Keywords. Bargaining. Walrasian Equilibrium. Price-setting.

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1. Introduction

Price-taking behavior is typically invoked as a necessary requirement to obtain the competitive outcome. In this paper, we propose a bargaining foundation for the Walrasian equilibrium in a small exchange economy where agents are not price-takers. The bargaining procedure we analyze relates to those studied in Binmore (1987) and Yildiz (2003). More specifically, in our set-up each agent alternatingly offers a price and a maximum amount to be exchanged, and the respondent either accepts and chooses the quantities to be traded at the terms of the offer, or rejects and makes an offer in the next period in which utilities are discounted. We show in this set-up that the competitive outcome obtains when bargaining frictions vanish, even without price-taking behavior. This convergence result holds for any standard exchange economy. Moreover, the outcome does not depend on details such as relative impatience and bargaining power or outside options. Paradoxically, by explicitly introducing price-setting as a strategic variable in an otherwise standard bargaining environment, the competitive outcome is restored. Price-taking is therefore not a necessary requirement for attaining a perfectly competitive outcome.

The main implication of the convergence result is that, as discounting frictions vanish, the bargaining outcome does not depend on the exact specification of time preferences. Instead, the bargaining outcome converges to a Walrasian allocation which is determined by the preferences and endowments of the agents. It seems natural after all that the bargaining outcome is not exclusively determined by relative patience (or by exogenous bargaining power in axiomatic Nash bargaining) as is, on the contrary, the case in the alternating-offer bargaining of Rubinstein (1982) or Ståhl (1972). There, the outcome depends on the relative impatience of the bargaining parties, with the relatively patient agent obtaining a proportionally larger share of the surplus.[1] Rather, in many economic environments the bargaining outcome may depend, at least in part, on preferences and endowments, for example on the degree of substitutability between the goods consumed.

The bargaining procedure with price offers that guarantees our convergence results *necessarily* involves maximum trade constraints. This indicates that the details of the bargaining procedure are important (see also Binmore (1987)). We show that the conditions for convergence obtained in Yildiz (2003) for a bargaining procedure over prices *without maximum trade constraints* are too strong in the sense that almost no economy satisfies the assumptions made in Yildiz (2003). For any economy of an open and dense subset of the set of standard exchange economies there will exist at least one equilibrium of the bargaining game without quantity constraints that converges to an inefficient allocation. The importance of maximum trade constraints is first established by Binmore (1987) in the context of axiomatic bargaining. He is the first in the modern bargaining literature to connect the competitive equilibrium to bargaining outcomes in two person economies. He presents

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[1] With discounting, the offering agent can extract some rents from the recipient because a counteroffer cannot be made until one period later. Smith and Stacchetti (2003) refer to this as a temporal monopoly by the offering agent. Even in the case when *both* agents become infinitely patient, the rate at which they do so will determine the final outcome. Binmore, Rubinstein and Wolinsky (1986) (see also Binmore (1987)) establishes that the alternating-offers bargaining solution approaches the Nash bargaining solution with utilities that reflect the incentives to settle and with the disagreement point properly chosen.

a modified Nash demand game with minimum prices and maximum quantity constraints and shows that the (large) set of Nash equilibria of this game includes the Walrasian allocation.[2]

In the next section, we lay out the model. In section 3, we present the main result of convergence to the Walrasian allocation. We obtain first the result for equilibria in which there is immediate acceptance by showing in Theorem 1 that, without discounting, every Stationary Subgame Perfect (SSP) equilibrium allocation is Walrasian and conversely. The convergence result in Theorem 2 follows then from the upper hemicontinuity of the equilibrium correspondence. In section 4 we study other equilibria. We establish first that SSP equilibria with delay do not exist, from which the convergence result in Theorem 3 for *every* SSP equilibrium follows. We study then non-stationary Subgame Perfect equilibria. Whenever there is a unique Walrasian allocation, Theorem 4 establishes that any non-stationary SP equilibrium converges to the Walrasian allocation. In Section 5 we consider the same bargaining procedure but without maximum trade constraints, as in Yildiz (2003). In this case, Theorem 5 shows the generic existence of asymptotically inefficient SSP equilibria, and Theorem 6 shows the impossibility of a unique SSP equilibrium converging to a Walrasian outcome. Section 6 provides a discussion with some examples: of inefficiency even in the limit of SSP equilibria if no maximum trade constraints are imposed, and of non-convergence in the absence of differentiability. Section 7 finishes with some concluding remarks. Proofs and some Propositions and Lemmas are relegated to the Appendix.

2. THE MODEL

Consider an exchange economy with two agents A and B, each with endowments e^A and e^B of n goods over which they have preferences represented by utility functions u^A and u^B satisfying Assumption A1.

Assumption A1.

(1) for all $i = A, B$, u^i is $\mathbb{R}_+$-valued, continuous in $\mathbb{R}^n_+$, differentiable in $\mathbb{R}^n_{++}$, monotonous in the sense that $Du^i(x) \in \mathbb{R}^n_{++}$ always, strictly differentiably quasi-concave in the sense that $D^2u^i(x)$ is always negative definite in the normal space of $Du^i(x)$, and well-behaved in the boundary in the sense that $(u^i)^{-1}(a) \subset \mathbb{R}^n_{++}$ for any $a \in u^i(\mathbb{R}^n_{++})$.

(2) $e^A, e^B \in \mathbb{R}^n_{++}$.

In general, for given endowments, the allocation is not Pareto efficient and therefore there exist gains from trade. The central issue in this paper is to address the question of how those gains from trade are realized. More specifically, we are interested in establishing whether all gains from trade are completely achieved (i.e. whether the outcome is efficient) and if so, which of all efficient allocations is obtained.

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[2] See also Serrano and Volij (2001) for the relation between axiomatic bargaining and Walrasian allocations.

Since the price-taking assumption is not easily justified in a two-person economy, we propose a bargaining procedure in which agents set prices that allow them to realize completely the gains from trade. More specifically, we consider an alternating-offers bargaining game in which, in any given period, one of the agents offers to the other a vector of relative prices at which he is willing to trade up to some maximum amount (the *maximum trade constraint* henceforth). Thus an offer by say A consists of a vector of prices p^A (in terms of good 1) and a quantity constraint q^A. Without loss of generality, we assume the quantity constraint is on the amount traded of good 1. Upon the reception of an offer, the recipient, i.e. B in this case, can either accept the offer or reject it. If she accepts, she then chooses her most preferred consumption $\tilde{x}^B(p^A, q^A)$ at the offered price, without violating the maximum trade constraint expressed in the offer, i.e. $\tilde{x}^B(p^A, q^A)$ solves

$$\begin{aligned} &\max_{x^B} u^B(x^B) \\ \text{s.t. } &p^A(x^B - e^B) \leq 0 \\ &|x_1^B - e_1^B| \leq q^A. \end{aligned} \tag{1}$$

If B rejects the offer, then B counter-offers another pair (p^B, q^B) with new prices and a new maximum trade constraint. The utility of both agents A and B is discounted for every iteration of the bargaining by positive discount factors δ^A and δ^B not bigger than 1. Not reaching an agreement entails a zero utility to both agents.

A stationary subgame perfect (SSP) equilibrium with immediate acceptance of this game consists of a pair of offers (p^A, q^A) and (p^B, q^B) such that, in every subgame where A is called to make an offer, A offers (p^A, q^A) and this offer solves

$$\begin{aligned} &\max_{p^A, q^A} \; u^A(e - \tilde{x}^B(p^A, q^A)) \\ \text{s.t. } &u^B(\tilde{x}^B(p^A, q^A)) \geq \delta^B u^B(e - \tilde{x}^A(p^B, q^B)) \end{aligned} \tag{2}$$

given (p^B, q^B) (where $e = e^A + e^B$ denotes the total endowments) and similarly for (p^B, q^B) given (p^A, q^A).

In effect, from subgame perfection, once B decides to accept A's offer, she will choose the consumption bundle $\tilde{x}^B(p^A, q^A)$ that maximizes her utility subject to the terms of the offer. Therefore, knowing that upon acceptance B chooses $\tilde{x}^B(p^A, q^A)$, A decides to make an offer that maximizes his utility of consuming $e - \tilde{x}^B(p^A, q^A)$, provided that the offer induces B to accept it. This requires that B obtains at least as much utility from accepting the current offer, i.e. $u^B(\tilde{x}^B(p^A, q^A))$, as she would get from rejecting the offer and waiting for her equilibrium offer (p^B, q^B) to be accepted in the next period, which gives her a utility $\delta^B u^B(e - \tilde{x}^A(p^B, q^B))$.

It turns out to be the case that there is no loss of generality in focusing on the SSP equilibria with immediate acceptance. In effect, we focus first in Section 3 on equilibria with immediate acceptance, and then we show in Section 4 that there are no SSP equilibria with delay whenever the agents are impatient.

3. The convergence result

In this section, we develop the argument that establishes the convergence to a Walrasian allocation of every SSP equilibrium with immediate acceptance of the bargaining over prices and maximum trades. The key insight of the argument is realizing that in exchanging price and quantity offers, the agents are actually bargaining over *some* allocations. In effect, given the subgame perfection of the equilibrium, any agent making an offer anticipates the optimal acceptance behavior by the recipient, and therefore an offer (p^A, q^A) amounts to offering the allocation $(e - \tilde{x}^B(p^A, q^A), \tilde{x}^B(p^A, q^A))$, where $\tilde{x}^B(p^A, q^A)$ is the consumption chosen by B given the prices p^A and the maximum trade constraint q^A.

This allows us to characterize the allocations that might be accepted at a SSP equilibrium with immediate acceptance. Note first that in the absence of a maximum trade constraint q^A (or equivalently, when the constraint is slack), B's response to A's offer is to choose her demand $x^B(p^A)$ at the prices p^A. Note also that by means of the maximum trade constraint q^A, agent A can prevent agent B from attaining $x^B(p^A)$, forcing her to a lesser trade. Nevertheless, in no instance can A force B to exchange more than necessary to attain her desired demand $x^B(p^A)$ at those prices. As a consequence, an offer by A that is responded with an optimal acceptance decision by B results in an allocation $(e - x^B, x^B)$ such that

$$Du^B(x^B)(x^B - e^B) \geq 0. \tag{3}$$

Condition (3) characterizes the set of solutions to maximization problems of the class (1) above.[3] It holds with equality if the maximum trade constraint does not effectively constrain B's choice and with strict inequality otherwise. A similar condition holds for offers made by B that might be accepted by A at a SSP equilibrium with immediate acceptance.

As a consequence a SSP equilibrium with immediate acceptance of the alternating-offers bargaining over prices and maximum trades can be characterized by two feasible allocations (x_A^A, x_A^B) and (x_B^A, x_B^B): (x_A^A, x_A^B) denotes the allocation to A and B resulting from A's offer, (x_B^A, x_B^B) results from B's offer. Then (x_A^A, x_A^B) solves

$$\begin{aligned} &\max_{x^A, x^B} \; u^A(x^A) \\ &Du^B(x^B)(x^B - e^B) \geq 0 \\ &u^B(x^B) \geq \delta^B u^B(x_B^B) \\ &x^A + x^B = e^A + e^B \end{aligned} \tag{4}$$

given (x_B^A, x_B^B), and likewise for (x_B^A, x_B^B) given (x_A^A, x_A^B). Condition (4) is thus equivalent to condition (2).

This characterization allows us to establish in Theorem 1 below that, for infinitely patient agents (i.e. $\delta^A = \delta^B = 1$), every Walrasian allocation is the allocation of a SSP equilibrium with immediate acceptance and conversely. A proof of this result is provided in the appendix.

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[3] See Lemma A1 in the appendix.

Theorem 1. *Every SSP equilibrium allocation with immediate acceptance of the alternating-offers bargaining over prices and maximum trades of an economy* $\{u^i, e^i\}_{i=A,B}$ *satisfying A1 is Walrasian, and conversely, whenever the agents are infinitely patient.*

Thus the sets of Walrasian allocations and SSP equilibrium allocations with immediate acceptance coincide when $\delta^A = \delta^B = 1$. Now the convergence towards a Walrasian allocation of the allocation of any SSP equilibrium with immediate acceptance as the agents become arbitrarily patient follows from the upper hemicontinuity of the correspondence from the discount factors to the SSP equilibrium allocations whenever the latter is not empty-valued in some neighborhood of $(\delta^A, \delta^B) = (1, 1)$.[4] Theorem 2 establishes this convergence, its proof is provided in the appendix.

Theorem 2. *Every SSP equilibrium allocation with immediate acceptance of the alternating-offers bargaining over prices and maximum trades of an economy* $\{e^i, u^i\}_{i=A,B}$ *satisfying A1 converges to a Walrasian allocation as the agents become arbitrarily patient.*

4. Delay and Non-stationary Strategies

Are there SSP equilibria with delay? Although in general bargaining environments with complete information, equilibria with delay in reaching an agreement may exist (for an exhaustive treatment, see Merlo and Wilson (1996), and also Sákovics (1993)), in the bargaining procedure considered here there is no SSP equilibrium with delay when the agents are impatient. In order to see this, we first extend the definition of a SSP equilibrium in (1) and (2) to allow for a delay in the acceptance of an offer. Without loss of generality, and in line with the argument above, we define the equilibrium in terms of the allocations x^A, x^B that result from the acceptance of offers p^A, q^A and p^B, q^B.

In general, a SSP equilibrium is characterized by two allocations (x^A_A, x^B_A) and (x^A_B, x^B_B) (that would result from the acceptances of A's and B's offers respectively) and acceptance rules according to which A rejects any offer that does not allow him to attain at least a utility $\delta^A u^A(x^A_A)$, and similarly for B, that satisfy:

(1) A's offer is rational: given B's offer, either A prefers B's acceptance to B's rejection, i.e.

$$u^A(x^A_A) > \delta^A u^A(x^A_B) \tag{5}$$

and therefore makes his most preferred offer acceptable to B, i.e. (x^A_A, x^B_A) solves

$$\begin{aligned} &\max_{x^A, x^B} u^A(x^A) \\ Du^B(x^B)&(x^B - e^B) \geq 0 \\ u^B(x^B) &\geq \delta^B u^B(x^B_B) \\ x^A + x^B &= e^A + e^B \end{aligned} \tag{6}$$

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[4] Proposition 1 in the appendix provides a sufficient condition for the nonempty-valuedness of the correspondence that is satisfied by a large class of utility functions that includes, among others, all the CES utility functions. Moreover, the condition is not necessary so that the nonempty-valuedness holds for an even wider set of preferences.

given (x_B^A, x_B^B), or else A prefers B's rejection to B's acceptance, i.e.

$$u^A(x_A^A) \leq \delta^A u^A(x_B^A) \tag{7}$$

and is accordingly making an unacceptable offer to B, i.e.

$$u^B(x_A^B) < \delta^B u^B(x_B^B) \tag{8}$$

and

(2) B's reply to A's offer is rational: B accepts if $u^B(x_A^B) \geq \delta^B u^B(x_B^B)$ and rejects otherwise,

and similarly for B's offer.

Consider now a candidate SSP equilibrium in which say B rejects and A accepts. In such equilibrium A prefers to make an offer that is rejected by B, and hence that gives A a smaller utility $u^A(x_A^A)$ than the discounted utility $\delta^A u^A(x_B^A)$ he derives from B's offer, i.e.

$$\delta^A u^A(x_B^A) \geq u^A(x_A^A). \tag{9}$$

Also A prefers to accept B's offer because it gives A a higher utility $u^A(x_B^A)$ than the discounted utility $\delta^A u^A(x_A^A)$ from his own offer, i.e.

$$u^A(x_B^A) \geq \delta^A u^A(x_A^A). \tag{10}$$

Then, for $\delta^A < 1$, this last inequality cannot be binding. That is to say, in a SSP equilibrium in which B rejects and A accepts, B's offer x^A leaves A more utility than what would be necessary to obtain A's acceptance. As a consequence, the second constraint of B's problem analogous to (6) is not binding, but since agent B's preferences are monotone, then necessarily the first constraint will be binding at x_B^A, i.e.

$$Du^A(x_B^A)(x_B^A - e^A) = 0. \tag{11}$$

This implies that there always exists a profitable deviation by A that prevents such (x_B^A, x_B^B) to be an allocation resulting from A's acceptance of B's offer at a SSP equilibrium. In effect, since (x_B^A, x_B^B) cannot be efficient,[5] then there would be room for agent A to deviate profitably making an offer resulting into another $({x'}_A^A, {x'}_A^B)$ that, when accepted by B, makes both A and B better off.

Thus since there is no SSP equilibria with delay, it trivially follows that all SSP equilibria converge to the Walrasian allocation, as stated in Theorem 3 below, which summarizes the main result of the paper.

Theorem 3. *Every SSP equilibrium allocation of the alternating-offers bargaining over prices and maximum trades of an economy* $\{u^i, e^i\}_{i=A,B}$ *satisfying A1 converges to a Walrasian allocation as the agents become arbitrarily patient.*

Key to this result is that the constraints on the offers made is a *maximum* trade constraint. When the bargaining procedure is modified to allow for minimum quantity constraints, SSP equilibria with delay do exist (see Dávila and Eeckhout

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[5] This is a consequence of the strictly differentiably quasi-concavity of the preferences (the details are provided in the proof of Lemma A3 in the Appendix).

(2002)). Maximum constraints not only seem more natural, they in addition provide a starker prediction, i.e. convergence to the Walrasian allocation, compared to minimum constraints.

Non-stationary Subgame Perfect Equilibria. So far, we have restricted attention to stationary strategies, therefore ruling out the use of non-stationary threats. Merlo and Wilson (1995) show that such non-stationary strategies can lead to a continuum of subgame perfect (SP) equilibria. In the light of such indeterminacy, stationarity is often invoked as a natural selection criterion because it acts as focal point within the set of SP equilibria, or because it requires a minimal number of states for automata to implement a SP equilibrium.

Nonetheless, for economies in which the Walrasian allocation is unique, it follows from Theorems 1 through 3 that every SSP equilibrium allocation converges to the unique Walrasian allocation. Theorem 4 below then follows from this convergence result and the results in Merlo and Wilson (1995). The proof is provided in the appendix.[6]

Theorem 4. *If an economy* $\{u^i, e^i\}_{i=A,B}$ *satisfying A1 has a unique Walrasian equilibrium, then all the SP equilibria of the bargaining over prices and maximum trades converge to the walrasian equilibrium.*

5. BARGAINING OVER PRICES ONLY

As the agents exchange price offers in the bargaining protocol considered above, the maximum trades they expressed in their offers are essential for the convergence result to obtain. In effect, if the agents bargain over prices only (as in Yildiz (2003)), then generically there exist SSP equilibria whose payoffs to the agents remain bounded away from the Pareto frontier, even in the limit as their discount factors δ^A and δ^B converge to 1. This bargaining model is essentially a special case of the one analyzed above. Acceptance is as before, and an offer now consists of a price p, which is a special case of a price offer with a maximum trade constraint where the trade constraint is slack. As a result, the model satisfies the equations above but without the quantity constraint ever binding.

The existence of an inefficient SSP equilibrium can be discerned by looking at how the SSP equilibrium payoffs behave in the space of utilities as δ^A and δ^B converge to 1 in this case.[7] Note first that at a SSP equilibrium necessarily the only constrains the agents face making their offers (namely, the acceptability of their offers) must be binding, i.e.

$$\begin{aligned} u^A(x^A(p^B)) - \delta^A u^A(e - x^B(p^A)) &= 0 \\ u^B(x^B(p^A)) - \delta^B u^B(e - x^A(p^B)) &= 0. \end{aligned} \tag{12}$$

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[6] We are grateful to Antonio Merlo for pointing us to their result in Merlo and Wilson (1995).

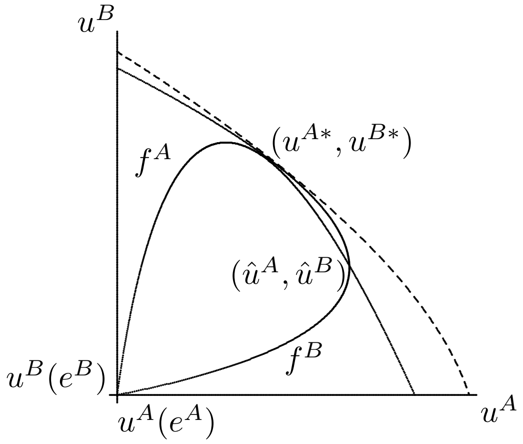
[7] The analysis in Yildiz (2003) is developed entirely in the space of utilities as well. As a matter of fact, even the assumptions in Yildiz (2003) are assumptions made directly on derived concepts in this space, such as the "offer curves" in the space of utilities, instead of being made on the fundamentals of the economy. Unfortunately, it turns out that the assumptions in Yildiz (2003) made this way are actually satisfied jointly only by a degenerate set of economies. Our analysis above has been made instead working in the space of allocations and taking into account explicitly the fundamentals of the economy in order to avoid this problem.

Equivalently, the payoffs of a SSP equilibrium of the bargaining over prices only must be intersections in the space of utilities of two curves $f^A_{\delta^B}, f^B_{\delta^A}$ parameterized by the relative price p and defined as

$$\begin{aligned} f^A_{\delta^B}(p) &= (u^A(x^A(p)), \delta^B u^B(e - x^A(p))) \\ f^B_{\delta^A}(p) &= (\delta^A u^A(e - x^B(p)), u^B(x^B(p))). \end{aligned} \tag{13}$$

These curves are, for discount factors δ^A and δ^B close to 1, slight continuous deformations of their counterparts f^A, f^B for $\delta^A, \delta^B = 1$ represented in Figure 1 below.

Figure 1



The typical pattern of f^A, for instance, is that as A's utility increases when we move away from the endowments point along A's offer curve, B's utility initially increases too, but eventually decreases.[8] And similarly for f^B with the roles of the axes reversed. The two curves f^A and f^B intersect at any profile of Walrasian utilities like (u^{A*}, u^{B*}) on the Pareto frontier (in dashes) in Figure 1. Also it follows from the differentiability of both f^A and f^B, and of the Pareto frontier itself that all the three curves are tangent at (u^{A*}, u^{B*}).[9]

The intersection of f^A and f^B at the Walrasian intersection (u^{A*}, u^{B*}) has another important property, illustrated in Figure 1 and stated in Lemma 1 below, that plays a crucial role in showing the existence of SSP equilibria of the alternating-offers bargaining over prices only that remain inefficient even in the limit. The proof is provided in the appendix.

Lemma 1. *For any generic economy $\{u^i, e^i\}_{i=A,B}$ satisfying A1,*[10] *the curves f^A and f^B intersect without crossing at any Walrasian intersection (u^{A*}, u^{B*}).*

A short discussion is at this point in order. If one assumes on the contrary that f^A and f^B do actually cross at a unique Walrasian intersection (u^{A*}, u^{B*}), then it can be readily proved that the alternating-offers bargaining over prices has only one SSP equilibrium that moreover necessarily converges to the Walrasian equilibrium

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[8] Note that the curve needs not be single-peaked in general, nor is this needed in the proofs.

[9] See Proposition 9 in Dávila-Eeckhout (2002) for a proof of this property.

[10] More specifically, for any economy within an open and dense set of the space of economies satisfying A1, with respect to the topology of C^n uniform convergence on compacts in the space of utility functions, for $n \geq 2$, and the usual topology in the space of endowments.

as δ^A and δ^B converge to 1. This has been established in Yildiz (2003) under his assumptions A3 (both monopolistic outcomes are dominated by some allocation attainable along an offer curve) and A4 (there is a unique crossing of f^A and f^B within the interval defined by the profiles of utilities attained at the monopolistic outcomes). Nevertheless, while each of the two assumptions A3 and A4 in Yildiz (2003) are not degenerate on their own, the requirement of both of them to hold simultaneously amounts to having a crossing of f^A and f^B at a Walrasian profile of utilities which, according to Lemma 1 above, makes them a degenerate set of assumptions.

Lemma 1 has strong implications for the existence of asymptotically inefficient SSP equilibria of the alternating-offers bargaining over prices. In effect, note for instance that Lemma 1, along with the behavior of f^A and f^B at the boundary, implies the existence of an intersection of f^A and f^B like $(\hat{u}^A, \hat{u}^B)$ in Figure 1. This intersection does not correspond to a Walrasian equilibrium, since it is inefficient. Note also that, as the discount factors δ^A and δ^B depart slightly from 1, by continuity a nearby intersection $(\hat{u}^A, \hat{u}^B)_{\delta^A,\delta^B}$ of $f^A_{\delta^B}$ and $f^B_{\delta^A}$ still exists. This intersection $(\hat{u}^A, \hat{u}^B)_{\delta^A,\delta^B}$ not only satisfies the necessary conditions for a SSP equilibrium of the bargaining over prices, but actually corresponds to such an equilibrium whenever both $f^A_{\delta^B}$ and $f^B_{\delta^A}$ have a negative slope there, which is the case for δ^A and δ^B close enough to 1, by continuity, for a non-empty open set of economies.[11] Finally, note that the intersection $(\hat{u}^A, \hat{u}^B)_{\delta^A,\delta^B}$ converges necessarily to $(\hat{u}^A, \hat{u}^B)$ as δ^A and δ^B converge to 1. As a consequence, the corresponding SSP equilibrium of the alternating-offers bargaining over prices is not only inefficient for every δ^A and δ^B close to 1, but it also remains bounded away from the Pareto frontier as δ^A and δ^B converge to 1. This argument is formalized in Proposition 2 in the appendix.

As a matter of fact, the existence of asymptotically inefficient SSP equilibria of the alternating-offers bargaining over prices is a generic property of these economies. Theorem 5 below establishes this and its proof is provided in the appendix.

Theorem 5. *Within any neighborhood of any exchange economy* $\{u^i, e^i\}_{i=A,B}$ *satisfying A1, there exists an open*[12] *set of economies with asymptotically inefficient SSP equilibria of the bargaining over prices.*

Notwithstanding, there may still exist SSP equilibria of the bargaining over prices that do converge to a Walrasian equilibrium as the discount factor δ^A and δ^B converge to 1, but if that is the case there will be a multiplicity of such equilibria. Whether there is or not SSP equilibria converging to a Walrasian outcome actually depends on how δ^A and δ^B approach 1: Theorem 6 below establishes this and its proof can be found in the appendix.

Theorem 6. *For any Walrasian allocation* x^* *of a generic economy* $\{u^i, e^i\}_{i\in\{A,B\}}$ *satisfying A1,*[13] *and any* $(\delta^A_n, \delta^B_n) \to (1, 1)$,

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[11] A crossing where either f^A or f^B is positively sloped leaves room for a mutually beneficial deviation that undoes the candidate equilibrium.

[12] With respect to the usual topology in the space of endowments, and the topology of C^1 uniform convergence on compacts in the space of utility functions.

[13] That is to say, for an open and dense set of economies satisfying A1, with respect to the usual topology in the space of endowments, and the topology of C^n uniform convergence on compacts in the space of utility functions, for all $n \geq 2$.

(1) *either there are multiple*[14] *SSP equilibrium allocations converging to* x^*,
(2) *or no SSP equilibrium allocation converges to* x^*.

Moreover none of the two cases (1) *and* (2) *is degenerate.*

6. Discussion

In this section, we provide first some intuition for the conjecture that, not only every SSP equilibrium outcome converges to a Walrasian outcome as stated in Theorem 3, but also every Walrasian outcome is reachable as a SSP equilibrium outcome as $\delta^A, \delta^B \to 1$. Then we provide some examples illustrating the need for some of the assumptions in order to obtain the results. The first example illustrates the necessity of the maximum trade constraints. Without maximum quantity constraints, we produce a robust SSP equilibrium that converges to an inefficient allocation. We then provide an example with non-differentiable utility functions where bargaining with quantity constraints does not necessary imply convergence to the Walrasian equilibrium, which stresses the importance of the differentiability conditions.

Is every Walrasian allocation reachable as a SSP equilibrium allocation as $\delta^A, \delta^B \to 1$. We have established in Theorem 3 that every SSP equilibrium allocation of the alternating-offers bargaining over prices and maximum trades must converge to a Walrasian allocation as $\delta^A, \delta^B \to 1$. However another question one might be interested in is whether every Walrasian allocation is reachable as a SSP equilibrium allocation this way. This question is however not addressed by the previous results. Still the conjecture that this conjecture holds true is supported by the intuition provided by the behavior of the SSP equilibrium payoffs as $\delta^A, \delta^B \to 1$.

¿From agent A's problem in equation (4) and the corresponding problem for B, it follows that at a SSP equilibrium the feasible allocations (x^A_A, x^B_A) and (x^A_B, x^B_B) resulting from the offers by A and B respectively, the second constraint of each agents problem must be binding, i.e. $u^A(x^A_B) = \delta^A u^A(x^A_A)$ and $u^B(x^B_A) = \delta^B u^B(x^B_B)$. That is to say $(u^A(x^A_B), \delta^B u^B(x^B_B))$ and $(\delta^A u^A(x^A_A), u^B(x^B_A))$ must denote the same point of the intersection of the sets $\tilde{f}^A_{\delta^B}$ and $\tilde{f}^B_{\delta^A}$ of utility profiles that can be attained through an acceptance by A or B respectively (see Figure 2 for the $\delta^A = \delta^B = 1$ case).[15]

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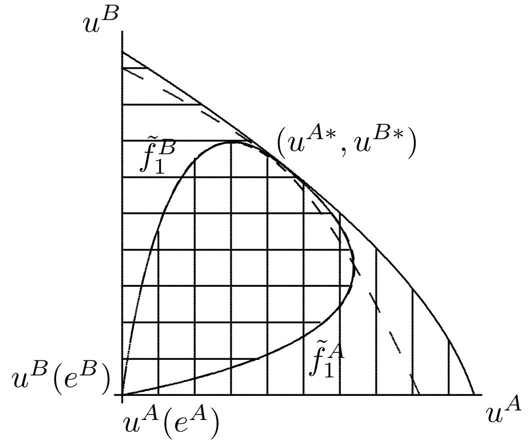
[14] An even number of them to be precise.

[15] Note that, in the case of, for instance, $\tilde{f}^A_1$ whenever A's desired trade at some prices p^B by B is smaller than B's desired trade at those same prices, there is no way in which B can obtain a bigger trade than that resulting from A's demand at the prices p^B. Hence the utilities in $\tilde{f}^A_1$ attainable through A's acceptance are bounded above by the profiles along A's offer curve to the left of the Walrasian profile (u^{A*}, u^{B*}). On the contrary, whenever A's desired trade is bigger than B's, efficiency can be imposed by B by means of the maximum trade constraint of his offer (which explains that to the right of (u^{A*}, u^{B*}) the upper boundary of $\tilde{f}^A_1$ is the Pareto frontier). And similarly for $\tilde{f}^B_1$.

Figure 2



In the case $\delta^A = \delta^B = 1$, the point $(u^A(x_B^A), \delta^B u^B(x_B^B)) = (\delta^A u^A(x_A^A), u^B(x_A^B))$ must be in the vertically and horizontally shaded area in Figure 2 that is the intersection of $\tilde{f}_1^A$ and $\tilde{f}_1^B$. Nevertheless, note that not all the utilities in that area can be SSP equilibrium payoffs. Every profile Pareto-dominated by some point in either $\tilde{f}_{\delta^B}^A$ or $\tilde{f}_{\delta^A}^B$ corresponds to a situation in which there is room for a mutually beneficial deviation by some agent. The only profiles of utilities that are not Pareto-dominated this way, and correspond hence to SSP equilibrium payoffs, are the crossings of the *boundaries* of $\tilde{f}_{\delta^B}^A$ and $\tilde{f}_{\delta^A}^B$. In the case $\delta^A = \delta^B = 1$, these crossings correspond to the Walrasian payoffs.[16] By continuity, for any δ^A and δ^B converging to 1 there exist an undominated crossing of the boundaries of $\tilde{f}_{\delta^B}^A$ and $\tilde{f}_{\delta^A}^B$ converging to each Walrasian equilibrium crossing of the boundaries of $\tilde{f}_1^A$ and $\tilde{f}_1^B$ that corresponds to SSP equilibrium payoffs for discount factors close enough to 1.

The necessity of maximum quantity constraints. We illustrate the existence of asymptotically inefficient SSP equilibria of the bargaining over prices without maximum quantity constraints in a simple Cobb-Douglas setup. Let $u^i(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$, for $i = A, B$. The total resources are $e = (1, 1)$ and the distribution of initial endowments between A and B is $e^A = (0.9, 0.3)$ and $e^B = (0.1, 0.7)$. In this example, the contract curve is the diagonal of the Edgeworth box.[17]

A SSP equilibrium with immediate acceptance must satisfy the necessary conditions in (12) above. When there is no discounting, p^A and p^B equal to the Walrasian relative price $p^* = 1$ leading to the allocation $x^{A*} = (0.6, 0.6)$, $x^{B*} = (0.4, 0.4)$ is a solution to equations (12). Figure 3 below shows another solution to the system (12) with $\delta^A = \delta^B = 1$, namely $(p^A, p^B) = (1.750, 1.333)$ leading to the allocations $\bar{x}$ and $\hat{x}$ on A's and B's offer curves respectively, with

$$\begin{aligned} \bar{x}^A &= (\bar{x}_1^A(p^B), \bar{x}_2^A(p^B)) = (0.5625, 0.75) \\ \hat{x}^A &= (e - \hat{x}_1^B(p^A), e - \hat{x}_2^B(p^A)) = (0.75, 0.5625) \end{aligned} \tag{14}$$

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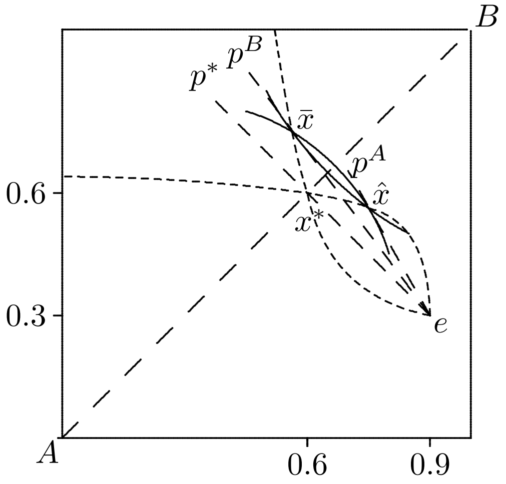
[16] Note that, while Lemma 1 established that f^A and f^B do not cross generically at the Walrasian payoffs, the boundaries of $\tilde{f}_1^A$ and $\tilde{f}_1^B$ do necessarily cross at (u^{A*}, u^{B*}).

[17] The fact that the result in Yildiz (2003) is non-generic translates here into the fact that only the economies with initial endowments on the anti-diagonal between the upper-left and lower-right corners of the Edgeworth box satisfy the assumptions made in Yildiz (2003). Any small deviation away from the anti-diagonal gives rise to asymptotically inefficient SSP equilibria.

and the complementary bundles for agent B. Note that, unlike the Walrasian solution, this other solution is not Pareto-efficient. Moreover, $(p^A, p^B) = (1.750, 1.333)$ are indeed SSP equilibrium prices since no agent can profitably deviate at any stage of the game.[18]

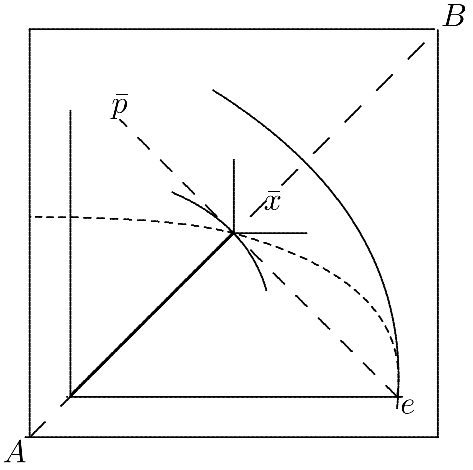
Figure 3



By continuity, for δ^A and δ^B close to 1, there exists as well a solution $(p^A_{\delta^A\delta^B}, p^B_{\delta^A\delta^B})$ to the system of equations (12) close to $(p^A, p^B) = (1.750, 1.333)$. That solution is still a SSP equilibrium. Note that as $\delta^A, \delta^B \to 1$, $(p^A_{\delta^A\delta^B}, p^B_{\delta^A\delta^B})$ converges to $(p^A, p^B) = (1.750, 1.333)$ and hence remains bounded away from efficiency.

A counter-example in the case of non-differentiability. Consider an economy represented in the Edgeworth box in Figure 4: $e^A = (0.9, 0.1)$, $e^B = (0.1, 0.9)$, $u^A(x_1, x_2) = \min\{x_1, x_2\}$, $u^B(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$. Its unique Walrasian equilibrium allocation is $\bar{x}^A = \bar{x}^B = \left(\frac{1}{2}, \frac{1}{2}\right)$, supported by the relative price $\bar{p} = 1$.

Figure 4



In this economy, the relevant part of A's offer curve coincides with the contract curve, which is the diagonal of the Edgeworth box. When making an offer, A can impose a maximum constraint on B and therefore any offer that would induce B to consume more than the Walrasian equilibrium amount of good 2 will lead to

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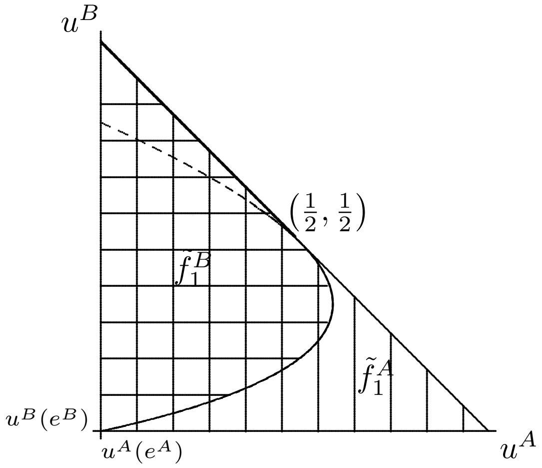
[18] This is a consequence of the fact that no allocation on the offer curves Pareto-improves upon $\hat{x}, \bar{x}$.

an offer accepted on the contract curve. Therefore, offers accepted by B exactly coincide with offers accepted by A. The thick line segment on the diagonal in Figure 4 represents those coinciding allocations. In the space of utilities in Figure 5 below, this translates into a continuum of undominated intersections of the boundaries of $\tilde{f}_1^A$ and $\tilde{f}_1^B$ corresponding to utility levels $u^A \in [0, 1/2], u^B \in [1/2, 1]$ such that $u^A + u^B = 1$. Each element in this continuum of intersections corresponds then to a SSP equilibrium with δ's equal to one.

Figure 5



With discounting, when $\delta^A = \delta^B < 1$ and therefore the discount factors converge to 1 at the same rate, there is still a continuum of SSP equilibria. Clearly these equilibria need not converge to the Walrasian one as δ approaches 1. For δ's converging to 1 with δ^A/δ^B bounded away from 1, the SSP equilibrium converges either to the Walrasian allocation or to the corner solution where B extracts all the surplus from trade.

7. Concluding Remarks

In this paper, we have proposed a simple bargaining procedure that achieves the competitive equilibrium allocation without assuming price-taking behavior. The procedure is commonly observed, in the sense that negotiating parties often bargain over a price with a quantity constraint, and then the quantity of trade is chosen separately. An interesting property of the main result of this paper is that, by always obtaining the Walrasian equilibrium, the outcome of the bargaining does not depend on specifics such as relative bargaining powers or impatience, but only on primitives, i.e. preferences and endowments.

In the context of the Nash demand game, Binmore (1987) stresses the importance of quantity constraints. Bargaining procedures with maximum quantity constraints capture the main aspects of several existing price setting mechanisms. For example, in commodity futures markets, the seller of future contracts will typically announce to a candidate buyer the price for the contract and how many contracts he has on offer. The candidate buyer can accept the price offer and choose the number of future contracts as long as it does not exceed the quantity constraint that was offered initially. The same is true for limit orders when selling stock. Your limit order guarantees a certain price for the stock, but you cannot be sure that the order will be filled. Only if there is sufficient demand at that price will your order be filled (either partially or completely). In addition, our bargaining procedure involves separation of the price-setting by the proposer from the quantity decision by the recipient. This price-quantity separation is obviously well known in negotiations and has common applications in several economic environments such as union-wage bargaining in the labor economics literature,[19] and standard buy-out provisions in two-person partnerships.[20]

Finally, Gale (1986) (see also Kunimoto and Serrano (2004)) establishes a bargaining foundation for the Walrasian equilibrium outcome in general exchange economies with a continuum of traders and random pairwise matching. In his paper, the possibility of being matched later on to another agent offering better terms drives the convergence to the competitive outcome. That argument does not apply in a small economy like ours. Therefore, our results contribute to extending a bargaining foundation for Walrasian equilibrium to economies with a small number of agents. Despite the (only two) agents being price-setters, the perfectly competitive outcome still obtains. We conjecture that our results extend to the case of an arbitrarily finite number of agents. In the extreme case in which the number of agents increases to a continuum and pairs are formed through random matching, then the bargaining procedure proposed in Gale (1986) leads to the Walrasian equilibrium outcome.

Appendix

Lemma A1. *If u and e satisfy A1 and $x \in \mathbb{R}^n_{++}$ solves*

$$\begin{aligned} \max u(x) \\ p(x-e) \leq 0 \\ \|x-e\| \leq q \end{aligned} \tag{15}$$

where $\|\cdot\|$ stands for the Euclidean norm,[21] *then*

$$Du(x)(x-e) \geq 0, \tag{16}$$

and, conversely, if x satisfies (16)*, then there exist p, q for which x solves* (15)*.*

Proof. Since x solves (15), then necessarily for some $\lambda, \mu \geq 0$

$$\begin{aligned} Du(x) &= \lambda p + \mu(x-e) \\ \lambda p(x-e) &= 0 \\ \mu[(x-e)^t(x-e) - q^2] &= 0 \end{aligned} \tag{17}$$

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[19] See Solow and MacDonald (1981) and Farber (1986) amongst others who study and document such bargaining over wages where the union negotiates the wage and the employer chooses the level of employment.

[20] Cramton, Gibbons and Klemperer (1987), and Fiesseler, Kittsteiner and Moldovanu (2003) model such buy-out provisions. When partners decide they want to separate, the provision prescribes that one partner chooses the price of the shares, and the other partner chooses the quantity traded, i.e. whether to buy or sell.

[21] In equation (1) the maximum trade constraint was expressed in terms of good 1. It is clear that any such constraint can be expressed equivalently as in (15).

Therefore

$$\begin{aligned} Du(x)(x-e) &= \lambda p(x-e) + \mu(x-e)^t(x-e) \\ &= \mu(x-e)^t(x-e) \geq 0 \end{aligned} \tag{18}$$

Conversely, assume $x \in \mathbb{R}^n_{++}$ satisfies (16). Assume, after relabelling if necessary, that $x_1 \neq e_1$. Then

$$\begin{aligned} \mu &= \frac{D_1u(x) - \lambda}{x_1 - e_1} \geq 0 \\ p_i &= \frac{1}{\lambda}[D_iu(x) - (x_i - e_i)\frac{D_1u(x) - \lambda}{x_1 - e_1}] \geq 0 \end{aligned} \tag{19}$$

for any λ such that $D_1u(x) - \frac{x_1 - e_1}{x_i - e_i}D_iu(x) < \lambda \leq D_1u(x)$ if $x_1 - e_1 > 0$, or for any λ such that $D_1u(x) - \frac{x_1 - e_1}{x_i - e_i}D_iu(x) \geq \lambda > D_1u(x)$ if $x_1 - e_1 < 0$, satisfy conditions (17) above as well as the constrains of (15). This first order conditions are sufficient for x to solve (15) given the assumptions on u. Q.E.D.

Proof of Theorem 1. Let (x^A_A, x^B_A) (respectively (x^A_B, x^B_B)) be the feasible allocation resulting from B's (resp. A's) acceptance of A's (resp. B's) offer of a price and maximum trade at a SSPE with immediate acceptance for infinitely patient agents, that is to say such that (x^A_A, x^B_A) solves

$$\begin{aligned} &\max_{x^A, x^B} u^A(x^A) \\ Du^B(\tilde{x}^B)&(\tilde{x}^B - e^B) \geq 0 \\ u^B(x^B) &\geq u^B(x^B_B) \\ x^A + x^B &= e^A + e^B \end{aligned} \tag{20}$$

given (x^A_B, x^B_B), and similarly for (x^A_B, x^B_B). Then necessarily there exist λ^A, μ^A, λ^B, $\mu^B \geq 0$ and ν^A_i, ν^B_i, for all $i = 1, \ldots, n$, such that the following $2n$ equations are satisfied

$$\begin{aligned} &\begin{pmatrix} Du^A(x^A_A) \\ 0 \end{pmatrix} + \lambda^A \begin{pmatrix} 0 \\ Du^B(x^B_A) \end{pmatrix} \\ &\qquad + \mu^A \begin{pmatrix} 0 \\ Du^B(x^B_A) + D^2u^B(x^B_A)(x^B_A - e^B) \end{pmatrix} + \sum_{i=1}^n \nu^A_i \begin{pmatrix} e_i \\ e_i \end{pmatrix} = 0 \end{aligned} \tag{21}$$

where e_i stands for the i-th vector of the canonical basis of $\mathbb{R}^n$, or equivalently

$$Du^A(x^A_A) = \lambda^A Du^B(x^B_A) + \mu^A \big[Du^B(x^B_A) + D^2u^B(x^B_A)(x^B_A - e^B)\big] \tag{22}$$

and similarly for$B$'s problem.

Assume that $(x^A_A, x^B_A) \neq (x^A_B, x^B_B)$. Since at a SSP equilibrium the second constraint in (20) is binding (and similarly for B's problem),[22] both allocations are

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[22] Since (x^A_A, x^B_A) satisfies the constraints of B's maximization problem (in particular $Du^A(x^A_A)(x^A_A - e_A) \geq 0$ since A will never choose at equilibrium to trade more than necessary to attain his demand at the implicit prices), necessarily $u^B(x^B_A) \leq u^B(x^B_B)$. Hence it cannot be that $u^B(x^B_A) > u^B(x^B_B)$.

on the same indifference surface for both agents but none of them is efficient, in particular, for some i, j

$$\frac{D_i u^A(x^A_A)}{D_j u^A(x^A_A)} > \frac{D_i u^B(x^B_A)}{D_j u^B(x^B_A)}. \tag{23}$$

But then (22) above cannot hold for a non-negative μ^A.[23] In effect, λ^A, μ^A must solve (22) above, and in particular

$$\begin{pmatrix} D_i u^B(x^B_A) & D_i u^B(x^B_A) + D_{ii} u^B(x^B_A)(x^B_{Ai} - e^A_i) + D_{ij} u^B(x^B_A)(x^B_{Aj} - e^A_j) \\ D_j u^B(x^B_A) & D_j u^B(x^B_A) + D_{ji} u^B(x^B_A)(x^B_{Aj} - e^A_j) + D_{jj} u^B(x^B_A)(x^B_{Aj} - e^A_j) \end{pmatrix} \cdot \begin{pmatrix} \lambda^A \\ \mu^A \end{pmatrix} = \begin{pmatrix} D_i u^A(x^A_A) \\ D_j u^A(x^A_A) \end{pmatrix} \tag{24}$$

and hence

$$\mu^A = \frac{\begin{vmatrix} D_i u^B(x^B_A) & D_i u^A(x^A_A) \\ D_j u^B(x^B_A) & D_j u^A(x^A_A) \end{vmatrix}}{\begin{vmatrix} D_i u^B(x^B_A) & D_i u^B(x^B_A) + D_{ii} u^B(x^B_A)(x^B_{Ai} - e^A_i) + D_{ij} u^B(x^B_A)(x^B_{Aj} - e^A_j) \\ D_j u^B(x^B_A) & D_j u^B(x^B_A) + D_{ji} u^B(x^B_A)(x^B_{Ai} - e^A_i) + D_{jj} u^B(x^B_A)(x^B_{Aj} - e^A_j) \end{vmatrix}}. \tag{25}$$

Since (23) implies that the numerator is positive, the denominator is strictly positive as well, which amounts to

$$\begin{pmatrix} -D_j u^B(x^B_A) & D_i u^B(x^B_A) \end{pmatrix} \begin{pmatrix} D_{ii} u^B(x^B_A) & D_{ij} u^B(x^B_A) \\ D_{ji} u^B(x^B_A) & D_{jj} u^B(x^B_A) \end{pmatrix} \begin{pmatrix} x^B_{Ai} - e^B_i \\ x^B_{Aj} - e^B_j \end{pmatrix} > 0. \tag{26}$$

But $D^2 u^B(x^B_A)$ is negative definite in the space orthogonal to $Du^B(x^B_A)$, and hence for any $(0, \ldots, 0, x^B_{Ai} - e^B_i, 0, \ldots, 0, x^B_{Aj} - e^B_j, 0, \ldots, 0)$ orthogonal to $Du^B(x^B_A)$, i.e. such that

$$D_i u^B(x^B_A)(x^B_{Ai} - e^B_i) + D_j u^B(x^B_A)(x^B_{Aj} - e^B_j) = 0 \tag{27}$$

or equivalently collinear to $(-D_j u^B(x^B_A), D_i u^B(x^B_A))$, the left-hand side of (26) should be negative!!

Therefore, at any SSP equilibrium both allocations coincide, i.e. $(x^A_A, x^B_A) = (x^A_B, x^B_B) = (x^A, x^B)$, whenever $\delta^A = \delta^B = 1$. Since x^B must solve

$$\begin{gathered} \max u^A(e - \tilde{x}^B) \\ Du^B(\tilde{x}^B)(\tilde{x}^B - e^B) \geq 0 \\ u^B(\tilde{x}^B) \geq u^B(e - x^A) \end{gathered} \tag{28}$$

given x^A, should the first constraint not be binding at x^B, then x^B would solve as well

$$\begin{gathered} \max u^A(e - \tilde{x}^B) \\ u^B(\tilde{x}^B) \geq u^B(e - x^A) \end{gathered} \tag{29}$$

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[23] If the inequality (23) holds in the opposite direction, it is B's FOCs which cannot hold for a non-negative μ^B and the following argument applies with the obvious changes.

given x^A, because u^A is strictly monotone and u^B is strictly quasi-concave. Hence for some $\lambda > 0$, $Du^A(x^A) = \lambda Du^B(x^B)$. Since $Du^B(x^B)(x^B - e^B) > 0$, then $\frac{1}{\lambda} Du^A(x^A)(x^A - e^A) < 0$ which contradicts that x^A solves

$$\begin{gathered} \max u^B(e - \tilde{x}^A) \\ Du^A(\tilde{x}^A)(\tilde{x}^A - e^A) \geq 0 \\ u^A(\tilde{x}^A) \geq u^A(e - x^B) \end{gathered} \tag{30}$$

given x^B. Therefore, necessarily $Du^B(x^B)(x^B - e^B) = 0$ and similarly $Du^B(x^A)(x^A - e^A) = 0$, i.e. at the allocation (x^A, x^B) each agent gets his demand at the implicit prices, and hence (x^A, x^B) is a Walrasian allocation.

Conversely, let x^A, x^B be a Walrasian allocation. Then necessarily it is feasible and efficient, i.e. x^A solves

$$\begin{gathered} \max_{\tilde{x}^A} u^B(e - \tilde{x}^A) \\ u^A(\tilde{x}^A) \geq u^A(e - x^B) \end{gathered} \tag{31}$$

given x^B, and gives to agent A his demand at the implicit prices. Therefore x^A trivially satisfies

$$Du^A(\tilde{x}^A)(\tilde{x}^A - e^A) \geq 0 \tag{32}$$

and hence x^A solves also

$$\begin{gathered} \max_{\tilde{x}^A} u^B(e - \tilde{x}^A) \\ u^A(\tilde{x}^A) \geq u^A(e - x^B) \\ Du^A(\tilde{x}^A)(\tilde{x}^A - e^A) \geq 0 \end{gathered} \tag{33}$$

given x^B, and similarly for A's problem, which implies that (x^A, x^B) is the allocation resulting from the acceptance by any agent of the offer received from the other agent at a SSP equilibrium with immediate acceptance. QED

Proof of Theorem 2. Since, from Theorem 1, every SSP equilibrium allocation with immediate acceptance is Walrasian for $\delta^A, \delta^B = 1$ and, as shown below, the correspondence associating to each (δ^A, δ^B) the allocations of SSP equilibria with immediate acceptance is everywhere upper hemicontinuous (in particular at $(\delta^A, \delta^B) = (1, 1)$), then the conclusion follows.

In effect, let Γ assign to every (δ^A, δ^B) the SSP equilibrium allocations with immediate acceptance, i.e.

$$\Gamma(\delta^A, \delta^B) = \left\{ (x^A, x^B) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \middle| \; (x^A, x^B) \in \Phi(x^A, x^B; \delta^A, \delta^B) \right\} \tag{34}$$

where

$$\Phi(x^A, x^B; \delta^A, \delta^B) = \Psi^A(x^A, x^B; \delta^A, \delta^B) \cap \Psi^B(x^A, x^B; \delta^A, \delta^B) \tag{35}$$

and

$$\begin{aligned}\Psi^A(x^A, x^B; \delta^A, \delta^B) = \arg \max_{\tilde{x}^A, \tilde{x}^B} & \ u^A(\tilde{x}^A) \\ Du^B(\tilde{x}^B)(\tilde{x}^B - e^B) & \geq 0 \\ u^B(\tilde{x}^B) & \geq \delta^B u^B(x^B) \\ \tilde{x}^A + \tilde{x}^B & = e^A + e^B \\ \text{given } & (x^A, x^B),\end{aligned} \tag{36}$$

and similarly for $\Psi^B(x^A, x^B; \delta^A, \delta^B)$.

¿From the Theorem of the Maximum Ψ^A is a compact-valued, upper hemicontinuous correspondence depending on all x^A, x^B, δ^A, and δ^B —since u^A is continuous in $\tilde{x}^A$ and trivially in all $\tilde{x}^B, x^A, x^B, \delta^A, \delta^B$ as well, and the constraints correspondence

$$\begin{aligned}\Omega^A(x^A, x^B, \delta^A, \delta^B) = \Big\{ (\tilde{x}^A, \tilde{x}^B) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \ | & Du^B(\tilde{x}^B)(\tilde{x}^B - e^B) \geq 0, \\ & u^B(\tilde{x}^B) \geq \delta^B u^B(x^B), \text{ and} \\ & \tilde{x}^A + \tilde{x}^B = e^A + e^B \Big\}\end{aligned} \tag{37}$$

is continuous and compact-valued— and similarly for Ψ^B. Hence, as an intersection of compact-valued, upper hemicontinuous correspondences, $\Phi(\cdot, \cdot; \delta^A, \delta^B)$ is compact-valued and upper hemicontinuous itself. Since $\Gamma(\delta^A, \delta^B)$ is the set of fixed points of $\Phi(\cdot, \cdot; \delta^A, \delta^B)$ and $\Phi(\cdot, \cdot; \delta^A, \delta^B)$ is compact-valued and upper hemicontinuous, then Γ is upper hemicontinuous itself (see Lemma A2 below). Q.E.D.

Lemma A2. *If X, Y are metric spaces and $\Phi \in \mathcal{P}(X)^{X \times Y}$ is compact-valued and upper hemicontinuous, then $\Gamma \in \mathcal{P}(X)^Y$ such that*

$$\Gamma(y) = \big\{ x \in X \mid x \in \Phi(x, y) \big\} \tag{38}$$

is upper hemicontinuous.

Proof. Assume that Γ is not upper hemicontinuous at some y. Then there exist $\{y_n\} \to y$, x and $\{x_n\} \to x$ such that $x_n \in \Gamma(y_n)$ for all $n \in \mathbb{N}$, while $x \notin \Gamma(y)$. That is to say, for all $n \in \mathbb{N}$, $x_n \in \Phi(x_n, y_n)$ while $x \notin \Phi(x, y)$. As a consequence, since Φ is compact-valued, then Φ is not upper hemicontinuous at (x, y)! Q.E.D.

Proposition 1. *If $\{u^i, e^i\}_{i=A,B}$ satisfies A1 and, for $i = A, B$, u^i is strongly concave,*[24] *then there exists a SSP equilibrium with immediate acceptance of the*

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[24] In the sense that,

$$\det \left\{ 2D^2 u^h(x) + \left[\sum_{k=1}^{n} D_{ikj} u^h(x)(x_k - e^h_k) \right]_{ij} \right\}$$

does not change sign. This guarantees that the offer curve (or surface in general) does not ever change curvature (i.e. has no inflexion points) and hence the constrained domain delimited by the offer curve is convex. This condition is satisfied whenever the substitution effect dominates largely the wealth effect, and in particular by every CES utility function.

alternating-offers bargaining over prices and maximum trades, for any $\delta^A, \delta^B \in [0,1]$.

Proof. A SSP equilibrium is characterized by two allocations, (x_A^A, x_A^B) and (x_B^A, x_B^B), resulting from the acceptance by B and A respectively of offers made by A and B such that

(1)

$$\begin{aligned}(x_A^A, x_A^B) &\in \arg\max_{x^A, x^B} u^A(x^A) \\ Du^B(x^B)(x^B - e^B) &\geq 0 \\ u^B(x^B) &\geq \delta^B u^B(x_B^B) \\ x^A + x^B &= e^A + e^B \\ \text{given } &(x_B^A, x_B^B),\end{aligned} \tag{39}$$

(2) and

$$\begin{aligned}(x_B^A, x_B^B) &\in \arg\max_{x^A, x^B} u^B(x^B) \\ Du^A(x^A)(x^A - e^A) &\geq 0 \\ u^A(x^A) &\geq \delta^A u^A(x_A^A) \\ x^A + x^B &= e^A + e^B \\ \text{given } &(x_A^A, x_A^B)\end{aligned} \tag{40}$$

Then letting the right-hand side of (39) being $\Upsilon^A(x_A^A, x_A^B, x_B^A, x_B^B)$ and the right-hand side of (40) being $\Upsilon^B(x_A^A, x_A^B, x_B^A, x_B^B)$ (note that both argmax correspondences depend trivially on *both* given allocations), a SSP equilibrium of the bargaining game is a fixed point of the correspondence $\Upsilon^A \times \Upsilon^B$ (abusing notation only slightly) that associates to every $(x_A^A, x_A^B, x_B^A, x_B^B)$ the set $\Upsilon^A(x_A^A, x_A^B, x_B^A, x_B^B) \times \Upsilon^B(x_A^A, x_A^B, x_B^A, x_B^B)$.

Since by the Theorem of the Maximum both Υ^A and Υ^B are compact-valued and upper hemicontinuous, they have closed graphs, and hence so does $\Upsilon^A \times \Upsilon^B$. Also, since u^A and u^B are strongly concave for both agents, then $Du^i(x^i)(x^i - e^i) \geq 0$, for $i = A, B$, defines a convex domain. Since the other constraints define also a convex domain, then Υ^A and Υ^B are both convex-valued, and therefore so is $\Upsilon^A \times \Upsilon^B$.

Since $\Upsilon^A \times \Upsilon^B$ is a convex-valued, closed graph correspondence defined on, and taking values in, the nonempty, compact, convex set

$$\begin{aligned}\big\{(x_A^A, x_A^B, x_B^A, x_B^B) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}_+^n \mid x_A^A + x_A^B &= e^A + e^B \\ x_B^A + x_B^B &= e^A + e^B\big\}\end{aligned} \tag{41}$$

then by Kakutani's Fixed Point Theorem a fixed point of $\Upsilon^A \times \Upsilon^B$ exists that corresponds to a SSP equilibrium of the bargaining game. Q.E.D.

Lemma A3. *If* $\{u^i, e^i\}_{i=A,B}$ *satisfies A1 and the agents are impatient (that is to say,* $\delta^A, \delta^B < 1$*), then there does not exist any SSP equilibrium with delay.*

Proof. Consider a candidate SSP equilibrium (p^A, q^A, p^B, q^B) in which, for instance, B rejects and A accepts. Let $(x^A_A, x^B_A) = (e - \tilde{x}^B(p^A, q^A), \tilde{x}^B(p^A, q^A))$ and $(x^A_B, x^B_B) = (\tilde{x}^A(p^B, q^B), e - \tilde{x}^A(p^B, q^B))$ be the allocations resulting from A's and B's offers if accepted. Then it must be the case that

(1) B's offer is rational, that is to say B is interested in A's accept acceptance since he obtains more utility this way than from A's offer one period later, i.e.

$$u^B(x^B_B) > \delta^B u^B(x^B_A) \tag{42}$$

and also B's offer is his most preferred acceptable to A, i.e.

$$\begin{aligned} (x^A_B, x^B_B) \in \arg \max_{x^A, x^B} \;& u^B(x^B) \\ Du^A(x^A)(x^A - e^A) &\geq 0 \\ u^A(x^A) &\geq \delta^A u^A(x^A_A) \\ x^A + x^B &= e^A + e^B \end{aligned} \tag{43}$$

given (x^A_A, x^B_A), which guarantees A's acceptance, and

(2) A's offer is rational, that is to say A is interested in B's rejection since A obtains more utility from B's offer one period later, i.e.

$$\delta^A u^A(x^A_B) \geq u^A(x^A_A) \tag{44}$$

and accordingly makes an unacceptable offer to B, i.e.

$$u^B(x^B_A) < \delta^B u^B(x^B_B) \tag{45}$$

which guarantees B's rejection.

Therefore, from (44) and the fact that (x^A_B, x^B_B) solves (43) above, it follows that whenever $\delta^A < 1$, necessarily

$$u^A(x^A_B) > \delta^A u^A(x^A_B) \geq u^A(x^A_A) > \delta^A u^A(x^A_A) \tag{46}$$

i.e. the second constraint in (43) is not binding. Since u^B is strictly monotone (and hence the solution cannot be interior), then necessarily the first constraint in (43) must be binding, i.e. $Du^A(x^A)(x^A - e^A) = 0$. As a consequence, $\tilde{x}^A(p^B, q^B) = x^A(p^B)$, that is to say q^B does not actually constrain A's demand (in the 2 goods case, (x^A_B, x^B_B) is on A's offer curve in the Edgeworth box).

Since u^A is strictly differentiably quasi-concave, then (x^A_B, x^B_B) is not efficient,[25]

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[25] In effect, since the normal direction to A's offer curve (manifold, in general) is $Du^A(x^A_B) + D^2u^A(x^A_B)(x^A_B - e_B)$, this is collinear to $Du^A(x^A_B)$ (which is necessary for $((x^A_B, x^B_B))$ to be efficient) only if

$$(1-r)Du^A(x^A_B) + D^2u^A(x^A_B)(x^A_B - e_B) = 0$$

for some $r > 0$. But since $Du^A(x^A_B)(x^A_B - e_B) = 0$ and u^A is strictly differentiably quasi-concave in the space normal a $Du^A(x^A_B)$, that would imply that

$$\begin{aligned} 0 > (x^A_B - e_B)^t D^2u^A(x^A_B)(x^A_B - e_B) = \\ (1-r)(x^A_B - e_B)^t Du^A(x^A_B) + (x^A_B - e_B)^t D^2u^A(x^A_B)(x^A_B - e_B) = \\ (x^A_B - e_B)^t\big[(1-r)Du^A(x^A_B) + (x^A_B - e_B)^t D^2u^A(x^A_B)(x^A_B - e_B)\big] = 0! \end{aligned}$$

and therefore there is room for A deviating making an offer that is Pareto improving with respect to x_B and that B would accept.[26] Q.E.D.

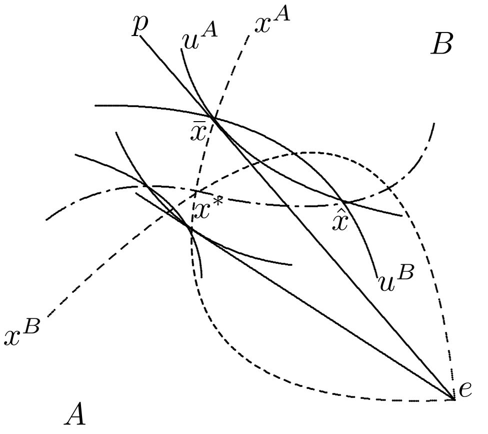
Proof of Theorem 4. Theorem 8 in Merlo and Wilson (1995) establishes that the extreme payoffs of the set of Subgame Perfect equilibria are stationary, and that all Subgame Perfect equilibria are contained within those extreme SSP payoffs. Since there is a unique Walrasian equilibrium and every SSP equilibrium converges to it, these extreme payoffs (and hence every SP equilibrium payoffs) converge to the Walrasian payoffs. The strict convexity of the preferences implies that the convergence takes place also in allocations and prices. Q.E.D.

Proof of Lemma 1. Let $\bar{x}$ be an allocation on A's offer curve close to the Walrasian allocation x^*. Let $\hat{x}$ be the allocation distinct from $\bar{x}$ giving A and B the same utilities as $\bar{x}$. Then $\bar{x}$ and $\hat{x}$ are characterized by the equations (in terms of A's consumptions)

$$\begin{aligned} u^A(\hat{x}^A) - u^A(\bar{x}^A) &= 0 \\ u^B(e^A + e^B - \hat{x}^A) - u^B(e^A + e^B - \bar{x}^A) &= 0 \\ \phi(\bar{x}^A) &= 0 \\ (p,1)(\bar{x}^A - e^A) &= 0 \end{aligned} \tag{47}$$

for some relative price p, where $\phi^A(\bar{x}^A) = Du^A(\bar{x}^A)(\bar{x}^A - e^A)$ and hence $\phi^A(x) = 0$ is the equation of A's offer curve. The system (47) defines implicitly $\hat{x}$ as a differentiable function of p in a neighbourhood of the Walrasian price p^*, so that as p varies or, equivalently, as $\bar{x}$ runs along A's offer curve, $\hat{x}$ follows a smooth path that goes through x^* for $p = p^*$, for which $\bar{x} = \hat{x} = x^*$ (see Figure 6).

Figure 6



In effect, the function that determines $\hat{x}^A$ for each p in the system (47) is the composition of the function $\bar{\xi}^A$ associating $\bar{x}^A$ to each p that is implicitly defined by the last two equations in (47) and the function $\hat{\xi}^A$ associating $\hat{x}^A$ to each $\bar{x}^A$ that is implicitly defined by the first two equations in (47). In order to see this,

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[26] At any rate, for weaker assumptions on u^A (e.g. just differentiable quasi-concavity), if it was efficient, then A could deviate offering himself B's offer instead, since he will accept it anyway later, saving the cost of the delay in reaching an agreement.

regarding $D\bar{\xi}^A(p)$ note first that in the Jacobian of the left-hand side of the last two equations in (47)

$$\begin{pmatrix} D_1\phi^A(\bar{x}^A) & D_2\phi^A(\bar{x}^A) & 0 \\ p & 1 & \bar{x}_1^A \end{pmatrix} \tag{48}$$

the first two columns are linearly independent, even at the Walrasian equilibrium allocation x^*,[27] and hence

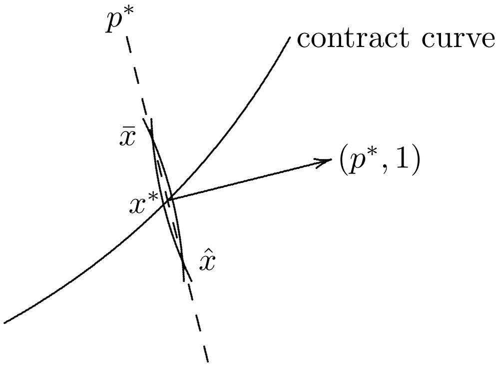
$$D\bar{\xi}^A(p) = -\begin{vmatrix} D_1\phi^A(\bar{x}^A) & D_2\phi^A(\bar{x}^A) \\ p & 1 \end{vmatrix}^{-1} \begin{pmatrix} -D_2\phi^A(\bar{x}^A)\bar{x}_1^A \\ D_1\phi^A(\bar{x}^A)\bar{x}_1^A \end{pmatrix}. \tag{49}$$

As for $D\hat{\xi}^A(x^{A*})$, note that, altough the Jacobian of the left-hand side of the first two equations in (47)

$$\begin{pmatrix} D_1u^A(\hat{x}^A) & D_2u^A(\hat{x}^A) & -D_1u^A(\bar{x}^A) & -D_2u^A(\bar{x}^A) \\ -D_1u^B(\hat{x}^B) & -D_2u^B(\hat{x}^B) & D_1u^B(\bar{x}^B) & D_2u^B(\bar{x}^B) \end{pmatrix}. \tag{50}$$

drops rank at the Walrasian allocation x^*, the first two equations in (47) still define $\hat{x}^A$ as a function of $\bar{x}^A$ since, for strictly convex preferences and any given point $\bar{x}$, there exists a unique $\hat{x}$ where the two indifference curves through $\bar{x}$ cross each other again (if $\bar{x}$ happens to be efficient, then $\hat{x}$ actually coincides with $\bar{x}$). This function is not only differentiable off the contract curve (where the Jacobian is full rank and the implicit function theorem does apply), but also at x^* since, as $\bar{x}$ departs slightly from an efficient allocation x^* on the contract curve, the lens formed by the two indifference curves going through $\bar{x}$ cross again (almost) at a point $\hat{x}$ across the contract curve in the direction of the line supporting x^* as a Walrasian equilibrium (see Figure 7).

Figure 7



The linear mapping approximating this function is

$$D\hat{\xi}^A(x^{A*}) = \begin{pmatrix} p^* & -1 \\ 1 & p^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -c^* \end{pmatrix} \begin{pmatrix} p^* & -1 \\ 1 & p^* \end{pmatrix}^{-1} \tag{51}$$

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[27] This is a consequence of the strictly differentiably quasi-concavity of u^A.

for some $c^* > 0$ that depends on the curvature of A's and B's indifference curves at x^*.[28] Therefore, since $\frac{d\hat{x}^A}{dp}(p^*) = D\hat{\xi}^A(x^{A*})D\bar{\xi}^A(p^*)$ it follows that

$$\frac{d\hat{x}_2^A}{d\hat{x}_1^A}(x_1^{A*}) = \frac{\frac{d\hat{x}_2^A}{dp}(p^*)}{\frac{d\hat{x}_1^A}{dp}(p^*)} = \frac{(1 - c^* p^{*2})D_1\phi^A(x^{A*}) - (1 + c^*)p^* D_2\phi^A(x^{A*})}{(c^* - p^{*2})D_2\phi^A(x^{A*}) + (1 + c^*)p^* D_1\phi^A(x^{A*})}. \tag{52}$$

If, as shown in Figure 6, the slope of the path followed by $\hat{x}$ is at x^* smaller than the slope of B's offer curve, then around the Walrasian allocation, for any given level of utility u^A close to $u^{A*} = u^A(x^{A*})$, agent B attains on B's offer curve a higher utility than on A's, and hence around (u^{A*}, u^{B*}) the curve f^B is above the curve f^A, as shown in Figure 1. And conversely if, on the contrary, the path followed by $\hat{x}$ had at x^* a slope bigger than B's offer curve. Only in the case in which the path followed by $\hat{x}$ had at x^* a slope equal to that of B's offer curve, i.e. only if

$$-\frac{D_1\phi^B(x^{B*})}{D_2\phi^B(x^{B*})} = \frac{(1 - c^* p^{*2})D_1\phi^A(x^{A*}) - (1 + c^*)p^* D_2\phi^A(x^{A*})}{(c^* - p^{*2})D_2\phi^A(x^{A*}) + (1 + c^*)p^* D_1\phi^A(x^{A*})} \tag{53}$$

could a crossing of f^A and f^B occur at (u^{A*}, u^{B*}). But the equation (53) imposes a constraint on the partial derivatives of order two of the utility functions at the Walrasian allocation x^* that is degenerate in the space of utility functions[29] with respect to any topology of C^n uniform convergence on compacts, for $n \geq 2$.[30] Q.E.D.

Proposition 2. *For any economy $\{u^i, e^i\}_{i \in \{A,B\}}$ of a non-empty open set of economies satisfying A1, and for any discount factors δ^A, δ^B close enough to 1, there exists a SSP equilibrium of the bargaining over prices. Moreover, that equilibrium remains bounded away from efficiency as $\delta^A, \delta^B \to 1$.*

Proof. An immediate corollary of Lemma 1 is that, generically in the space of economies, there exists an inefficient intersection $(\hat{u}^A, \hat{u}^B)$ of the curves f^A and f^B. In effect, assume that every intersection of f^A and f^B is efficient. Then the behavior of f^A and f^B at the boundaries implies that at one of those efficient intersections these curves must cross, which by Lemma 1 above happens only for the complement

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[28] In words, $D\hat{\xi}^A(x^{A*})$ consists of the composition of (i) a change to an orthogonal basis containing the price vector $(p^*, 1)$, (ii) a jump across the first axis of that basis, and (iii) the undoing of the change of basis.

[29] The perturbation need not always be made in the space of utility functions. For instance, in the case of the symmetric Cobb-Douglas example we provide in Section 6 this condition is satisfied only for initial endowments on the anti-diagonal of the Edgeworth box, i.e. in a closed and nowhere dense subset of endowments space for the given Cobb-Douglas utility functions.

[30] Note that in the equation (53) appear equilibrium values of the endogenous variables. This may deserve a comment. In general, an equation to be fulfilled by equilibrium values needs not impose any constraint on the primitives of the economy, as for instance is the case for the Walras Law (as a condition on the primitives, the Walras Law happens to be an identity, and is hence satisfied whichever the primitives are). Notwithstanding, any equation in the equilibrium values of an economy actually *is* a condition on the primitives of the economy that *needs not* be an identity on the primitives either. In our particular case, equation (53) above *is not* satisfied by an open and dense set of primitives for the economy as this Lemma 1 establishes.

of an open and dense set of economies. By continuity, this inefficient intersection $(\hat{u}^A, \hat{u}^B)$ remains for discount factors δ^A and δ^B close to 1, and corresponds to two inefficient allocations $\hat{x}, \bar{x}$ that are SSP equilibrium allocations if both curves f^A and f^B are downward-sloped at $(\hat{u}^A, \hat{u}^B) = (u^A(\hat{x}^A), u^B(\hat{x}^B)) = (u^A(\bar{x}^A), u^B(\bar{x}^B))$.[31]

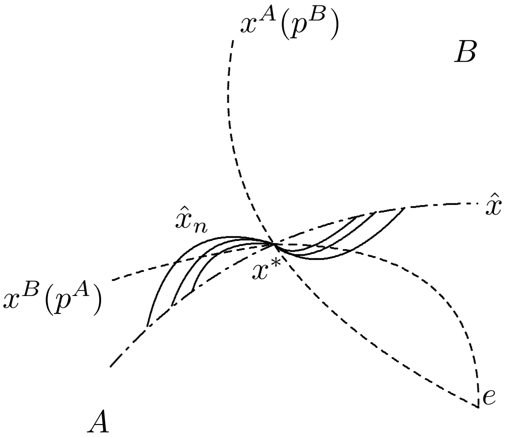
Note then that if in Figure 5 above the slope $\frac{d\hat{x}_2^A}{d\hat{x}_1^A}$ of the path followed by $\hat{x}$ in the system (47) is close to the slope $-\frac{D_1\phi^B(x^{B*})}{D_2\phi^B(x^{B*})}$ of B's offer curve at a Walrasian allocation x^* supported by prices p^*, then the slopes of f^A and f^B at $(\hat{u}^A, \hat{u}^B)$ would be close to their common negative slope at the Walrasian allocation profile (u^{A*}, u^{B*}), and hence they will be negative also. But from (52) in Lemma 1, $\big(-\frac{D_1\phi^A(x^{A*})}{D_2\phi^A(x^{A*})}, \frac{d\hat{x}_2^A}{d\hat{x}_1^A}(x_1^{A*})\big)$ is on the graph of

$$g(z) = \frac{(1 - c^* p^{*2})z + (1 + c^*)p^*}{(1 + c^*)p^* z - (c^* - p^{*2})} \tag{54}$$

(see Figure 9, for the case $c^* = 1$ and $1 < p^{*2}$, in the proof of Theorem 5 below). Hence $\frac{d\hat{x}_2^A}{d\hat{x}_1^A}(x_1^{A*})$ is close to $-\frac{D_1\phi^B(x^{B*})}{D_2\phi^B(x^{B*})}$ whenever the profile of offer curves slopes $\big(-\frac{D_1\phi^A(x^{A*})}{D_2\phi^A(x^{A*})}, -\frac{D_1\phi^B(x^{B*})}{D_2\phi^B(x^{B*})}\big)$ is close to the graph of g, which is satisfied by a non-empty open set of economies. Q.E.D.

Proof of Theorem 5. Given a Walrasian equilibrium allocation $x*$ of an economy $\{u^i, e^i\}_{i=A,B}$, consider a sequence $\{u_n^A\}_n$ such that (i) the corresponding sequence of paths followed by $\{\hat{x}_n\}_n$ as defined in Lemma 1 above converges pointwise to the path followed by $\hat{x}$ of $\{u^i, e^i\}_{i=A,B}$ around x^*, and (ii) all $\hat{x}_n$ have a common slope at x^* that reverses its order with respect to the slope of B's offer curve x^B at x^* so that each $\hat{x}_n$ intersects B's offer curve (see Figure 8).

Figure 8



The pointwise convergence of $\{\hat{x}_n\}$ to $\hat{x}$ guarantees the pointwise convergence within a compact of the associated offer curves[32] $\{x_n^A(p^B)\}$ to $x^A(p^B)$. Also the (piecewise)

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[31] Remember that the negative slope at $(u^A(\hat{x}^A), u^B(\hat{x}^B)) = (u^A(\bar{x}^A), u^B(\bar{x}^B))$ amounts to the non-existence of an allocation on the offer curve of either agent that Pareto-improves upon $\bar{x}$ and $\hat{x}$. Should any of the curves of profiles of utilities be upward-sloped, then a mutually beneficial counter-offer could be made to the agent whose curve of profiles of utilities along his offer curve is upward-sloped.

[32] Not depicted in Figure 7 for the sake of readability.

monotone and pointwise convergence of $\left\{x_n^A(p^B)\right\}$ within a compact guarantees that their convergence to $x^A(p^B)$ is uniform indeed. As a consequence, the utility functions u_n^A generating these offer curves $x_n^A(p^B)$ converge in the topology of C^1 convergence on compacts towards the utility function u^A that generates the offer curve $x^A(p^B)$.

For such a sequence $\{u_n^A\}_n$ to exist, it suffices that the slope $\frac{d\hat{x}_2^A}{d\hat{x}_1^A}(x_1^{A*})$ of $\hat{x}$ at x^* (the right-hand side of (55) below)[33] may be made equal to the slope of B's offer curve at x^* (the left-hand side of (55) below) without inducing new Walrasian equilibria, i.e.

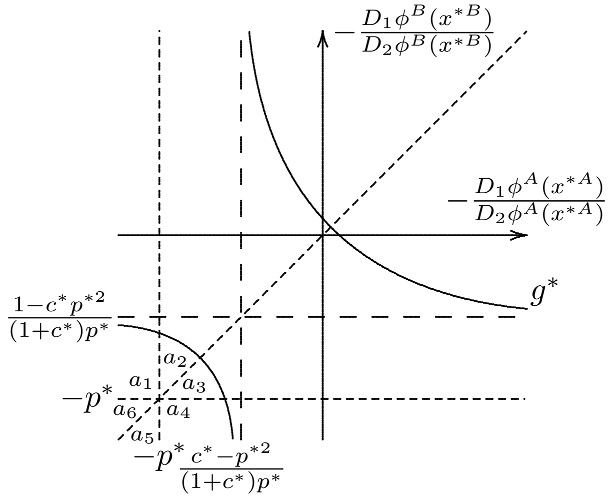
$$-\frac{D_1\phi^B(x^{B*})}{D_2\phi^B(x^{B*})} = \frac{(1-c^*p^{*2})D_1\phi^A(x^{A*}) - (1+c^*)p^*D_2\phi^A(x^{A*})}{(1+c^*)p^*D_1\phi^A(x^{A*}) + (c^*-p^{*2})D_2\phi^A(x^{A*})}. \tag{55}$$

This is possible because the slope of A's offer curve can be perturbed as little as required in the C^1 topology for u^A in such a way that the pair of offer curves slopes at the Walrasian allocation $\left(-\frac{D_1\phi^A(x^{A*})}{D_2\phi^A(x^{A*})}, -\frac{D_1\phi^B(x^{B*})}{D_2\phi^B(x^{B*})}\right)$ is on the graph of

$$g(z) = \frac{(1-c^*p^{*2})z + (1+c^*)p^*}{(1+c^*)p^*z - (c^*-p^{*2})} \tag{56}$$

in Figure 9 below (for the case $c^* = 1$ and $1 < p^{*2}$)[34] without ever crossing the boundaries (in short dashes) between the regions a_i, $i = 1, \ldots, 4$, which would imply new crossings of the offer curves that would correspond to new Walrasian equilibria (excluding a_5 and a_6 where A's and B's demand are simultaneously upward-sloped for both goods).[35]

Figure 9



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[33] An injective function of the slope of A's offer curve at x^* with range of $\mathbb{R} \setminus \{\frac{1-c^*p^{*2}}{(1+c^*)p^*}\}$.

[34] The relevant property is that, since for any $c^*, p^* > 0$ it holds true that $-p^* < \frac{c^*-p^{*2}}{(1+c^*)p^*}$ and $-p^* < \frac{1-c^*p^{*2}}{(1+c^*)p^*}$ always, the asymptotes of g^* (and hence g^* itself) intersect every region a_i in Figure 8, except for a_5, a_6 see footnote 33 about these excluded cases.

[35] That is to say, in a_5, a_6, for $i = A, B$, it holds $\frac{dx_1^i}{dp} > 0$ and $\frac{dx_2^i}{dp^{-1}} > 0$ simultaneously for some range of prices. We think of this case in which demand increases for the good that is becoming more expensive and decreases for the good that is becoming cheaper as a non-observed pathological case. Note that this does not prevent backward-bending offer curves, and hence that any good may be inferior for some range of prices. It just excludes the possibility of both goods being inferior for the *same* range of prices.

Far enough in the sequence $\{\hat{x}_n\}$, A's marginal rate of substitution at the intersection of $\hat{x}_n$ with B's offer curve is close to p^*, and hence not bigger than the slope of B's offer curve at x^*. By continuity, the same is true for δ^A and δ^B close to 1. This guarantees that this intersection corresponds to a SSP equilibrium. Q.E.D.

Proof of Theorem 6. Note first that since a Walrasian allocation is efficient, then f^B is invertible around the profile of Walrasian utilities (u^{A*}, u^{B*}). Hence so is $\delta^A f^B$ around u^{A*} for δ^A close enough to 1.

For given δ^A, δ^B close to 1, should $(\delta^A f^B)^{-1}(u^{A*})$ be smaller (respectively bigger) than $\delta^B f^A(u^{A*})$, and $f^A(u^A) \leq (f^B)^{-1}(u^{A*})$ for every u^A close enough to u^{A*}, then there exist two other (respectively, no) intersections of $\delta^B f^A$ and $\delta^A f^B$.[36]

Now, clearly $\delta^B f^A(u^{A*}) = \delta^B u^{B*}$. As for $(\delta^A f^B)^{-1}(u^{A*})$, let $\tilde{f}^B(u^B, \delta^A) = \delta^A f^B(u^B)$. Linearizing $\tilde{f}^B$ around $(u^{B*}, 1)$ it follows that $(\delta^A f^B)^{-1}(u^{A*})$ is the level of utility u^B for B such that $0 \approx f^{B\prime}(u^{B*})(u^B - u^{B*}) + f^B(u^{B*})(\delta^A - 1)$ i.e.

$$(\delta^A f^B)^{-1}(u^{A*}) \approx u^{B*} + \frac{u^{A*}}{f^{B\prime}(u^{B*})}(1 - \delta^A). \tag{57}$$

Therefore $(\delta^A f^B)^{-1}(u^{A*}) < \delta^B f^A(u^{A*})$ holds for δ^A, δ^B smaller but close to 1 if, and only if,

$$u^{B*} + \frac{u^{A*}}{f^{B\prime}(u^{B*})}(1 - \delta^A) < \delta^B u^{B*} \tag{58}$$

i.e. if, and only if,

$$\frac{u^{B*}}{u^{A*}} < -\frac{1}{f^{B\prime}(u^{B*})} \frac{1 - \delta^A}{1 - \delta^B}. \tag{59}$$

Note that the range of values taken by $\frac{1-\delta^A}{1-\delta^B}$ in every neighborhood of $(\delta^A, \delta^B) = (1, 1)$ in $(0, 1) \times (0, 1)$ is $\mathbb{R}_{++}$. Therefore there always exist discount factors δ^A, δ^B arbitrarily close to 1 for which the condition (59) holds, as well as discount factors δ^A, δ^B arbitrarily close to 1 for which the reversed inequality holds. Since, generically, either $f^A(u^A) \leq (f^B)^{-1}(u^{A*})$ or $f^A(u^A) \geq (f^B)^{-1}(u^{A*})$ holds for all u^A close enough to u^{A*}, the conclusion follows.

Q.E.D.

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[36] Similarly $(\delta^B f_B)^{-1}(u^{A*}) > \delta^A f_A(u^{A*})$ (respectively $<$) along with $f_A(u^A) \geq f_B^{-1}(u^{A*})$ guarantees the existence of two other (respectively, no) intersections, for δ^A, δ^B close enough to 1.

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