A Dynamic Pricing Model under Duopoly Competition

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We introduce and analyze a multi-period, finite horizon, no-replenishment, dynamic pricing model under duopoly competition between two retailers who sell an identical good. According to our duopoly competition model, a single consumer, whose valuation is uniformly distributed on \([0, 1]\), visits only one of the two retailers in each period. If the price posted by the retailer visited by the consumer is below his valuation, he purchases the good. Otherwise, in the next period the consumer visits the other retailer. We show that, as compared to the corresponding single store monopoly model, under our duopoly competition: (i) prices decline exponentially rather than linearly, (ii) the initial price converges to 0.543, rather than 1, and (iii) the system profit loss due to competition may be up to 29.6% of the monopolistic profit. It is also shown that the pricing policy does not change when the model is extended to \(N\) similar consumers.

The model is further extended into a more general valuation distribution, into the capacitated case, both under similar and identical consumers, and into the centralized case.

**Key words**: Dynamic Pricing; Revenue Management; Duopoly Competition; Zigzag Competition; Markdowns

**History**:  

1. **Introduction**

Markdowns, which are used by retailers for various of reasons (Pashigian and Bowen, 1991), became so prevalent over the years that a new coin was termed in the apparel industry - “markdown money”\(^1\). Indeed, as Top of the Net (2002) reports that American retailers are losing more than $200 billion a year due to markdowns\(^2\) and 78% of all apparel that is currently sold at American

\(^1\)“Markdown money” is the amount stores subtract from suppliers’ checks because, they say, the clothes had to be marked down to sell them (New York Times, 2005ab).

\(^2\)An analysis of retail and economic trends by the US Census Bureau and the National Retail Federation (NRF), as cited by Top of the Net (2002).
The ubiquity of markdowns is also reflected from the New York Times (2002) article on pricing: “Like all retailers, Saks [an upscale fashion retailer] faces the problem of deciding when to start marking down prices, and at what prices, so the company can glean as much profit as it can, without discounting so early that the company sells out and alienates customers, or so late that the company looks like a museum for unwanted merchandise”. The question, then, is not whether to markdown, but when and how.

There is a vast literature which addresses markdowns, and, in general, dynamic pricing and revenue management, in a finite horizon, fixed inventory (i.e., no replenishment) setting. The various models that were introduced to analyze this setting differ mainly in the manner with which they model demand: consumers’ arrival and their valuations. In Lazear (1986) consumers are myopic and identical, all having a common valuation for the product which is uniformly distributed; Besanko and Winston (1990) have considered the case of N strategic, non-identical consumers, whose valuations are uniformly distributed; in Gallego and van Ryzin (1994) a price-dependent stochastic demand is modelled as a Poisson process; Bitran and Mondschein (1997) have extended Gallego and van Ryzin’s model to the case where customers have heterogeneous valuations, and Zhao and Zheng (2000) have studied a more general model where both the intensity of the customer arrival process and the reservation price distribution are time dependent. For recent extensive reviews of pricing models, the reader is referred, e.g., to Elmaghraby and Keskinocak (2003), Bitran and Caldentey’s (2003), and Chan et al. (2004).

Competition has a major effect on pricing. Indeed, as Bitran and Caldentey (2003) conclude: “Including market competition is another important extension to the model. [...] price competition among retailers is today the main driver in their selection of a particular pricing policy”. Similarly, Elmaghraby and Keskinocak (2003) have acknowledged that in the NR-I\(^4\) markets literature, competition is one of the missing bricks, and Chan et al. (2004) have admitted that “The reality is that most companies are not monopolies, and many specifically compete on product prices or service differentiation, so these effects need to be considered when establishing coordinated pricing and production policies”.

Nevertheless, in spite of the recognition that competition has a significant effect on prices, until quite recently, the effect of competition in the context of revenue management, where inventory

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\(^3\) STS market research study, as cited by Top of the Net (2002).

\(^4\) No Replenishment (NR) of inventory with Independent (I) demand over time, where consumers are either Myopic (M) or Strategic (S).
is fixed, has received relatively little attention.\textsuperscript{5} Recent papers addressing competition include Perakis and Sood (2006), who employ a robust optimization approach to study a multi-period, fixed inventory, dynamic pricing model under competition; Gallego and Hu (2006) who extend Gallego and van Ryzin (1994) to oligopolistic competition by modifying the demand intensity (non-homogeneous Poisson process) to depend on the prices set by all players; In Talluri (2003), prices are pre-determined but the competing retailers choose to offer only a subset of pre-designed products in each period; Lin and Sidbari (2004) consider a consumer choice model to study a competition between firms who sell substitutable products and face a consumer arrival which follows a Bernoulli process; Friesz et al. (2005) study a competitive model among service providers on a network which incorporates demand dynamics (differential evolutionary game); In Xu and Hopp (2004) retailers set inventory levels in the first stage and compete on prices in the second stage. In their model consumers are homogeneous and their arrival is either piecewise deterministic or follows a Brownian motion; Levin et al. (2006) consider an oligopolistic competition over a finite population of strategic consumers having valuations which follow a random utility model for differentiated products. Our approach to evaluate the effect of competition is different, as elaborated below, and the results and insights we derive are substantial and new.

Our main objective in this paper is to incorporate competition in a finite, multi-period, fixed inventory dynamic pricing setting, and to evaluate the effect of competition on prices and profits. In our base model, two profit-maximizing retailers, each of whom can satisfy the entire market, compete over prices, and face a single consumer whose valuation is uniformly distributed over \([0,1]\).

The myopic\textsuperscript{6} consumer, who does not have full price information, has to visit the retail stores in order to observe the prices\textsuperscript{7}. We neglect search costs, but assume that due to these costs the consumer is limited to a single store visit per period. At the store the consumer observes the posted price and if it is below his valuation he purchases the good. Otherwise, he leaves the store and

\textsuperscript{5}We note, however, that, starting with Bertrand and Edgeworth, price competition has been studied extensively in the economics literature (see, e.g., Tirole (1988), Shapiro (1989), and chapter 8 in Talluri and van Ryzin (2004) and the references therein), in the marketing literature (see, e.g., Moorthy (1993) and Eliashberg and Chatterjee (1985) and also in the International Journal of Research in Marketing’s special issue on competition and marketing (2001)), and in operations (see, e.g., Talluri and van Ryzin (2004), Bitran and Caldentey (2003), and Chan et al. (2004)).

\textsuperscript{6}One may suggest that due to the popularity of markdowns, consumers learn to expect them, and may also postpone their purchasing decisions even if the posted price is below their valuations. However, it is commonly assumed in the literature that consumers are myopic and buy the good as soon as the price drops below their valuations, since, as Bitran and Mondschein (1997) note, consumers “have partial information about inventories, which prevent them, to some extent, from acting strategically.” Thus, the ubiquity of markdowns coupled with the assumption that consumers are myopic imply that if the price tag exceeds a consumer’s valuation, he will return to check the price in the market in the next period.

\textsuperscript{7}This is a common assumption in the literature. Bitran and Mondschein (1997), for example, state that “customers [...] have little information about current prices before going to the store”.
in the following period visits the other retailer.\textsuperscript{8} Thus, the consumer keeps visiting the stores in a zigzag manner to check if prices have dropped below his valuation, at which point he purchases the good. We refer to this type of competition as zigzag competition. Such competition exposes the two retailers to the difficulties of pricing correctly in a competitive market. If the price is set too low, potential profit may be lost, while if it is set too high, the product may not be sold.

We first demonstrate in this paper that in our base model, wherein there is a single consumer, the effect of zigzag competition is quite dramatic. Indeed, not only are prices strictly decreasing, as in Lazear\textsuperscript{9}, but they decrease exponentially rather than linearly. Moreover, the initial price is increasing in the selling horizon and converges very fast to 0.543, as compared to 1 in Lazear’s model, and every period, except the last two, it decreases by a factor of almost 0.543. Thus, within very few periods the price in our competition model drops nearly to zero, which may explain the short life cycle of some products in competitive markets. The profit loss of the system due to the introduction of zigzag competition is shown to converge to 0.147, which represents a loss of about 29.6\% of the profit, and the profit loss of a monopolist due to the introduction of such competition is shown to converge to 64.78\% of his pre-competition profit. Finally, we find that all the results still hold when there are $N$ similar\textsuperscript{10} consumers.

To investigate the robustness of our results, we briefly analyze the effect of zigzag competition when some of our assumptions are modified or extended. More specifically, we consider the following cases: (i) the valuation of the good by consumers has a power distribution on $[0, 1]$, (ii) the system is capacitated in the sense that the retailers cannot satisfy the entire demand in the market, and (iii) consumers are identical rather than similar, i.e., all consumers have the same valuation for the product which is uniformly distributed. We note that the proper evaluation of cases (i) and (ii) has necessitated the analysis of corresponding monopoly case.

The analysis of the effect of zigzag competition with a power distribution has revealed that the nature of the exponential decline of prices depends on the parameter, $q$, $-1 < q \leq \infty$, of the distribution, which for $q = 0$, coincides with the uniform distribution. Specifically, the decline of prices is most severe when $q \rightarrow -1$, and it is most moderate, yet exponential when $T \rightarrow \infty$, when

\textsuperscript{8}In an ensuing paper (Granot, Granot, and Mantin, 2006) we analyze the considerably more complex case, wherein the consumer may return to the same store with some arbitrary probability (see §3.3). This return probability can serve as a proxy for the intensity of competition in the market. If it is one, there is no competition and the model is reduced to two local monopolists, while if it is zero, as is the case in the zigzag competition which is considered in this paper, then there is an intense competition in the market.

\textsuperscript{9}Since Lazear’s model can be perceived as a monopoly facing a single consumer over $T$ periods, we compare our results with those obtained by Lazear.

\textsuperscript{10}That is, the consumers’ valuations are independently drawn from the same uniform distribution.
granot, granot, and mantin: a dynamic pricing model under duopoly competition
article submitted to management science; manuscript no. (please, provide the manuscript number!)

\[ q \to \infty \], i.e., when the valuations by the consumers are very likely to be very close to one. The
analysis of the capacitated case, when the retailers cannot satisfy the entire demand, demonstrates
that zigzag competition still has a major impact on prices and profits. However, the decline in
prices and profits become more moderate when the system is more constraint, i.e., the retailers
can supply a small fraction of the market, or when the number of periods increases.

Prices in our zigzag competition model with identical consumers are shown to decline about as
much as they do with similar consumers. Yet, a quantitative difference between these two cases
is uncovered. Specifically, while there is a unique pure-strategy SPNE for prices in the similar
consumers case, when consumers are identical, there are multiple, and, even in some cases a contin-
umum of pure-strategy SPNE. In particular, it is shown that for the two-period case with identical
consumers there are precisely two pure-strategy SPNE in prices, both of which are non-symmetric.

The analysis of the capacitated case with identical consumers has revealed that in some instances,
when the system is very constrained, the profit of the duopoly system may exceed the profit of the
single store monopoly. However, by definition, this profit is lower than the profit of a centralized
two-store monopoly, which is the right reference for comparison, and, in return, is subsequently
studied in the paper.

The two-store monopoly is similar to the duopoly competition setting with one main difference:
the chain is controlled by a central planner who is setting possibly different prices in both retail
stores simultaneously. We find that the profit of the two-store monopoly could be strictly higher
than the one-store monopoly when the consumers are identical. Indeed, it is shown that for a fixed
number of consumers, the profit of the centralized system is increasing in the number of consumers
who encounter the higher price, as long as the number of consumers who encounter the lower
price is not zero. By contrast, it is shown that when the consumers are not identical, a centralized
two-store monopoly system does not have any advantage over the single-store monopoly.

Our main contribution in this paper, aside for our model of zigzag competition, is our ability
to evaluate and quantify the effect of competition on prices and profits. Specifically, we show that
zigzag competition exerts significant pressure on prices, resulting with an exponentially declining
price trajectory, and that its introduction reduces the monopolist profit quite dramatically. Further,
the effect of such competition is shown to be robust in the sense that it essentially holds (i) for
a large class of product valuation distribution, (ii) when the retailers cannot satisfy the entire
market demand, and (iii) for both identical and similar consumers.

The rest of the paper is arranged as follows. In Section 2 we introduce and analyze the basic zigzag
competition model, first with a single consumer and, subsequently, with \( N \) similar consumers.
Section 3 considers several extensions to the model: a more general distribution of valuation, capacitated system, and briefly discusses a more general visit pattern of consumers between stores. In Section 4 we study the behavior of the model when consumers are identical rather than similar under both the capacitated and uncapacitated cases. Section 5 investigates the case of a two-store monopoly and Section 6 concludes with a brief summary and some discussion on future work.

2. The Model
2.1. The Basic Model

Two competing retailers, each of whom has in stock only one unit of an identical good, encounter a single myopic consumer whose valuation for that good is $V$. That is, the consumer is willing to pay up to $V$ for the good, but no more, and he purchases the good once he observes a price which is below his valuation. The retailers do not know $V$ with certainty, but they do have prior knowledge of the density of $V$, denoted $f(V)$, with distribution function $F(V)$. The two retailers are risk-neutral and seek to maximize their expected profits. They have $T$ periods to sell their good to the consumer, and in each of these periods they can post a new price.

The consumer does not have any knowledge about the posted prices in the market in each period. To obtain that information he needs to visit each retail store and observe the posted price. We neglect search costs, but assume that the consumer is limited to a single retail store visit per period.

We introduce in this paper a new type of competition between the retailers to be referred to as zigzag competition. In zigzag competition, the retailers compete to sell their product to the single consumer who visits only one store per period. In period $t$ he observes a price $R_t$ at the retailer he visits. If this price is below his valuation, he purchases the good. Otherwise he does not purchase the good, and in the following period he visits the other retailer with certainty. Figure 1 demonstrates the zigzagging pattern of the consumer across the two retail stores.

We refer to the retailer who encounters (resp., does not encounter) the consumer in the first period as Retailer 1 (resp., 2). Thus, in an odd (resp., even) period, if no purchase has occurred before that period, the consumer visits Retailer 1 (resp., 2).

In each period the price posted at the retail store not visited by the consumer is not relevant, and no assumption is needed for a retailer’s knowledge about the other retailer’s posted prices. However, since both retailers solve the same model, they can compute the prices that the other

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11 Switching behavior of consumers has been also considered in the operations literature, e.g., by Zhao and Atkins (2003), Netessine and Shumsky (2005), and in Netessine et al. (2006). However, in their models switching from one retailer to another may occur due to stockouts, i.e., a consumer might switch to a competing retailer only if the demanded product at the visited location is out of stock.
Figure 1  The zigzagging pattern and the observed prices

retailer is posting. If the good is not sold in period $t$ at price $R_t$, the retailers can immediately infer that $V < R_t$.

We denote Retailer 1’s (resp., Retailer 2’s) expected profit-to-go function at period $t$ as $\pi_1^t$ (resp., $\pi_2^t$). If $t$ is odd, Retailer 1 is visited by the consumer in that period, and we have:

$$\pi_1^t = R_t[1 - F_t(R_t)] + R_{t+2}[1 - F_{t+2}(R_{t+2})]F_{t+1}(R_{t+1})F_t(R_t) + \cdots$$

$$= \sum_{i=0}^{\lfloor \frac{T-t-1}{2} \rfloor} \left( R_{t+2i}[1 - F_{t+2i}(R_{t+2i})] \prod_{j=t}^{t+2i-1} F_j(R_j) \right),$$

where the first term in $\pi_1^t$ is the expected profit from a sale in period $t$. It is equal to the price charged in period $t$ times the probability that the good sells in period $t$, given the information from period $t - 1$; the second term is the conditional expected profit from a sale in period $t + 2$, and so on.

Similarly, the expected profit-to-go of Retailer 2, who does not encounter the consumer in that period (since $t$ is odd), can be written as:

$$\pi_2^t = R_{t+1}[1 - F_{t+1}(R_{t+1})]F_t(R_t) + R_{t+3}[1 - F_{t+3}(R_{t+3})]F_{t+2}(R_{t+2})F_{t+1}(R_{t+1})F_t(R_t) + \cdots$$

$$= \sum_{i=0}^{\lfloor \frac{T-t-1}{2} \rfloor} \left( R_{t+2i+1}[1 - F_{t+2i+1}(R_{t+2i+1})] \prod_{j=t}^{t+2i} F_j(R_j) \right).$$

The profit-to-go functions from (1) and (2) can be expressed in a recursive form. If $t$ is odd,

$$\pi_1^t = R_t[1 - F_t(R_t)] + \pi_{t+1}^1 F_t(R_t) \quad \text{and} \quad \pi_2^t = \pi_{t+1}^2 F_t(R_t).$$

Thus, Retailer 1 sets prices, at odd periods, so as to:

$$\text{Max} \quad \pi_1^t = \text{Max}_{R_t} \left( R_t[1 - F_t(R_t)] + \pi_{t+1}^1 F_t(R_t) \right),$$
and similarly, at even periods, Retailer 2 solves:

$$\text{Max } \pi_t^2 = \text{Max } \frac{R_t}{1 - F_2(R_t)} = \pi_{t+1}^2 F_2(R_t).$$

## 2.2. Uniformly Distributed Valuation

For simplicity, suppose first that the prior on the valuation \( V \) is uniformly distributed between zero and one. By Bayes’ Theorem, this implies that the posterior distribution carried from period \( t \) to \( t + 1 \) is uniform between zero and \( R_t \), so that \( F_{t+1}(V) = \frac{V}{R_t} \).

To illustrate, we solve the problem, backwards, for the last four periods of the zigzag competition. In Table 1 we display the retail prices for both the zigzag competition model and the monopoly model for \( T = 4 \). The price ratio at period \( t \) is \( \frac{R_t}{R_{t-1}} \). Observe from Table 1 that, under zigzag competition, prices decline much more rapidly and the initial price is much lower compared to the monopoly model.

<table>
<thead>
<tr>
<th>Period</th>
<th>Price Ratio</th>
<th>Retailer 1’s Price</th>
<th>Retailer 2’s Price</th>
<th>( \pi_t^1 )</th>
<th>( \pi_t^2 )</th>
<th>Price Ratio</th>
<th>Price</th>
<th>( \pi_t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{R_t}{1 - S_{t-1}} )</td>
<td>( R_t \approx 0.538 )</td>
<td>Not observed</td>
<td>( \approx 0.269 )</td>
<td>( \approx 0.077 )</td>
<td>( \frac{2}{7} )</td>
<td>0.8</td>
<td>0.4</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{R_t}{1 - S_{t-2}} )</td>
<td>Not observed</td>
<td>( R_2 \approx 0.287 )</td>
<td>( \approx 0.011R_2 )</td>
<td>( \approx 0.266R_1 )</td>
<td>( \frac{2}{7} )</td>
<td>0.6</td>
<td>0.375R_1</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{R_t}{1 - S_{t-3}} )</td>
<td>( R_3 \approx 0.143 )</td>
<td>Not observed</td>
<td>( 0.25R_2 )</td>
<td>( 0.0625R_3 )</td>
<td>( \frac{2}{7} )</td>
<td>0.4</td>
<td>0.33R_2</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{R_t}{1 - S_{t-4}} )</td>
<td>Not observed</td>
<td>( R_4 \approx 0.0717 )</td>
<td>( 0 )</td>
<td>( 0.25 )</td>
<td>( \frac{2}{7} )</td>
<td>0.2</td>
<td>0.25R_3</td>
</tr>
</tbody>
</table>

**Table 1** Price ratios, prices, and profits under zigzag competition and monopoly settings

### Lemma 1

The profit-to-go in each period is an affine function of the price that is set in the previous period. That is, \( \pi_t^1 = S_t R_{t-1}, \) \( i = 1, 2, \) where \( S_t \) are scalars. Moreover, at odd (resp., even) periods, the values of \( S_t^1 \) and \( S_t^2 \) can be expressed recursively as follows: \( S_t^1 = \frac{1}{4(1 - S_{t+1})^2} \) (resp., \( S_t^1 = \frac{S_{t+1}}{4(1 - S_{t+1})^2} \)), and \( S_t^2 = \frac{S_{t+1}}{4(1 - S_{t+1})^2} \) (resp., \( S_t^2 = \frac{1}{4(1 - S_{t+1})^2} \)).

**Proof.** The proof is by induction. From Table 1 it can be verified that the recursions hold for the final periods. Assume the Lemma holds for \( t + 1 \), and we will show that it holds for an odd \( t \). Now, Retailer 1’s profit-to-go is \( \pi_t^1 = R_t(1 - \frac{R_t}{R_{t-1}}) + \pi_{t+1}^1 \frac{R_t}{R_{t-1}} = R_t(1 - \frac{R_t}{R_{t-1}}) + S_t^1 R_{t-1} \frac{R_t}{R_{t-1}} \), where the last equality follows from the induction hypothesis. Setting the derivative of \( \pi_t^1 \) with respect to \( R_t \) to zero results with

$$R_t = \frac{1}{2(1 - S_{t+1})} R_{t-1}. \tag{4}$$

Using (4) we rewrite \( \pi_t^1 \) and \( \pi_t^2 \) to obtain: \( \pi_t^1 = \frac{S_{t+1}}{4(1 - S_{t+1})} R_{t-1} \) and \( \pi_t^2 = \frac{S_{t+1}^2}{4(1 - S_{t+1})^2} R_{t-1}. \) Thus,

$$S_t^1 = \frac{1}{4(1 - S_{t+1})}, \quad \text{and} \quad S_t^2 = \frac{S_{t+1}^2}{4(1 - S_{t+1})^2}. \tag{5}$$
and due to the induction hypothesis $S^1_t$ and $S^2_t$ are scalars as well. The proof for an even $t$ follows in a similar way. $\Box$

Note that the scalars $S^1_t$ and $S^2_t$ have a dimension of sales, and will thus be referred to as the normalized sales-to-go in period $t$ of Retailers 1 and 2, respectively, with respect to the price that is set in the previous period, $R_{t-1}$.

Let $\{s^1_t\}$ (resp., $\{s^2_t\}$) denote the sequence of normalized sales-to-go of the retailer who encounters (resp., does not encounter) the consumer in period $t$. That is, for an odd $t$, $s^1_t = S^1_t = \frac{1}{4(1-s^2_{t+1})^2}$ (resp., $s^2_t = S^2_t = \frac{s^1_t}{4(1-s^2_{t+1})}$), and for an even $t$, $s^1_t = S^2_t$ (resp., $s^2_t = S^1_t$). Thus, for all $t$,

$$s^1_t = \frac{1}{4(1-s^2_{t+1})} \quad \text{and} \quad s^2_t = \frac{s^1_{t+1}}{4(1-s^2_{t+1})^2}. \quad (6)$$

Note that in both $\{s^1_t\}$ and $\{s^2_t\}$ sequences, as well as all other sequences considered in the sequel, the sequences progress from $t=T$ to $t=1$. Thus, e.g., the first element in $\{s^1_t\}$ is $s^1_T$ and the last element is $s^1_1$.

**Lemma 2.** The normalized sales-to-go sequences $\{s^1_t\}$ and $\{s^2_t\}$ are bounded sequences. Specifically, $\{s^1_t\}$ (resp., $\{s^2_t\}$) is bounded from below by $s^1_T = \frac{1}{4}$ (resp., $s^2_T = 0$) and from above by $\overline{s^1_T} = \frac{3}{2} \left( \frac{19+3\sqrt{33}}{6} \right) \frac{1}{2} \approx 0.271$ (resp., $\overline{s^2_T} = \frac{5}{6} - \frac{1}{6} (19+3\sqrt{33}) \frac{1}{2} - \frac{3}{2} \frac{1}{(19+3\sqrt{33})^2} \approx 0.0803$).

**Proof.** The proof is by induction. The induction base is satisfied for the last two periods as $s^1_T = \frac{1}{4}$, $s^2_T = 0$, and $s^2_{T-1} = \frac{1}{10}$. Suppose that $\frac{1}{4} \leq s^1_{t+1} < s^1_t$, $0 \leq s^2_{t+1} \leq s^2_t$ for some $t$. As $s^1_t = \frac{1}{4(1-s^2_{t+1})}$, the lowest value for $s^1_t$ is obtained if the lower bound value of $s^2_t$ is used. Thus, $s^1_t = \frac{1}{4(1-s^2_{t+1})} \geq \frac{1}{4(1-0)} = \frac{1}{4} = s^1_1$, where the last inequality is satisfied by the induction hypothesis. The largest value for $s^1_t$ is obtained if the upper bound value of the sequence of $s^2_t$ is used. We have $s^1_t = \frac{1}{4(1-s^2_{t+1})} \leq \frac{1}{4(1-s^2_T)}$, as follows from the induction hypothesis. Thus, $s^1_1 \leq \frac{3}{2} \left( \frac{19+3\sqrt{33}}{6} \frac{1}{2} \right)$.

As required, we have shown that $s^1_1 \leq s^1_t \leq s^1_T$. The proof for the boundedness of $\{s^2_t\}$ follows in a similar way. $\Box$

**Proposition 1.** The sequences of normalized sales-to-go, $\{s^1_t\}$ and $\{s^2_t\}$, are monotone convergent sequences as the number of periods, $T$, goes to infinity, while $t$ goes to 1.

**Proof.** The proof is based on the Monotone Convergence Theorem. The two sequences of the normalized sales, $\{s^1_t\}$ and $\{s^2_t\}$, are bounded from above as follows from Lemma 2. Next, we show that the sequences are increasing.

12 These bounds are obtained by omitting the subscripts from (6) and solving for $s^1$ and $s^2$. 


Proposition 2. The normalized sales-to-go of the retailer who encounters the consumer in period $t$ is strictly larger than the normalized sales-to-go of the retailer who does not encounter the consumer in that period.\footnote{Proofs which are not presented in the body of the paper are provided in the appendix.}

Proposition 3. The normalized sales-to-go of a monopolist is at least as large as the combined normalized sales-to-go of both retailers in the zigzag competition model.

Note that for $t = 1$ the normalized sales-to-go of the retailers coincide with their expected profits. Therefore, Proposition 3 implies:

Corollary 1. A monopolist’s profit is at least as large as the combined profits of both retailers in the zigzag competition model.

Let us next consider the sequence of price ratios, $\{B_t\}$, defined as

$$B_t \equiv \frac{R_t}{R_{t-1}} = \frac{1}{2(1-s_{t+1}^2)}.$$  \hspace{1cm} (7)

From (6) and (7) one can observe that $s_t^1 = \frac{1}{2} B_t$ and $s_t^2 = 4s_{t+1}^1(s_t^1)^2$. Thus, we find that

$$B_t = \frac{1}{2 - (B_{t+1})^2 B_{t+2}}.$$  \hspace{1cm} (8)

Proposition 4. The sequence of price ratios, $\{B_t\}$, is increasing and each element therein is bounded from below by 0.5 and from above by\footnote{In Proposition 5 we show that $\{B_t\}$ converges to 0.543 and thus each element in the sequence is bounded from above by the convergence value.} 0.55.

From Proposition 4 we obtain:

Theorem 1. Under zigzag competition the prices decrease exponentially.

Proof. For every $t$, $B_t = \frac{R_t}{R_{t-1}} \leq B_1 \leq 0.55$. Thus, since $R_0 = 1$, $R_t \leq (0.55)^{t-1}$.

To illustrate the behavior of the model, consider an example with 20 periods. The price ratios and the prices that are set under zigzag competition in each of the 20 periods and the prices in the monopoly setting are plotted in Figure 2(a), whereas the profit-to-go of the retailers are plotted in Figure 2(b). We observe that:

$\{s_t^1\}$ is increasing if $\frac{s_t^1}{s_{t+1}^1} = \frac{1}{4s_{t+1}^1(1-s_{t+1}^1)} \geq 1$, which certainly holds (by Lemma 2) as $0 < 4s_{t+1}^1(1-s_{t+1}^1) \leq 1$. The proof that $\{s_t^2\}$ is increasing is by induction. We have that $s_{t+1}^2/s_{t+2}^2 = \frac{s_{t+1}^1}{s_{t+2}^1} \geq 1$, which certainly holds (by Lemma 2) as $0 < 4s_{t+1}^1(1-s_{t+1}^1) \leq 1$.
• The price ratio under the zigzag competition does not change much over most of the selling season. As it is solved recursively, it slightly increases from a value of 0.5 at $T = 20$ to a value of about 0.543 at $t = 1$, which is dramatically lower than the initial price set in the corresponding monopoly model, $\frac{20}{21}$. In other words, the fear of losing a sale due to competition significantly suppresses the initial price. Since the price ratio is equal to the initial price had that period been the first period of the selling horizon, we conclude that the initial price is decreasing with perishability (measured by the length of the selling horizon), though only marginally.

• As proven by Theorem 1, the prices that are set over the selling horizon under zigzag competition are decreasing exponentially. They approach zero fairly quickly. By contrast, in a monopoly market, prices decline linearly.

• The profit-to-go of Retailers 1 and 2 also decrease exponentially with some spikes. As the price declines quickly, the conditional expected valuation of the consumer declines as well, and there is less profit to be made. Thus, the largest part of the profit is achieved in the early periods, and later periods of the selling season contribute very marginally to the total profit. The spikes occur since a retailer who does not encounter the consumer in period $t$ will encounter him only with some probability in period $t + 1$.

![Prices and price ratio](image1.png)
![Retailers' profit-to-go](image2.png)

Figure 2 A 20-period model: price, price ratio and profit-to-go

To illustrate the last point, let us examine the retailers expected profit over a range of selling horizons, as demonstrated in Figure 3(a), wherein the selling horizon, $T$, ranges from $T = 1$ up to $T = 50$. Observe that Retailer 1’s expected profit equals 0.25 in a single and two-period horizons and then converges quickly to about 0.271. Similarly, Retailer 2’s expected profit equals zero in a single period model (as he does not encounter the consumer at all), rises to 0.0625 in a two-period horizon and then converges quickly to about 0.08. Thus, we may conclude that longer selling
horizons do not contribute much to the expected profit of each of the retailers. Consequently, the system’s profit converges quickly to about 0.35. It can be observed from Figure 3(b) that the monopolist’s profit \( \frac{T}{2(T+1)} \) converges slower but to a higher value. Thus, the larger the value of \( T \) is, for \( T > 1 \), the larger is the profit advantage of a monopoly over the the zigzag competitors.

Consumer’s surplus measures the difference between the consumer’s valuation and the price charged, if the consumer purchased the good. Thus, under both settings, the consumer’s surplus equals \( \sum_{t=1}^{T} \frac{1}{2}(R_{t-1} - R_{t})^2 \). Figure 3(c) contrasts the consumer’s surplus under zigzag competition and under monopoly. With longer horizons, the monopolist can more efficiently price the good and extract more of the surplus from the consumer and, therefore, the consumer’s surplus in the monopoly setting is decreasing in \( T \). However, in the zigzag competition the two competing retailers are mainly engaged in price cutting competition and, therefore, the consumer’s surplus first increases from 0.125 to 0.156, and then it slightly decreases (since the initial price also marginally increases) and quickly converges to 0.147.

![Figure 3](image-url)  
(a)Retailers’ profits  
(b)System’s profit and profit loss  
(c)Consumer’s surplus  
Figure 3  
Profit, profit loss, and consumer surplus in monopoly and duopoly under zigzag competition

### 2.3. Values at Convergence

We first provide sharp bounds for the initial price.

**Proposition 5.** The initial price, \( R_1 \), is bounded from below by \( \frac{1}{2} \) (if \( T=1 \) or 2) and monotonically converges to \( \frac{1}{3} \left( (17 + 3\sqrt{33})^{\frac{1}{3}} - \frac{2}{3} \right) \frac{1}{3} = 0.543 \) as \( T \to \infty \).

Recall from Theorem 1 that under zigzag competition prices decrease exponentially. From Proposition 5 the price ratio converges to 0.543. Thus, the exponential decline of prices is further bounded as follows: \( R_t \leq (0.543)^{t-1} \).
Proposition 6. The expected profit of Retailer 1 (resp., 2) is increasing in $T$. Further, it is bounded from below by $\frac{1}{4}$ (resp., 0) if $T = 1$, and monotonically converges to about 0.2718 (resp., 0.0803), as $T \to \infty$.

To measure the effect of the zigzag competition we compare the system’s profit under zigzag competition and under monopoly. The monopolist’s profit is increasing in $T$, is bounded from below by $\frac{1}{4}$ when $T = 1$ and converges to $\frac{1}{2}$ as $T \to \infty$. Under zigzag competition, the system’s profit is increasing in $T$, is bounded from below by $\frac{1}{4}$ and converges to about 0.3522. Thus, the system’s profit loss due to zigzag competition, at convergence, is $\frac{1}{2} - 0.3522 \approx 0.1477$.

Proposition 7. The system’s profit loss due to the zigzag competition is bounded from below by zero (when $T = 1$), and it converges to about 0.1477 (when $T \to \infty$), which represents a profit loss of about 29.6%.

If we assume that in the zigzag competition model both retailers have equal probabilities to be visited by the consumer at the first period then we have:

Proposition 8. The profit loss of a monopolist retailer due to the introduction of zigzag competition with a second retailer is bounded from below by $\frac{1}{8}$ when $T = 1$ and converges to about 0.3239, which represents a loss of about 64.78% of the profit.

2.4. $N$ Consumers

In this subsection we extend the model to $N$ similar consumers, whose valuations are drawn independently from a uniform distribution on [0,1], and each of the retailers can satisfy the entire demand in the market, i.e., each has in stock $N$ units of the good. The consumers do not all necessarily zigzag as a single group. That is, in the first period $N^1$ (resp., $N^2$) consumers visit Retailer 1 (resp., 2), with $N^1 + N^2 = N$, and, as before, the consumers maintain the zigzagging search pattern. Let us denote by $N^i_t$, $i = 1, 2$, the number of consumers who visit Retailer $i$ in period $t$. Thus, $N^i_t \leq N^j_{t-1}$, $i \neq j = 1, 2$, with $N^i_t = N^i$. Let $\pi^i_{(N^i_t, N^j_t)}$ denote the profit-to-go of Retailer $i$, $i = 1, 2$, in period $t$ when there are $N^i_t + N^j_t$ consumers in the system in period $t$, and let $R^i_t$ denote the respective price in that period.

Since the consumers’ valuations may differ, it is possible that some consumers will make a purchase in a certain period while the others will continue to zigzag. Let $P_n^{N^i_t}$, $n = 0, ..., N^i_t$, $i = 1, 2$, denote the probability that Retailer $i$, who is visited by $N^i_t$ consumers, sells $n$ units in that period. Thus, in period $t$, $P_n^{N^i_t} = \frac{N^i_t!}{n!(N^i_t-n)!}[F_t(R^i_t)]^{N^i_t-n}[1 - F_t(R^i_t)]^n$. 


Proposition 9. The profit-to-go of Retailer $i$, $i = 1, 2$, in period $t$ when there are $N_i^t + N_j^t$ similar consumers in the system, such that $N_i^t$ (resp., $N_j^t$) of them visit Retailer $i$ (resp., $j$) in that period, can be expressed as $\pi^t_{(N_i^t, N_j^t)} = N_i^t \pi_1^t + N_j^t \pi_2^t$, $i \neq j = 1, 2$, if $t$ is odd, and $\pi^t_{(N_i^t, N_j^t)} = N_i^t \pi_2^t + N_j^t \pi_1^t$, $i \neq j = 1, 2$, if $t$ is even, where $\pi_1^t$ and $\pi_2^t$ are the profit-to-go functions of Retailers 1 and 2, respectively, in the model with a single consumer.

Thus, the resulting pricing policy is the same as in the single consumer case. In other words, the pricing policy is independent of the number of consumers in the system, as long as consumers are similar and each of the retailers can satisfy the entire market demand.

3. Extensions to the Model

So far we have studied the model under the assumptions that consumers’ valuations are uniformly distributed and that the two competing retailers can satisfy the entire demand in the market. Hereby, we relax these two assumptions. In Section 3.1 we study the basic setting while assuming that the single consumer’s valuation follows a more general distribution, and in Section 3.2 we investigate the capacitated case. Section 3.3 briefly discusses other visit patterns of consumers between the two competing stores.

3.1. Different Distributions

In the basic model previously considered we extend the valuation of the single consumer from uniform to a power distribution. That is, $V$, follows a power distribution, with a p.d.f. of the form $(q + 1)V^q$, $q > -1$, $0 \leq V \leq 1$. Thus, $F(0) = 0$ and $F(1) = 1$. Note that when $q = 0$, the power distribution coincides with the uniform distribution. To evaluate the effect of competition, we first develop the expressions for the monopolistic case (Section 3.1.1), then we elaborate the duopolistic case (Section 3.1.2), and we conclude with a short discussion on this effect (Section 3.1.3).

3.1.1. Monopoly under Power Distribution

One can show that when the monopolist is facing a single consumer, his profit-to-go in each period is an affine function of the price that is set in the previous period. That is, $\pi_t = S_t R_{t-1}$, where $S_t$ is a scalar, which can be expressed recursively as

$$S_t = \frac{q + 1}{(q + 2)(1 - S_{t+1})(q + 2))^{1/(q+1)}}. \quad (9)$$

This result is proved by induction, as in period $t$ the profit-to-go is given by $\pi_t = R_t(1 - F_t(R_t)) + F_t(R_t)S_{t+1}R_t = R_t(1 - \frac{R_t}{R_{t-1}})^{q+1} + \frac{R_t}{R_{t-1}}^{q+1}S_{t+1}R_t$. Taking the derivative w.r.t. $R_t$ to zero, we obtain $R_t = \frac{q+1}{((1-S_{t+1})(q+2))^{1/(q+1)}}R_{t-1}$. Rewriting the profit results with $\pi_t = \frac{q+1}{(q+2)((1-S_{t+1})(q+2))^{1/(q+1)}}R_{t-1}$. 

By induction it can be shown that the sequence of \( \{ S_t \} \) is backwards increasing in \( t \) and is bounded from above by \( S^* \), which, using (9), is the value of \( S \) that solves \( S((1 - S)(q + 2))^{1/(q+1)} = \frac{q+1}{q+2} \). Therefore, the sequence of \( \{ S_t \} \) is a convergent sequence which converges to \( S^* \). At convergence, the value of the normalized sales-to-go coincides with the expected profit of the monopolist. For example, if \( q = 2 \) (resp., \(-\frac{1}{3} \) and \( 6 \)), then the monopolist’s expected profit converges to 0.75 (resp., \( \frac{1}{3} \) and 0.875). Let \( B_t \equiv \frac{R_t}{R_{t-1}} = \frac{1}{((1-S_{t+1})(q+2))^{1/(q+1)}} \). Noting that \( S_t = \frac{q+1}{q+2} B_t \), one can express \( B_t \) in a recursive way,\[
B_t = \frac{1}{((1-S_{t+1})(q+2))^{1/(q+1)}}.
\]
The sequence of \( \{ B_t \} \) is a convergent sequence, which converges, not surprisingly, to 1. It may be worth noting that for \( q = 0 \) (resp., \( q > 0 \)) the decline is linear (resp., slower) than linear.

### 3.1.2. Duopoly under Power Distribution

In a duopoly, it can be shown, as before, that the profit-to-go expressions in each period are affine functions of the price that is set in the previous period. For Retailer \( i \) (resp., \( j \)) who encounters (resp., does not encounter) the consumer in period \( t \), \( S_t^i = \frac{q+1}{q+2} \left( \frac{1}{(1-S_{t+1})^{(q+2)}} \right)^{1/(q+1)} \) and \( S_t^j = S_{t+1}^j \left( \frac{1}{(1-S_{t+1})^{(q+2)}} \right)^{1/(q+1)} \). Let us denote by \( \{ s_t^i \} \) (resp., \( \{ s_t^j \} \)) the sequence of normalized sales-to-go of the retailer who encounters (resp., does not encounter) the consumer in period \( t \). Thus, \( s_t^i = \frac{q+1}{q+2} \left[ \frac{1}{(1-s_{t+1}^i)^{(q+2)}} \right]^{1/(q+1)} \) and \( s_t^j = s_{t+1}^j \left[ \frac{1}{(1-s_{t+1}^j)^{(q+2)}} \right]^{1/(q+1)} \). Let \( B_t \equiv \frac{R_t}{R_{t-1}} = \left[ \frac{1}{(1-s_{t+1}^i)^{(q+2)}} \right]^{1/(q+1)} \), and note that \( s_t^i = \left( \frac{q+2}{q+1} \right)^{q+2} s_{t+1}^i \) and \( s_t^j = \frac{q+1}{q+2} B_t \). Thus,\[
B_t = \left[ \frac{1}{(1-S_{t+1}^i)^{(q+2)} B_{t+2}^i (q + 2)} \right]^{1/(q+1)}.
\]
By induction one can show that \( \{ B_t \} \) is backwards increasing in \( t \) and that it is bounded from below by \( \left( \frac{1}{q+2} \right)^{1/(q+1)} \) and from above by \( B^* \), which is real, satisfies \( \left( \frac{1}{q+2} \right)^{1/(q+1)} \leq B < 1 \), and solves \( B(q + 2 - (q + 1)(B)^{q+3})^{1/(q+1)} = 1 \). Thus, based on the Monotone Convergence Theorem, \( \{ B_t \} \) is a convergent sequence, which converges to \( B^* \).

### 3.1.3. Effect of Competition under Power Distribution

As \( q \) increases the center of the valuation distribution shifts to the right. For values of \( q \) very close to \(-1\) it is very likely that the consumer has a very low valuation for the product, and therefore the monopolist drops prices fairly quickly to increase the number of search points with low prices. Yet, the monopolist does not give up the opportunity of searching for the chance the consumer has a high valuation. As a consequence, we can see that, for example, when \( q = -0.5 \), even though there is 50% chance that the consumer’s valuation is less than 0.25, the first seven prices the monopolist sets in a 10-period
selling horizon are higher than 0.25 (Figure 4(a)). So the monopolist is trading the opportunity of a high gain with low probability and a low gain associated with high probability. On the other hand the competing retailers do not have the luxury of spending search points on high prices and due to competition they mark down prices fairly aggressively. For the same example, when \( q = -0.5 \), only the first two prices posted by the retailers exceed 0.25 (Figure 4(a)). As \( q \) increases and the center of valuation distribution shifts to the right, the consumer is more likely to have a high valuation for the product and, therefore, both the monopolist and the duopoly’s retailers set higher prices in each period. In a sense, both systems slow down the price search, but since the monopolist is not subject to competition, he can focus the search only on the “promising” area and, thus, be more successful. Figure 4 demonstrates the shifting of the price decline over a 10-period selling horizon for different \( q \) values and Figure 5 exhibits the expected system profit (both for a monopoly and a duopoly) as a function of the selling horizon for different values of \( q \). In Figure 5 the difference between the plots is the system profit loss due to competition, which is increasing in \( T \), the length of the selling horizon, but can be observed to decline percentage-wise as \( q \) increases. In other words, asymptotically, as \( q \to \infty \), the percent loss due to competition goes to zero.

![Figure 4](image-url)  
**Figure 4** Prices in a 10-period selling horizon when valuation follows a power distribution

![Figure 5](image-url)  
**Figure 5** System profit for different selling horizons when valuation follows a power distribution
3.2. Limited Capacity

So far we assumed that both retailers can satisfy the market. In this section we extend the model by analyzing the capacitated case. That is, we analyze the case where each of the retailers cannot satisfy the entire market demand, the consumers are similar and their valuations are uniformly distributed. First, in Section 3.2.1 we develop the monopolistic setting, as to have a base case for comparison. Then, in Section 3.2.2, we study the duopolistic competition. We also briefly investigate the role of information in the capacitated setting.

3.2.1. Monopoly under Limited Capacity

The single unit case

When a monopolist who is facing \(N\) similar consumers has only one unit of good in stock to sell over a selling horizon of \(T\) periods, his profit-to-go in each period can be expressed as an affine function of the preceding price. That is, it can be proved by induction that \(\pi_t = S_t R_{t-1}\), where \(S_t\) is a scalar, which can be expressed recursively as \(S_t = \frac{N}{(1-S_{t+1})^{1/N} (N+1)^{(N+1)/N}}\). It can be further proved:

**Proposition 10.** The sequence \(\{S_t\}\) is a monotone convergent sequence as the number of periods, \(T\), goes to infinity, while \(t\) goes to 1.

Omitting subscripts we solve \(S = N(1-S)^{-1/N} (N + 1)^{-1/(N+1)}\), for which the solution is \(S^* = \frac{N}{N+1}\), which represents the monopolist’s expected profit at convergence. This result is not surprising, as at convergence the monopolist should obtain a profit which equals the expected value of the highest valuation amongst all consumers.

Since \(\pi_t = R_t \left(1 - \left(\frac{R_t}{R_{t-1}}\right)^N\right) + \left(\frac{R_t}{R_{t-1}}\right)^N S_{t+1} R_t\), it follows that \(\pi_t = \left(\frac{1}{(1-S_{t+1})^{1/N} (N+1)}\right)^\frac{1}{N}\). Figure 6 illustrates the prices set by a monopolist having a single unit in stock over a selling horizon of 20 periods for a varying values of \(N\).

\(K\) units in stock

Now assume that the monopolist who is facing \(N\) similar consumers has \(K\) units of the good in stock. If in period \(t-1\) a price \(R_{t-1}\) is set and not all goods are purchased, then all consumers who do not purchase the good have valuations below \(R_{t-1}\). Let \(P_{n_t}^{N_t}\) denote the probability that \(n_t\) of the \(N_t\) consumers who visit the monopolist in period \(t\) have valuations which exceed the posted price \(R_t\), i.e., \(P_{n_t}^{N_t} = \frac{N_t!}{n_t!(N_t-n_t)!} \left(\frac{R_t}{R_{t-1}}\right)^{N_t-n_t} (1 - \frac{R_t}{R_{t-1}})^{n_t}\). Note, that (i) if \(n_t < N_t\), then \(N_{t+1} = N_t - n_t\), and if \(n_t \geq N_t\), then \(N_{t+1} = 0\), (ii) \(N_1 = N\), and (iii) \(R_0 = 1\). The monopolist profit in a two-period selling horizon is

\[
\pi = \sum_{n_1=0}^{K-1} P_{n_1}^{N_1} (n_1 R_1 + \sum_{n_2=0}^{K-n_1-1} (P_{n_2}^{N_2-n_1} n_2 R_2^{n_1}) + \sum_{n_2=K-n_1}^{N-n_1} P_{n_2}^{N_2-n_1} (K-n_1) R_2^{n_1}) + \sum_{n_1=0}^{N} P_{n_1}^{N} K R_1\]


where $R_{2}^{N-n_1}$ is the price the monopolist sets in period 2 given that $n_1$ consumers purchase the good in the first period. The optimal prices are found by solving this equation backwards, first for all second period prices, $R_{2}^{N-n_1}$, $n_1 = 0 \ldots K - 1$, and then for the first period price, $R_1$.

Due to the complexity of the expressions for the optimal prices, we do not provide a general closed-form solutions for $R_1$ and $R_2$ for any $N$ and $K$. However, we can demonstrate their main characteristics. As Figure 7 illustrates that as $N$ increases, both $R_1$, $R_2$, and the corresponding profit increase as well. Similarly, Figure 8(a) illustrates, as $K$ increases, the optimal prices when $N = 10$, wherein the solid line represents $R_1$ and the triangles below represent the second period prices in a deceasing order of leftover inventory. For example, when $K = 5$, the uppermost triangle
is the second period price when the monopolist has one unit remaining in stock (i.e., he sold 4 units in the first period), and the lowermost triangle corresponds to the case of 5 units in stock (in other words, nothing was sold in the first period). Observe that when $K = N = 10$, for any positive leftover inventory, the monopolist sets the same price. Note that as Figure 8(b) demonstrates, as $K$ increases the expected profit increases as well, yet, the profit per unit stocked is decreasing. This reflects a decreasing incentive to stock inventory when there is a positive holding cost. Indeed, so far we have assumed zero wholesale price (or production cost) and zero holding cost. However, for wholesale prices larger than zero, the corresponding optimal stocking level may be less than $N$. For example, with $N = 10$ if the wholesale price is 0.2 (resp., 0.5), the optimal stocking level is $K = 6$ (resp., 3) with a corresponding profit of 0.31 (resp., 0.18) per unit.

![Figure 8](image-url)

(a) Prices (b) Expected Profit

**Figure 8** Prices a monopolist sets in a 2-period model and the corresponding profit when $N = 10$ for varying values of $K$

### 3.2.2. Duopoly under Limited Capacity

The capacitated duopoly case is fairly easy to formulate but solutions are not simple to obtain as expressions involve polynomial of high degrees. Thus, let us first consider the simple two-period case, wherein each retailer has one unit of good in stock and is visited by one consumer in the first period. In other words, each of the retailers can satisfy half the market (and together they can satisfy the entire market). The expression of the expected profit of Retailer $i$ is given by

$$
\pi^i = (1 - R^i_1) R^i_1 + R^i_1 R^j_1 (R^i_2 - \frac{R^i_1}{R^j_1}), \quad i \neq j = 1, 2,
$$

where the first term, $(1 - R^i_1) R^i_1$, is the expected profit in the first period when Retailer $i$ sets a price $R^i_1$, and the second term reflects the conditional expected profit in the second period. Specifically,
$R^i_1 R^j_1$ is the probability that Retailer $i$ is not out of stock and that the consumer who has visited Retailer $j$ in the first period has not purchased the good, and $R^i_2 (1 - R^i_1/R^j_1)$ is the corresponding expected profit given that event.

Solving backwards, we find that $R^i_1 = R^j_1 = 4 - 2 \sqrt{3} \approx 0.535$ and $R^i_2 = R^j_2 = \frac{1}{2} R^i_1 = 2 - \sqrt{3}$, with a corresponding expected profit of $28 - 16 \sqrt{3} \approx 0.287$ for each of them. By comparison, the corresponding prices set by a monopolist who is facing two consumers and is holding one (or two)\(^{15}\) unit(s) of good in stock are 0.736 and 0.425 (resp., 0.666 and 0.333) in the first and second period respectively, and the associated profit is 0.49 (resp., 0.666). Thus, duopoly competition results with lower prices than the monopoly, as expected, and a loss of 41\% (resp., 13.8\%) of the profit from a retailer’s (resp., system’s) perspective.

Extending this setting to a three-period selling horizon, Retailer $i$’s profit is given by

$$\pi^i = (1 - R^i_1) R^i_1 + R^i_1 R^j_1 (1 - R^i_2/R^j_1) + R^i_2 R^j_2 (1 - R^i_3/R^j_2) + R^i_1 (1 - R^i_1) R^i_{m,3} (1 - R^i_3/R^j_1), \quad i \neq j = 1, 2,$$

where the first two terms are as before. The third term, $R^i_3 R^j_2 (1 - R^i_3/R^j_2)$, is the conditional expected profit in the third period if both retailers have not sold the good yet, and the last term, $R^i_1 (1 - R^i_1) R^i_{m,3} (1 - R^i_3/R^j_1)$, reflects the expected profit in the third period when Retailer $j$ sells the good in the first period while Retailer $i$ does not. In that case, Retailer $i$ is not visited at all in the second period and may realize that he has become a monopoly, and the price that he sets in that case is denoted by $R^i_{m,3}$.

To find the optimal solution for this setting, we solve the model backwards. $R^i_1$ is found numerically (by using the best reaction functions of the retailers) to be about 0.6207. Correspondingly, $R^i_1 = 0.332$, $R^i_1 = 0.166$, $R^i_{m,3} = 0.31$, with an expected profit of 0.34 for each retailer. This is compared with prices of 0.809, 0.59, and 0.34 (resp., 0.75, 0.5, and 0.25) a monopolist sets over the three periods when he faces two consumers and has one (resp., two) unit(s) of good in stock, and an expected profit of 0.539 (resp., 0.75). Thus, when the horizon extends to three periods, duopoly competition keeps exerting pressure on prices and profits. Yet, this pressure is decreasing, and the loss incurred to the profit from a retailer’s (resp., system’s) perspective is 36.9\% (resp., 9.3\%), which is slightly lower than in the two-period horizon.

Next, we consider the case wherein each retailer has one unit of good in stock and is visited initially by two consumers. That is, each retailer can satisfy a quarter of the entire demand and together they can satisfy half the market. In that case informational issues arise.

\(^{15}\)We provide these two comparisons as to provide retailer’s perspective (who can satisfy half the market) and a system’s perspective (which satisfies the entire demand). Retailer’s perspective reflects the case where an identical retailer is introduced into the monopolistic market and consumers are equally distributed between the two competing retailers in the first period. The system’s perspective represents a situation where both retailers could satisfy the entire demand together, so in a sense, the only “inefficiency” is due to competition.
No Sales Information

When Retailer $i$ cannot observe the sales at other location, he sets a price $R_i^1$ in the first period which leads to an expected profit of $R_i^1(1 - (R_i^1)^2)$ in the first period. With probability (w.p.) $(R_i^1)^2$, Retailer $i$ doesn’t sell at all in the first period and sets a price $R_i^2$ in the second period. W.p. $(R_i^1)^2$ two consumers arrive and w.p. $1 - (R_i^1)^2$ at least one of them buys the good. If only one consumer arrives to Retailer $i$, then w.p. $2R_i^1(1 - R_i^1)$ his valuation is below $R_i^1$ and he buys the good w.p. $1 - R_i^2$, and w.p. $(1 - R_i^1)^2$ his valuation exceeds $R_i^1$ and he buys the good w.p. $1$ (if $R_i^2 \leq R_i^1$). Assuming $R_i^2 \leq R_i^1$ (and this can be verified to hold), the profit of Retailer $i$ (after simplifying) is given by:

$$
\pi^i = R_i^1(1 - (R_i^1)^2) + (R_i^1)^2R_i^2\left(((R_i^1)^2 - (R_i^2)^2) + 2(1 - R_i^1)(R_i^1 - R_i^2) + (1 - R_i^2)^2\right).
$$

Solving backwards, we find that $R_i^2 = \frac{1}{4}(2R_i^1 - 2 + \sqrt{4(R_i^1)^2 - 8R_i^1 + 7})$ and we find numerically that $R_i^1 = R_i^2 \approx 0.657$. Thus, $R_i^1 = R_i^2 \approx 0.393$ and each retailer’s expected profit is about 0.471. These prices are lower than the prices set by a monopolist holding one (resp., two) unit(s) of good in inventory and facing four consumers, in which case the prices the monopolist sets are 0.809 in the first period and 0.54 in the second period (resp., 0.74 in the first period, 0.43 and 0.466 in the second period if he sells nothing or one unit in the first period, respectively). Yet, these prices are higher than in the previous case wherein each retailer was visited by a single consumer. Consequently, the loss incurred to the profit from a retailer’s (resp., system’s) perspective is 27.2% (resp., 11.1%), which is dramatically lower than the profit loss when each retailer was visited by a single consumer, which was 41% (resp., 13.8%).

With Sales Information

We further assume that each retailer may observe the sales level at the other retailer. As each retailer has one unit in stock it means that they simply need to observe whether the other retailer is out of stock or not. Now, a retailer may set different prices in the second period, based on the observation made at the end of the first period. Let $R_i^{1(1)}$ (resp., $R_i^{1(2)}$) denote the price Retailer $i$ sets in the second period if only one consumer (resp., two consumers) visit(s) him in the second period. Assuming $R_i^2 \leq R_i^1$ (and this can be verified to hold), the profit of Retailer $i$ is given by:

$$
\pi^i = R_i^1(1 - (R_i^1)^2) + (R_i^1)^2\left(R_i^{1(2)}((R_i^1)^2 - (R_i^2)^2) + R_i^{1(1)}(2(1 - R_i^1)(R_i^1 - R_i^2) + (1 - R_i^2)^2)\right).
$$

Solving backwards, we find that the second period prices are $R_i^{1(1)} = \frac{R_i^1 + 1}{4}$ and $R_i^{1(2)} = \sqrt{3R_i^1} / 3$. The first period price, $R_i^1$, is approximately 0.658. This price is only marginally higher than the price set by the retailers in the case of no sales information. The corresponding second period prices are
0.414 if one consumer arrives and 0.3799 if two consumers arrive, and the corresponding profit is 0.4714. The value of information in that case increases the expected profit only by about 0.04%.

Additional observations are presented in Table 2 below.

### 3.2.3. Effect of Competition under Limited Capacity

Let us briefly make several qualitative observations regarding the effect of competition in the most elementary capacitated case wherein each retailer has only one unit of good in stock. We have observed that, similar to the uncapacitated case, competition in the capacitated case dramatically suppresses prices, which results with a substantial decline in profit. As the number of consumers in the system increases, while inventory levels are fixed, the competitive pressure diminishes, and as a result prices are set closer to the monopolistic case and correspondingly, the profit loss is lower. Also, as the selling horizon extends, the competitive pressure declines. Table 2 below summarizes prices that are set and profit per consumer achieved in a two-period selling horizon. The first (resp., second) monopolistic observation in each case is to provide the retailer’s (resp., system’s) perspective, as was discussed earlier.

Indeed, here we have used an elementary model, wherein each retailer has only one unit of good in stock, but it can be extended to other situations where retailers have more than one unit in stock.

<table>
<thead>
<tr>
<th>Two Consumers in the system</th>
<th>Setting</th>
<th>$R^i_1$</th>
<th>$R^i_2$</th>
<th>Profit per unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Duopoly</td>
<td>0.535</td>
<td>0.268</td>
<td>0.287</td>
<td></td>
</tr>
<tr>
<td>Monopoly: 1 unit in stock</td>
<td>0.736</td>
<td>0.425</td>
<td>0.49</td>
<td></td>
</tr>
<tr>
<td>Monopoly: 2 units in stock</td>
<td>0.66</td>
<td>0.33</td>
<td>0.33</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Four Consumers in the system</th>
<th>Setting</th>
<th>$R^i_1$</th>
<th>$R^i_2$</th>
<th>Profit per unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Duopoly: no sales info.</td>
<td>0.657</td>
<td>0.392</td>
<td>0.4712</td>
<td></td>
</tr>
<tr>
<td>Duopoly: with sales info.</td>
<td>0.658</td>
<td>0.414, 0.38</td>
<td>0.4714</td>
<td></td>
</tr>
<tr>
<td>Monopoly: 1 unit in stock</td>
<td>0.809</td>
<td>0.54</td>
<td>0.647</td>
<td></td>
</tr>
<tr>
<td>Monopoly: 2 units in stock</td>
<td>0.74</td>
<td>0.466, 0.43</td>
<td>0.53</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Six Consumers in the system</th>
<th>Setting</th>
<th>$R^i_1$</th>
<th>$R^i_2$</th>
<th>Profit per unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Duopoly: no sales info.</td>
<td>0.726</td>
<td>0.478</td>
<td>0.576</td>
<td></td>
</tr>
<tr>
<td>Duopoly: with sales info.</td>
<td>0.726</td>
<td>0.5, 0.457</td>
<td>0.577</td>
<td></td>
</tr>
<tr>
<td>Monopoly: 1 unit in stock</td>
<td>0.849</td>
<td>0.614</td>
<td>0.728</td>
<td></td>
</tr>
<tr>
<td>Monopoly: 2 units in stock</td>
<td>0.7911</td>
<td>0.698, 0.644</td>
<td>0.634</td>
<td></td>
</tr>
</tbody>
</table>

*All duopoly cases refer to the single unit in stock at each retailer.*

**Table 2** Summary of the capacitated case in a two-period selling horizon
3.3. Other Visit Patterns

Our zigzag competition model assumes that if a consumer visits one of the retailers and observes a price that exceeds his valuation, then in the ensuing period he visits the competing retailer. In Granot, Granot and Mantin (GGM) (2006) we generalize this visit pattern and acknowledge that a consumer may return to the same retailer in the ensuing period even if the current price he observes exceeds his valuation. Explicitly, we assume therein that the consumer returns to the same retailer in the ensuing period with probability $P$ or switches to the competing retailer with the complementary probability $1 - P$, where $P$ reflects market conditions or consumer’s experience at the store. In GGM (2006) it is found, among other results, that the exponential decline of prices is quite robust as it holds for fairly general visiting patterns by consumers and when the retailers are far from being symmetric.

4. Identical Consumers

In this section we study pricing schemes when all consumers are identical, with a common valuation uniformly distributed on $[0, 1]$ and the retailers cannot necessarily satisfy the entire demand. Let us consider only the symmetric case. That is, we assume that each retailer is visited by $N$ consumers in the first period and has $K$ units of goods in stock. We further assume retailers have sales information, i.e., they can observe whether consumers have purchased the good or not at the competing retailer. We find that in the identical consumers case there may exist only non-symmetric pure strategy SPNE as well as a continuum of equilibria.

4.1. Uncapacitated Duopoly

4.1.1. Two-Period Selling Horizon

Assume that $R_i^1 \leq R_j^1$, and $2N \leq K$. Then, the profit functions for Retailers $i$ and $j$ can be written as $\pi^i = NR_i^1(1 - R_i^1) + (R_j^1 - R_i^1)R_i^1N + R_i^1NR_j^2(1 - \frac{R_i^2}{R_j^1})$, and $\pi^j = NR_j^1(1 - R_j^1) + R_i^1NR_j^2(1 - \frac{R_i^2}{R_j^1})$. Solving backwards, we find that $R_i^2 = R_j^2 = \frac{1}{2}R_i^1$. Solving for the first period, we have $R_i^1 = \min(R_j^1, \frac{2(1+R_i^1)}{7})$ and $R_i^1 = \max(\frac{1}{2}, R_i^1)$. Thus, we have that the pure-strategy SPNE profile is $R_i^1 = \frac{3}{7}$, $R_j^1 = \frac{1}{2}$, and $R_i^2 = R_j^2 = \frac{3}{14}$. Correspondingly, the profits of the retailers are shown in Table 3. Obviously, the symmetric case in which $R_i^1 = \frac{1}{2}$, $R_j^1 = \frac{3}{7}$, $R_i^2 = R_j^2 = \frac{3}{14}$ is also an SPNE.

<table>
<thead>
<tr>
<th>Retailer $j$ sets</th>
<th>$\frac{1}{2}$</th>
<th>$\frac{3}{7}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Retailer $i$ sets $\frac{1}{2}$</td>
<td>$\frac{5}{16}N \approx 0.312N$, $\frac{5}{16}N$, $\frac{29}{98}N$, $\frac{9}{28}N$</td>
<td>$\frac{29}{98}N$, $\frac{9}{28}N$</td>
</tr>
<tr>
<td>Retailer $i$ sets $\frac{3}{7}$</td>
<td>$\frac{3}{7}N \approx 0.32N$, $\frac{29}{98}N \approx 0.296N$, $\frac{37}{156}N \approx 0.291N$, $\frac{37}{156}N$, $\frac{57}{196}N \approx 0.296N$</td>
<td></td>
</tr>
</tbody>
</table>

Table 3 Two-period expected profits for the two retailers
In view of Table 3 we can conclude:

**Corollary 2.** In an uncapacitated two-period selling horizon with identical consumers, there exists precisely two pure-strategy SPNE both of which are non-symmetric.

Recall that in the corresponding uncapacitated case when consumers are similar, the pure-strategy SPNE is unique and symmetric, and is such that each retailer sets the price of $\frac{1}{2}$ in the first period, and $\frac{1}{4}$ in the second period. Thus, on average, a duopoly system facing identical consumers reduces the first period price by 7%. Correspondingly, the system’s profit drops from $\frac{2}{3}N$ (since it faces $2N$ consumers) to $\frac{121}{196}N$, which represents a decline of almost 7.4%. Thus, in the uncapacitated case, identical consumers increase the competitive pressure on the two retailers.

Thus, we have:

**Corollary 3.** In an uncapacitated two-period selling horizon equilibrium prices and corresponding profits are lower when consumers are identical.

### 4.1.2. Longer Selling Horizons

We can extend the above analysis to longer selling horizons. However, with longer selling horizons the number of possible SPNE grows quickly. Let us consider the two possible such SPNE wherein (i) one of the retailers always sets the lower price while the other sets the higher price and (ii) the retailers alternate in these roles.

*Retailer i sets the lower price in all periods* The profit functions in period $t$ are $\pi_i^t = NR_i^t(R_{i-1}^t - R_j^t) + (R_i^t - R_j^t)R_j^t N + S_{i+1}^t N (R_i^t)^2$ and $\pi_j^t = NR_j^t(R_{i-1}^t - R_j^t) + S_{i+1}^t N (R_i^t)^2$, where $S_i^t$ (resp., $S_j^t$) is the normalized profit-to-go of Retailer $i$ (resp., $j$) in period $t$. We can show by induction that $S_i^t = \frac{9}{16(2 - S_{i+1}^t)}$ and $S_j^t = \frac{16 - 16S_{i+1}^t + 4(S_{i+1}^t)^2 + 9S_{i+1}^t}{16(2 - S_{i+1}^t)^2}$. The corresponding prices are $R_i^t = \frac{3}{4(2 - S_{i+1}^t)}R_{i-1}^t$ and $R_j^t = \frac{1}{2}R_{i-1}^t$, and the price ratios are $B_i^t = \frac{3}{8 - 3S_{i+1}^t} = \frac{3}{8 - 3S_{i+1}^t}$ and $B_j^t = \frac{1}{2}$. It can be shown that $\{B_i^t\}$ is a convergent sequence for $t < T - 1$, which converges to 0.451. Therefore, prices decline exponentially at convergence (that is, $T \to \infty$), $R_i^t = 0.451$, $R_j^t = 0.5$, $\pi_i = 0.338N$ and $\pi_j = 0.3139N$. This represents a system profit loss of 34.8% due to competition, when compared with the monopolist’s profit, which converges to $N$ if he holds $2N$ units in stock.

Recall that the price ratio sequence in the corresponding uncapacitated case when consumers are similar is also backwards increasing and bounded from below by $\frac{1}{2}$, which is greater than or equal to $B_i^t$ (which in turn is greater or equal to $B_i^t$), for any $t$, and strictly greater for $t < T - 1$. This leads us to the following conclusion:

**Corollary 4.** In the uncapacitated case, if consumers are identical and the same retailer constantly sets the lower price at each period, the resulting prices at each period are lower or equal to the prices when consumers are similar.
Alternating Lower Price The analysis for that case is similar to the preceding case. Indeed, it can be shown that, at convergence, the lower (resp., higher) price in the first period is 0.445 (resp., 0.5), Corollary 4 holds in this case as well, and prices decline exponentially.

4.2. Capacitated Duopoly

Let \( \alpha \equiv \frac{K}{N} \).

**Theorem 2.** In a capacitated duopoly with two-period selling horizon and identical consumers, when \( \alpha > \frac{2}{3} \) or \( \alpha < \frac{2}{3} \) there exist two pure-strategy SPNE, both of which are non-symmetric; when \( \frac{2}{3} < \alpha < \frac{2}{3} \) there is a continuum of symmetric pure-strategy SPNE; when \( \alpha = \frac{2}{3} \) or \( \alpha = \frac{3}{2} \) there exist a unique pure-strategy SPNE.

The equilibrium prices, as a function of \( \alpha \), are summarized in Table 4 and illustrated in Figure 9, wherein the lower and upper bound on prices are plotted. Figure 9 demonstrates the general declining trend of prices as \( \alpha \) increases from 0.5 to 2. When \( \alpha \leq 0.5 \), each retailer can satisfy up to a quarter of the entire demand. In that case, if only Retailer \( i \) sells in the first period, then Retailer \( j \) can sell his entire inventory in the second period. This results with the lowest pressure on prices and Retailer \( i \) (resp., \( j \)) sets a price of \( \frac{3}{4} \) (resp., \( \frac{5}{4} \)) in the first period. As \( \alpha \) increases from \( \frac{1}{2} \) to 1, if only Retailer \( i \) sells in the first period, then in the second period Retailer \( j \) can satisfy all leftover demand, but less than his entire inventory. As a result we observe a decrease in Retailer \( j \)'s initial price. At \( \alpha = \frac{2}{3} \) the prices the two retailers set coincide with the monopoly pricing. This equilibrium pricing, which is not unique, can be maintained until \( \alpha = 1 \), i.e., as long as the two retailers can satisfy the entire demand together. When \( 1 \leq \alpha \leq 1.5 \), there exists an equilibrium in symmetric pricing, where both retailers set a first period price of 0.5, which coincides with the two-period pricing under competition when there is only a single consumer in the system.

### Table 4

<table>
<thead>
<tr>
<th>( 2 \leq \alpha )</th>
<th>( R_1^i )</th>
<th>( R_1^j )</th>
<th>( \pi^i )</th>
<th>( \pi^j )</th>
<th>System’s profit</th>
<th>Profit loss, PL (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{2}{3} \leq \alpha \leq 2 )</td>
<td>( \frac{2}{3} )</td>
<td>( \frac{2}{3} N )</td>
<td>( \frac{2}{3} N )</td>
<td>( \frac{2}{3} N )</td>
<td>( \frac{2}{3} N )</td>
<td>( -\frac{5}{3} )</td>
</tr>
<tr>
<td>( \frac{1}{2} \leq \alpha &lt; \frac{2}{3} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} N )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} N )</td>
<td>( -\frac{1}{2} )</td>
</tr>
<tr>
<td>( \frac{1}{3} \leq \alpha &lt; \frac{1}{2} )</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{1}{3} N )</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{1}{3} N )</td>
<td>( -\frac{1}{3} )</td>
</tr>
<tr>
<td>( \frac{1}{4} \leq \alpha &lt; \frac{1}{3} )</td>
<td>( \frac{1}{4} )</td>
<td>( \frac{1}{4} )</td>
<td>( \frac{1}{4} N )</td>
<td>( \frac{1}{4} )</td>
<td>( \frac{1}{4} N )</td>
<td>( -\frac{1}{4} )</td>
</tr>
<tr>
<td>( \frac{1}{5} \leq \alpha &lt; \frac{1}{4} )</td>
<td>( \frac{1}{5} )</td>
<td>( \frac{1}{5} )</td>
<td>( \frac{1}{5} N )</td>
<td>( \frac{1}{5} )</td>
<td>( \frac{1}{5} N )</td>
<td>( -\frac{1}{5} )</td>
</tr>
</tbody>
</table>

Table 4 also provides the system’s profit and profit loss as compared to the profit obtained by a monopoly facing \( 2N \) consumers and having \( 2K \) units in stock, and note that, when \( \alpha \geq 1 \)
(resp., $\alpha \leq 1$), the monopolist profit is $\frac{2}{3}N$ (resp., $\frac{2}{3}K$). The profit loss (gain) of the system due to competition is illustrated in Figure 10(a). In general, the competitive pressure is intensifying in the share of the market retailers can satisfy. In other words, for a fixed stocking level, as the number of consumers in the system increases, the competitive pressure diminishes. Alternatively, when the number of consumers in the system is fixed, the competitive pressure diminishes as the stocking levels decrease. These results are supported by the general declining trend of the initial prices posted by the two competing retailers and by the general increasing trend of the profit loss. In view of Table 4 and Figure 10(a), one may be tempted to conclude that in a two-period selling horizon, when $\alpha < \frac{2}{3}$ a duopoly is better off than a monopoly. Thus, it seems that when the share of market retailers can satisfy, i.e., $\alpha < \frac{2}{3}$, one retailer takes an advantage of the “additional” observation and sets a slightly higher price, which rewards him with a slightly higher expected profit. However this conclusion is true only when a two-store duopoly is compared with a single-store monopoly.
Clearly, a two-store monopoly may do better than a single store-monopoly or a two-store duopoly. Indeed, at worst, when \( \alpha < \frac{2}{3} \), the centralized planner of a two-store monopoly would set the duopoly prices and would not suffer any profit loss. To enable the right analysis, in the next section we develop the model for a two-store monopoly. As Figure 10(b) demonstrates, a centralized two-store monopoly (with each retail store facing \( N \) consumers in the first period) can do at least as good as a duopolistic system in a symmetric setting. These results, however, are discussed in the next section, where a model for a centralized two-store monopoly is developed.

5. Centralized Zigzag with \( N \) Consumers

As was recognized by Elamghraby and Keskinocak (2003), pricing by chain stores is not very common in the literature. Bitran et al. (1998) have developed pricing heuristics for a retail chain with multiple stores, each facing an independent demand stream modeled as a time dependent Poisson arrival process, by constraining the price to be the same in all stores.

We assume in this section that the two retail stores are owned by a central planner who is setting the prices in both stores simultaneously to maximize profit. The \( N \) consumers are divided into two groups of sizes \( N^1 \) and \( N^2 \), with \( N^1 + N^2 = N \), such that \( N^1 \) (resp., \( N^2 \)) visit Retail Store 1 (resp., 2) in the first period, with \( N^1 \geq N^2 \). We also assume that the system can satisfy the entire demand experienced at each of the periods, and that the central planner knows \( N^1 \) and \( N^2 \).

If the central planner restricts the prices in both retail stores to be equal, the system reduces to a single retailer system. However, allowing for possibly different prices in the two retail stores, which is a common practice if the two stores have different brand names, provides the centralized system with up to twice as many observations on the consumers’ valuations. In this case, a centralized system may be able to sell the goods closer to the consumers’ valuations and, in turn, increase its profit.

In this section we show that the two-store centralized system facing similar consumer can do no better than a single store monopoly. In contrast, however, we show that a two-store centralized system facing identical consumers can achieve an expected profit that exceeds the profit obtained by the single store monopoly (as was studied by Lazear). Thus, the system’s profit loss due to competition, as derived in Section 4 when consumers are identical, could be higher.

\[ \text{Yet, this cannot be viewed as a single-store monopoly with twice as many observations, since at each period two prices are set simultaneously.} \]
5.1. Centralized Zigzag with $N$ Similar Consumers

Since the valuations of the $N$ similar consumers are possibly different, it may happen that some of them purchase the good earlier than others. The question that arises is whether setting possibly different prices at the two retail stores can increase the system’s profit. If the central planner sets a price $R_1^t$ (resp., $R_2^t$) at Retail Store 1 (resp., 2), then he can only infer that the valuations of all the consumers who visit Retail Store 1 (resp., 2) in period $t$ and do not purchase the good are below $R_1^t$ (resp., $R_2^t$). Since the valuations of the consumers are independent of each other, this information bears no relevance to the valuations of the consumers who visit Retail Store 2 (resp., 1) in period $t$ and do not purchase the item. This leads us to the following conclusion:

**Proposition 11.** A two-store monopoly facing $N$ similar consumers who zigzag between the two stores has no advantage over a single-store monopoly, as both post the same prices and yield the same expected profit, regardless of the valuation distribution and the distribution of the $N$ consumers between the two stores.

5.2. Centralized Zigzag with $N$ Identical Consumers

5.2.1. Uncapacitated Setting

Suppose that all $N$ consumers have the same valuation, which is drawn from a uniform distribution over $[0,1]$. It is shown in Proposition 13 that the best policy for the centralized planner in each period is to set at the location which is visited by the larger group of consumers a price that is higher than the price at the other location. Below, we provide some intuition for this result.

Let $R_{N1}^t$ (resp., $R_{N2}^t$) denote the price that is observed by the group of $N^1$ (resp., $N^2$) consumers in period $t$, and assume that $R_{N1}^t \geq R_{N2}^t$. Now, three cases may arise in period $t$:

*Case (i):* None of the consumers buys the good as their common valuation is below $R_{N2}^t$, and all $N$ consumers proceed to the next period following the zigzag pattern;

*Case (ii):* All $N$ consumers buy the good and the selling season ends;

*Case (iii):* Only the $N^2$ consumers who observe the lower price, $R_{N2}^t$, buy the good, while the $N^1$ consumers who observe the higher price, $R_{N1}^t$, do not buy.

In the latter case the central planner can conclude that the valuation of the remaining $N^1$ consumers is distributed uniformly between $R_{N2}^t$ and $R_{N1}^t$, and from period $t+1$ on, the two-store system can be viewed as a single retailer system (since there is only a single price observation in each period). Since $N^1 \geq N^2$, if $R_{N2}^t \leq V < R_{N1}^t$, the system profit is higher when $R_{N1}^t \geq R_{N2}^t$ (rather than $R_{N1}^t \leq R_{N2}^t$), as a lower price, $R_{N2}^t$, is charged from each of the $N^2$ consumers, and
possibly a higher price from the larger group of $N^1$ consumers. By the Appendix, if Case (iii) occurs in period $t$, and
\[ R_t^{N^1} \leq (T - t + 1) R_t^{N^2}, \tag{10} \]
then the centralized system profit-to-go in period $t + 1$ is $N^1 R_t^{N^1} R_t^{N^2} (T - t + 1) + R_t^{N^2} (T - t + 1)$, and the price in each of the following periods is $R_t = \frac{(T - t) R_{t+1}^{N^1} + R_{t+1}^{N^2} (T - t + 1)}{(T - t + 1)}$, $t = t + 1, \ldots, T$.

The dynamic programming problem of maximizing the expected profit is solved backwards. In period $t$, the profit-to-go of the system, as long as $N^2$ is not zero\(^{17}\), is:
\[ \pi_t = N^1 R_t^{N^1} [1 - F_t(R_t^{N^1})] + N^2 R_t^{N^2} [1 - F_t(R_t^{N^2})] + F_t(R_t^{N^1}) \pi_{t+1} + [F_t(R_t^{N^1}) - F_t(R_t^{N^2})] N^1 R_t^{N^1} (T - t - 1) + R_t^{N^2} (T - t + 1) \frac{2}{2(T - t)}, \tag{11} \]
where the first and second terms are the expected profits of the retail stores that meet the groups of $N^1$ and $N^2$ consumers, respectively, in period $t$. The third term is the conditional expected profit-to-go of the system in period $t + 1$ if no consumer makes a purchase in period $t$. The fourth term is the conditional expected profit-to-go of the system in period $t + 1$ if only the group with $N^2$ consumers makes a purchase (as they face a lower price). Display (11) is valid if (10) holds, which is verified to be the case in Lemma 5 below.

We shall start with a simple illustration by solving for the last three periods. Figure 11 provides the expressions of the centralized system’s profit-to-go and the prices set at both retail stores, and also illustrates the zigzagging pattern of the $N^1$ (solid arrows) and the $N^2$ consumers (dashed arrows), assuming $T - 2$ is odd. Observe that the prices and profit-to-go in each period are expressed as an affine function of the lower price from the previous period, which is the price that is observed by the $N^2$ consumers.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
\textbf{Period T-2:} & \textbf{Profit-to-go, $\pi_t$} & \textbf{Price at Retail Store 1} & \textbf{Price at Retail Store 2} \\
\hline
\text{N} & \text{N} & \text{N} & \text{N} \\
\text{\( \frac{N(3N^2 + 6N^1 - N^1N + N^1)}{4(N^2)^2 + 16N^1N + 16N^2 - 16N^1 + 11N^1N + 16N^2 - 8N^1} \)} & \text{N} & \text{N} \\
\hline
\text{\( \frac{N(4N - N^1)}{4(N^2 + 2N)} \)} & \text{N} & \text{N} \\
\hline
\text{\( \frac{1}{4}NR_{t-1}^{N^1} \)} & \text{N} & \text{N} \\
\hline
\text{\( \frac{1}{2}NR_{t-2}^{N^1} \)} & \text{N} & \text{N} \\
\hline
\end{tabular}
\caption{Centralized system profit-to-go and prices at both retail stores.}
\end{table}

Continuing with the illustration, let us assume that $T = 3$, $N^1 = 2$, and $N^2 = 1$. Using the expressions from Figure 11, as long as no purchase is made, the central planner sets the prices

\(^{17}\)The case of $\{N^1 = N, N^2 = 0\}$ coincides with the single store monopoly case and can be solved by setting $\{N^1 = 0, N^2 = N\}$. \hfill\text{29}
\{0.886, 0.66\}, \{0.377, 0.518\}, and \{0.188, 0.188\} at periods 1, 2, and 3, respectively, at Retail Store 1 and 2, respectively. However, if a single purchase occurs in the first (resp., second) period, then the ensuing price(s) are 0.773 and 0.66 (resp., 0.377). This pricing results with an expected profit of 1.217, which is 8.1\% higher than the expected profit of a single-store monopoly (1.125).

The central planner can also sub-optimally solve the problem by setting \(R_{N1} \leq R_{N2}\) (rather than \(R_{N2} \leq R_{N1}\)) which results with an expected profit of 1.165, which is still higher than that of the single-store monopoly, but less than the optimal solution.

Similar to the basic model with a single consumer we have:

**Lemma 3.** The profit-to-go in period \(t\) can be expressed as the normalized sales-to-go in period \(t\) times the lower price that was set in the preceding period. That is, \(\pi_t = S_t R_{N2}^t - 1\), where \(S_t\) is a scalar which can be expressed recursively as

\[
S_t = \frac{(N)^2 + (T - t)((N)^2 - 2N^1_{t+1})}{2((T - t)(2N^2 + N^1 - 2S_{t+1}) + 2(N - S_{t+1})).}
\]  

\(12\)

**Lemma 4.** The sequence of normalized sales-to-go, \(\{S_t\}\), is bounded from above by \(\frac{1}{2}N\).

The following is to verify that (10) holds:

**Lemma 5.** For each period \(t\), as long as no sale has occurred, \(R_{N1}^t \leq (T - t + 1)R_{N2}^t\).

**Proposition 12.** The sequence of the system’s normalized sales-to-go, \(\{S_t\}\), is a monotone convergent sequence as the number of periods, \(T\), goes to infinity while \(t\) goes to 1.

**Proposition 13.** The normalized sales-to-go of the centralized system is increasing with \(N^1\), as long as \(N^2\) is not zero, when \(N\) is kept constant.

Note that for \(t = 1\), \(\pi_1 = S_1\), i.e., in the first period the normalized sales-to-go coincides with the system’s expected profit. Therefore, Proposition 13 implies:

**Corollary 5.** The expected profit of the centralized system is increasing with \(N^1\), when \(N\) is kept constant, as long as \(N^2\) is not zero.

By Proposition 12 \(\{S_t\}\) is a monotone convergent sequence. To find the convergence value of \(\{S_t\}\) we omit the subscripts in (12) and solve the equation for \(S\), whose unique solution is \(S = \frac{1}{2}N\). As expected, the centralized system will extract the entire surplus from the consumers if the number of periods is large enough. Note, that the convergence value of \(\{S_t\}\) is independent of the distribution of the consumers across the two retail stores. Thus, we can conclude:

\(^{18}\) One can also show that the sequences of the price ratios, \(B_1^t\) and \(B_2^t\), which are defined by \(B_1^t \equiv \frac{R_{N1}^t}{R_{N2}^t}\) and \(B_2^t \equiv \frac{R_{N2}^t}{R_{N1}^t}\), monotonically converge as the number of periods, \(T\), goes to infinity while \(t\) goes to 1, to \(B_1 = B_2 = 1\).
Corollary 6. In the centralized system the expected profit converges to $\frac{1}{2}N$ as $T \to \infty$.

The centralized system’s expected profit equals to that of the single-store monopoly either when $T = 1$ (resulting with an expected profit of $\frac{1}{4}N$) or when $T$ goes to infinity (with an expected profit of $\frac{1}{2}N$). The centralized system is superior to the single-store monopoly in the intermediate cases, i.e., when $1 < T < \infty$. However, for any $T$, when $\{N^1 = 0, N^2 = N\}$, the centralized system profit coincides with that of single-store monopoly, whereas the highest profit is achieved by the centralized system when $\{N^1 = N - 1, N^2 = 1\}$ with $N \to \infty$ (Corollary 5).

The left graph of Figure 12 shows the upper and lower bounds on the profit per consumer in the two-store monopoly as a function of the selling horizon. The right graph of Figure 12 provides their ratio, which measures the maximum advantage of the two-store monopoly versus the single-store monopoly. This can also be viewed as the value of additional observations regarding the consumers’ common valuation. As can be observed, this profit advantage is maximized at $T = 3$, at which point it reaches a value of 14.28%. Yet, this advantage may be achieved only when there is a very large number of consumers where all but one visit Retail Store 1 in the first period.

We conclude this subsection with a note. A main assumption of the two-store monopoly model is that the central planner knows the distribution of the consumers across the two retail stores. However, if the realization of the distribution of consumers across the two retail stores differs from the distribution the central planner has assumed, then the system’s expected profit may be lower than the corresponding profit gained by the single store monopoly (which sets linearly decreasing prices). For example, in a two-period model, if the central planner expects $\{N^1 = 9, N^2 = 1\}$ and sets the prices accordingly, but the actual realization is $\{N^1 = 1, N^2 = 9\}$, then he achieves, in expectation, a profit of only 0.323 per consumer, which is lower than 0.333, which he could have achieved had he set the linearly decreasing prices.
5.2.2. Capacitated Setting Consider the two-period selling horizon under the capacitated setting. The capacitated setting refers to the case where the inventory stocked at each of the retail stores is not sufficient to satisfy the entire market. That is, there are \( N^i \) ad \( N^j \) consumers in the market who visit Retail Stores \( i \) and \( j \), respectively, in the first period, and the inventory stocked at each of these locations, \( K^i \) and \( K^j \), is less than \( N^i + N^j \). Assuming \( R_1^i \leq R_1^j \), the corresponding expected profit of the centralized two-store monopoly is given by:

\[
\pi = R_1^i \left( 1 - R_1^i \right) \min(K^i, N^i) + R_1^j \left( 1 - R_1^j \right) \min(K^j, N^j) \\
+ \left( R_1^i - R_1^j \right) R_2^i \left( \min(N^j, K^j - \min(K^i, N^i)) + \min(K^j, N^i - \min(K^i, N^i)) \right) \\
+ R_1^j \left( \min(K^i, N^i) R_2^j \left( 1 - \frac{R_2^j}{R_1^j} \right) + \min(K^j, N^i) R_2^j \left( 1 - \frac{R_2^j}{R_1^j} \right) \right),
\]

where the first two terms represent the first period profit. The third term is the expected profit in the second period if in the first period a sale has occurred at Retail Store \( i \), where a possibly lower price is posted. The corresponding second period price in that case is \( R_1^j \). Specifically, given a sale at Retail Store \( i \), the centralized planner infers that the common valuation is distributed \( U[R_1^i, R_1^j] \), and, by the Appendix, sets \( \max \left( \frac{1}{2} R_1^i, R_1^j \right) \). Solving the unconstrained model we find that \( R_1^i > \frac{1}{2} R_1^j \). Therefore, given a sale at Retail Store \( i \), the centralized planner sets the second period price in that case to \( R_1^i \). The last term in that expression is the expected profit when neither stores sells in the first period.

For simplicity, assume \( K^i = K^j \equiv K \) and \( N^i = N^j \equiv N \) and let \( \alpha \equiv \frac{K}{N} \). Solving backwards, we find that the second period price when neither stores sells is \( \frac{1}{2} R_1^i \), and we obtain that the optimal first period prices and corresponding profit are

\[
\{ R_1^i, R_1^j \} = \begin{cases} \left\{ \frac{3}{8}, \frac{5}{8} \right\}, & \text{if } \alpha \leq \frac{1}{2}, \\
\left\{ \frac{2}{3}, \frac{2}{3} \right\}, & \text{if } \frac{1}{2} \leq \alpha \leq \frac{2}{3}, \\
\left\{ \frac{2}{3}, \frac{5}{8} \right\}, & \text{if } \frac{2}{3} \leq \alpha \leq \frac{3}{4}, \\
\left\{ \frac{3}{8}, \frac{3}{8} \right\}, & \text{if } \alpha > \frac{3}{4}, \\
\left\{ \frac{3}{8}, \frac{5}{8} \right\}, & \text{if } \alpha > \frac{3}{4}, \\
\left\{ \frac{3}{8}, \frac{3}{8} \right\}, & \text{if } \alpha > \frac{3}{4}, \\
\end{cases}
\]

\[
\pi = \begin{cases} \frac{7}{10} K, & \text{if } \alpha < \frac{1}{2}, \\
\frac{\alpha^2 (\alpha - 4)}{2 (3 \alpha^2 - 6 \alpha + 1)} N, & \text{if } \frac{1}{2} \leq \alpha \leq \frac{2}{3}, \\
\frac{7}{3} K, & \text{if } \frac{2}{3} \leq \alpha \leq \frac{3}{4}, \\
\frac{1 - 4 \alpha}{2 (3 \alpha^2 - 6 \alpha + 1)} N, & \text{if } \frac{3}{4} \leq \alpha \leq 2, \\
\frac{7}{10} N, & \text{Otherwise}, \\
\end{cases}
\]

respectively.

6. Summary and Future Work

In this paper we have presented and analyzed a multi-period model of competition between two retailers with fixed inventories, under the assumption that each of the consumers visits only one of the retailers in any given period. If the posted price is above the consumer’s valuation, which is assumed to be uniformly distributed, in the following period he visits the competing retailer.
This model setting was also studied when (i) consumers’ valuations have a power distribution, (ii) retailers may not be able to satisfy the entire demand (capacitated setting), (iii) consumers are identical rather than similar, and (iv) the two retail stores are owned by a central planner, who may elect to post different prices in his two stores.

Our main conclusion, which was found to be quite robust, is that competition exerts significant pressure on prices and that it reduces, very significantly, the pre-competition profit of the monopolist. Our results also lead to several managerial insights. Specifically, under zigzag competition the initial price should be dramatically lower compared to the initial price set by a virtual monopolist. A firm which ignores competition, may overprice in the first period, and, as a result, may lose a substantial part of the expected profit. Similarly, retailers under competition should mark down quite substantially. Indeed, in our uncapacitated model, retailers mark down the price roughly by half in consecutive periods, which may explain the short life cycles of some products in competitive markets.

Additionally, our finding suggest that it is important for retailers to find the level of demand in the market before embarking on their markdown policy. Specifically, if the system is capacitated, in the sense that the retailers can only satisfy part of the market demand, then the effect of competition on prices is more moderate. Finally, we find that some of our results depend on whether consumers are similar or identical, or, to some extent the level of consumers’ homogeneity. In both cases, zigzag competition leads to roughly equal exponentially declining price trajectories. However, by contrast with the case of similar consumers, when consumers are identical there are multiple SPNE in prices, some of which are non symmetric, and it is not clear which equilibrium prices will be realized in the market. Additionally, by contrast with the case of similar consumers, a centralized retail chain consisting of two stores, can benefit by posting different prices in the two stores. Indeed, pricing according to our proposed scheme could gain up to about 14% more relative to the expected profit under identical pricing at both stores. The main managerial insight stemming from the analysis of the two-store retail chain is that when consumers are identical, it is best for the planner to keep a small test market in every period to test a lower price for the good.

Our model sets the ground for a richer context model wherein retailers may choose initially their inventory levels and then compete over prices over the selling horizon. Clearly, additional variants of the model can be explored, such as allowing for replenishment during the selling horizon, incorporating holding and wholesale costs as well as depreciation of products. Another possible extension is to study the impact of loyal consumers (e.g., frequent flyer programs or captive consumers), or to let the two competing retailers engage in marketing efforts in order to raise their
initial market share. In this case the number of consumers that visit each of the retail stores in the first period is a function of the marketing efforts of both retailers. Finally, as mentioned in Subsection 3.3, an extension of our zigzag competition model, which allows for more general visit pattern by consumers, is pursued in our sequel paper (GGM, 2006). Such an extension allows us to study, among other things, the effect of different levels of competition intensity on prices and profits.

Acknowledgments
The authors would like to thank Professor Charles B. Weinberg and two anonymous referees for their valuable comments. This research was partially supported by Natural Sciences and Engineering Research Council of Canada (NSERC) grants and a Social Sciences and Humanities Research Council of Canada (SSHRC) grant.

References


Appendix. Single Store Monopoly with a Consumer’s Valuation drawn from a Uniform $[a,b]$ Distribution.

Lazear (1986) studied the single store monopoly case with a consumer whose valuation is assumed to be drawn from a Uniform $[0,1]$ distribution. Hereby, we generalize his model and assume that the consumer’s valuation is drawn from a uniform distribution over $[a,b]$. In that case, if the consumer has not purchased the good by period $t$, then the probability that his valuation is below the price posted in period $t$, $R_t$, is given by $F_t(R_t) = \frac{F(R_t)}{R_{t-1} - a} = \frac{R_t - a}{R_{t-1} - a}$. The profit-to-go in period $t$ is given by $\pi_t = R_t[1 - F_t(R_t)] + F_t(R_t)R_{t+1}[1 - F_{t+1}(R_{t+1})] + F_t(R_t)F_{t+1}(R_{t+1})R_{t+2}[1 - F_{t+2}(R_{t+2})] + \cdots$. By solving the model backwards, we show:

**Proposition 14.** In a single store monopoly with a selling season of $T$ periods and a single consumer whose valuation is drawn from a Uniform $[a,b]$ distribution, both the price, $R_t$, and the profit-to-go, $\pi_t$, in each period can be written in a recursive form as follows:

$$R_t = \begin{cases} \frac{T-t+1}{T-t+2} R_{t-1} + a, & \text{if } R_{t-1} > (T-t+2)a, \\ \frac{T-t+1}{T-t} R_{t-1} + a, & \text{Otherwise,} \end{cases}$$

$$\pi_t = \begin{cases} \frac{(T-t+1)(R_{t-1})^2}{2(T-t+2)(R_{t-1} - a)}, & \text{if } R_{t-1} > (T-t+2)a, \\ \frac{(T-t+1)(R_{t-1})^2}{2(T-t+1)(T-t+2)}(R_{t-1} - a)^2, & \text{Otherwise.} \end{cases}$$

**Proof.** By induction. To verify that the Proposition holds for $t = T$, we solve: $\pi_T = \max \{ R_T[1 - F_T(R_T)] \} = \max \{ R_T[1 - \frac{F(R_T)}{R_T}] \}$. We take the derivative w.r.t. $R_T$ and equate to zero: $\frac{\partial \pi_T}{\partial R_T} = 0 \Rightarrow R_T = \max(a, \frac{1}{2} R_{T-1})$, or, $R_T = \begin{cases} \frac{1}{2} R_{T-1}, & \text{if } R_{T-1} > 2a, \\ a, & \text{Otherwise,} \end{cases}$ and $\pi_T = \begin{cases} \frac{(R_{T-1})^2}{4(R_{T-1} - a)^2}, & \text{if } R_{T-1} > 2a, \\ \frac{a}{R_{T-1} - a}, & \text{Otherwise.} \end{cases}$

Assume the Proposition holds for period $t$. We have to show it holds for $t-1$. In the first case, when $R_{t-1} > (T-t+2)a$, we solve $\max \{ \pi_{t-1} \} = \max \{ R_{t-1}[1 - F_{t-1}(R_{t-1})] + \pi_{t-1}F_{t-1}(R_{t-1}) \} = \max \{ R_{t-1} \frac{R_{t-1} - 2R_{t-1} + a}{2(T-t+3)(R_{t-1} - a)} \}$, which results, after taking the derivative w.r.t $R_{t-1}$ and setting to zero, with:

$$R_{t-1} = \frac{T-t+2}{T-t+3} R_{t-1} + a$$

and $\pi_{t-1} = \frac{(T-t+1)(R_{t-1})^2}{2(T-t+3)(R_{t-1} - a)^2}$ as required.

When $R_{t-1} \leq (T-t+2)a$, we solve $\max \{ \pi_{t-1} \} = \max \{ R_{t-1}[1 - F_{t-1}(R_{t-1})] + \pi_{t-1}F_{t-1}(R_{t-1}) \} = \max \{ R_{t-1} \frac{R_{t-1} - 2R_{t-1} + a}{2(T-t+3)(R_{t-1} - a)} \}$. Equating the derivative w.r.t. $R_{t-1}$ to zero and solving for $R_{t-1}$ gives us: $R_{t-1} = \frac{(T-t+1)(R_{t-1})^2}{T-t+3}$ and the corresponding profit-to-go is: $\pi_{t-1} = \frac{R_{t-1} - 2(T-t+1) + a}{2(T-t+3)}$, as required.

The total expected profit of the retailer, which coincides with his profit-to-go in period 1, is given by:

$$\pi = \begin{cases} \frac{T}{2} + \frac{b^2}{2(T-t+1) + a}(T-t+1), & \text{if } b > (T+1)a, \\ \frac{T}{2} + \frac{b^2}{2(T-t+1) + a}(T-t+1), & \text{Otherwise,} \end{cases}$$

and the price in period $t$ is given by:

$$R_t = \begin{cases} \frac{T-t+1}{T-t+2} R_{t-1} + a, & \text{if } b > (T+1)a, \\ \frac{T-t+1}{T-t+2} R_{t-1} + a, & \text{Otherwise.} \end{cases}$$

**Proof of Proposition 2:** By contradiction, assume that $s_{t+1}^1 = \frac{1}{4(1-s_{t+1}^2)} \leq s_{t+1}^2 = \frac{s_{t+1}^2}{4(1-s_{t+1}^2)^2}$. Rearranging yields: $1 \leq s_{t+1}^1 + s_{t+1}^2$, which is a contradiction by Lemma 2. $\square$
Proof of Proposition 3: It can be shown that the normalized sales-to-go of the single retailer in Lazear’s model, $s_t^i$, can be expressed in a recursive form as follows: $s_t^i = \frac{1}{4(1-s_{t-1}^i)}$.

By induction we prove that $s_t^i \geq s_t^1 + s_t^2$. Observe that $s_t^2 = \frac{1}{4} = s_t^1 + s_t^2$, and that $s_t^2 = \frac{1}{3} > \frac{1}{16} + \frac{1}{16} = s_t^2_{t-1} + s_t^2_{t-1}$. Assume that $s_{t+1}^i \geq s_{t+1}^1 + s_{t+1}^2$, and we will show that $s_{t+1}^i = \frac{1}{4(1-s_{t+1}^i)} \geq \frac{1}{4(1-s_{t+2}^i)} + \frac{s_{t+1}^2}{4(1-s_{t+2}^i)} = s_t^1 + s_t^2$. The last inequality holds if and only if $s_{t+1}^i \geq \frac{s_{t+1}^1 + s_{t+1}^2 + (s_t^2 - s^2_{t+1})^2}{1 + s_{t+1}^1 - s_{t+1}^2}$, which holds, since we have $s_{t+1}^i \geq s_{t+1}^1 + s_{t+1}^2$. Observe that $\pi_t^i$ for the inductive step can be shown using (6) to be 0.2718 and 0.0803, respectively.

Proof of Proposition 4: Since $B_T = B_{T-1} = \frac{1}{2}$, and $B_T = \frac{8}{15}$, the assertion holds for the first three elements of the sequence, and using (8), one can use induction to establish the bounds of 0.5 and 0.55 for every element in the sequence. Thus, it remains to prove that $\{B_t\}$ is increasing for all $t$, $t \leq T - 1$. The proof of this part is also by induction. We have $B_T - B_{T-1} = \frac{1}{2} = 1$, and $B_T - B_{T-2} = \frac{8}{15} = 1 \frac{1}{15} > 1$. Assume that $\frac{B_{t-2}}{B_{t-1}} = 1$ and $\frac{B_{t-1}}{B_t} \geq 1$. To complete the induction proof, we need to show that $\frac{B_{t-1}}{B_t} > 1$. By contradiction, assume that $\frac{B_{t-1}}{B_t} < 1$. Then, by substituting the expressions of $B_{t-1}$ and $B_{t-2}$, using (8), we obtain: $B_{t-1} - B_t = \frac{2-B_t}{B_t} < 1$. Since the denominator is positive, the last inequality can be rewritten as follows: $(\frac{B_{t-1}}{B_t})^2 \frac{B_{t-1}}{B_t} < 1$, which contradicts the induction assumption as both ratios are larger or equal to one. Thus we conclude that $\frac{B_{t-1}}{B_t} \geq 1$. □

Proof of Proposition 5: Since $s_t^1$ (resp., $s_t^2$) coincides with the expected profit of Retailer 1 (resp., 2), it suffices to prove the Proposition by using $\{s_t^1\}$ and $\{s_t^2\}$. By Proposition 1, $\{s_t^1\}$ and $\{s_t^2\}$ are monotone convergent sequences and their bounds are provided in Lemma 2. The convergence values of $\{s_t^1\}$ and $\{s_t^2\}$ can be shown using (6) to be 0.2718 and 0.0803, respectively. □

Proof of Proposition 6: The proof is by induction. For the initial step, $T$ may be even or odd, and for the inductive step $t$ may also be even or odd. We prove the Proposition for the case of $T$ even and the inductive $t$ is odd. The proof of the other cases is similar.

For $t = T$, the assertion holds since $\pi_T^{N^1, N^2} = P_T^{N^2} R_T + 2 P_T^{N^2} R_T + \cdots + N_T^{N^2} P_T^{N^2} R_T = R_T^{N_T^{N^2}} \sum_{n=1}^{N_T^{N^2}} \left( n P_n^{N_T^{N^2}} \right) = R_T^{N_T^{N^2}} [1 - F_T(R_T^2)] = N_T^{N_T^{N^2}}.$

Next, since the inductive $t$ is odd, $t + 1$ is even. Thus, assume that $\pi_{t+1}^{N^1, N_T^{N^2}} = N^1_{t+1} \pi^2_{t+1} + N^{1}_{t+1} \pi^1_{t+1}$. To complete the induction proof we need to show that $\pi_{t+1}^{N^1, N^2} = \pi_{t+1}^{N^1} + \pi_{t+1}^{N^2}$ holds as well:

$$\pi_{t+1}^{N^1, N^2} = \sum_{n=0}^{N^1_{t+1}} \left( P_n^{N^1} (n R_T + \sum_{n=0}^{N^2_{t+1}} P_n^{N^2} (N^1_{t+1} - n) \pi^2_{t+1} + (N^1_{t+1} - n) \pi^1_{t+1}) \right)$$

$$= \sum_{n=0}^{N^1_{t+1}} \left( P_n^{N^1} (n R_T + \sum_{n=0}^{N^2_{t+1}} P_n^{N^2} (N^1_{t+1} - n) \pi^2_{t+1} + (N^1_{t+1} - n) \pi^1_{t+1}) \right)$$
have

where the second equality follows by the induction step and the last one follows from the recursive expressions in (3). □

Proof of Theorem 2: Hereby we analyze the three possible cases.

Case 1: $1 \leq \alpha \leq 2$ (i.e., $N \leq K \leq 2N$)

The profit expressions are $\pi^i = NR_1^i(1-R_1^i) + (R_1^i - R_1^i)R_1^i(K-N) + R_1^iNR_2^i(1-R_1^i) + R_1^iNR_2^i(1-R_1^i)$ and $\pi^j = NR_1^j(1-R_1^j) + R_1^jNR_2^j(1-R_1^j)$. Solving backwards, we find that $R_1^2 = R_1^2 = \frac{1}{2} R_1^1$. Solving for the first period, we have $R_1^1 = \min(R_1^1, \frac{2(N+R_1^1(K-N))}{4N-K})$ and $R_1^1 = \max(\frac{1}{2}, R_1^1)$. Thus, we have that for $\frac{1}{2} \leq \alpha \leq 2$ the pure-strategy SPNE is $R_1^1 = \frac{1}{2} R_1^1$, $R_1^2 = \frac{1}{2}$, $R_1^2 = \frac{1}{2}$, and the symmetric SPNE wherein $i$ and $j$ are exchanged, and for $1 \leq \alpha \leq \frac{3}{2}$ any $\frac{1}{2} \leq R_1^1 \leq \frac{3}{2}$ and $R_1^2 = \frac{1}{2}$ is an SPNE.

Case 2: $\frac{1}{2} \leq \alpha \leq 1$ (i.e., $\frac{1}{2} N \leq K \leq N$)

The profit functions are $\pi^i = K R_1^1(1-R_1^1) + R_1^1KR_2^1(1-R_1^1)$ and $\pi^j = K R_1^j(1-R_1^j) + (R_1^j - R_1^j)R_1^j (N-K) + R_1^jKR_2^j(1-R_1^j)$. As before, $R_1^2 = R_1^2 = \frac{1}{2} R_1^1$. Solving for the first period, we have $R_1^1 = \min(R_1^1, \frac{3}{2})$ and $R_1^1 = \max(\frac{1}{2} + \frac{K(1-\alpha)}{2a}, R_1^1)$. Thus, for $\frac{1}{2} \leq \alpha \leq \frac{3}{2}$, the SPNE is $R_1^1 = \frac{1}{2}$, $R_1^2 = \frac{3}{2}$, and $R_1^2 = \frac{3}{2}$; and for $\frac{3}{2} \leq \alpha \leq 1$ the SPNE is $R_1^1 = \frac{1}{2}$, $R_1^2 = \frac{3}{2}$, and $R_1^2 = \frac{3}{2}$.

Case 3: $\alpha \leq \frac{1}{2}$ (i.e., $K \leq \frac{1}{2} N$)

The profit functions are $\pi^i = K R_1^1(1-R_1^1) + R_1^1KR_2^1(1-R_1^1)$ and $\pi^j = K R_1^j(1-R_1^j) + (R_1^j - R_1^j)R_1^j (N-K) + R_1^jKR_2^j(1-R_1^j)$. As before, $R_1^2 = R_1^2 = \frac{1}{2} R_1^1$. That is, the price is the same is in Case 3 when $\alpha = \frac{1}{2}$. Therefore, $R_1^1 = \frac{1}{2}$, $R_1^2 = \frac{1}{2}$, and $R_1^2 = \frac{1}{2}$.

Proof of Lemma 3: By induction. From Figure 11 it is verified that the Lemma holds for the last three periods. Assume the Lemma holds for period $t+1$ and we will show that it holds also for period $t$. Setting the derivatives of $\pi_t$ (as given in display (11)) with respect to $R_t^{N^1}$ and $R_t^{N^2}$ to zero and solving simultaneously, results with:

$$R_t^{N^2} = \frac{N + (T-t)N}{(T-t)(2N^2 + N^3 - 2N+1) + 2(N - S_{t+1})} R_{t+1}^{N^1},$$

(13)
Since the denominator of the right hand side is positive,\( S_t = \frac{N^2 + (T - t)(N^2 - 2S_{t+1}) + N}{(T-t)(2N^2 + N^2 - 2S_{t+1}) + 2(N - S_{t+1})} R_t^{N^2}. \) (14)

Plugging (13) and (14) into (11) we find that: \( \pi_t = -\frac{N^2 + (T - t)(N^2 - 2S_{t+1}) + N}{2((T-t)(2N^2 + N^2 - 2S_{t+1}) + 2(N - S_{t+1}))} R_t^{N^2}. \) Thus, \( S_t = \frac{1}{2} \frac{(N^2 + (T - t)(N^2 - 2S_{t+1}) + N}{(T-t)(2N^2 + N^2 - 2S_{t+1}) + 2(N - S_{t+1})}, \) which is a scalar, and the proof is complete. □

**Proof of Lemma 4:** By induction. In the last period the normalized sales-to-go is \( S_T = \frac{1}{2} N < \frac{1}{2} N. \) Assume the Lemma holds for period \( t + 1 \) and it remains to show that \( S_t \leq \frac{1}{2} N. \) By contradiction, assume that \( S_t > \frac{1}{2} N. \) Since the denominator of \( S_t \), as given in (12), is positive, then, it can be shown that \( S_t = \frac{1}{2} N = \frac{(2S_{t+1} - N)(N^2 + N^2(T-t))}{2[(T-t)(2N^2 + N^2 - 2S_{t+1}) + 2(N - S_{t+1})]} \leq 0, \) which contradicts the assumption that \( S_t > \frac{1}{2} N. \) □

**Proof of Lemma 5:** Using (13) and (14), \( R_t^{N^2} = (T - t + 1)R_t^{N^2} \) holds if \( \frac{(N^2 + (T - t)(N^2 - 2S_{t+1}) + N}{(T-t)(2N^2 + N^2 - 2S_{t+1}) + 2(N - S_{t+1})} R_t^{N^2} \leq \frac{(N^2 + (T-t)(N^2 - 2S_{t+1}) + N}{(T-t)(2N^2 + N^2 - 2S_{t+1}) + 2(N - S_{t+1})} R_t^{N^2}. \) Since the denominator in both sides is positive (follows from Lemma4), the last inequality holds only if \( (T-t)(2N^2 + N^2 - 2S_{t+1}) + N \leq (T-t)(N + (T-t)N^2), \) or, equivalently, \( 0 \leq (T-t)(N + (T-t)N^2), \) which certainly holds. □

**Proof of Proposition 12:** By Lemma 4, \( \{S_t\} \) is bounded from above. It remains to show that for each \( t, \frac{S_t}{S_{t+1}} \geq 1. \) By contradiction, assume that \( \frac{S_t}{S_{t+1}} < 1. \) Using (12), \( \frac{S_t}{S_{t+1}} = \frac{(N^2 + (T-t)(N^2 - 2S_{t+1}) + N}{2S_{t+1}((T-t)(2N^2 + N^2 - 2S_{t+1}) + 2(N - S_{t+1}))} \). Since the denominator of the right hand side is positive, \( \frac{S_t}{S_{t+1}} < 1 \) implies that \( (T-t)(N - 2S_{t+1})^2 < 0, \) which is a contradiction. □

**Proof of Proposition 13:** \( S_T = \frac{1}{2} N \) and thus \( \frac{\partial S_T}{\partial N} = 0, \) since \( N \) is constant. Assume that the assertion holds for \( t + 1. \) That is, \( \frac{\partial S_{t+1}}{\partial N^2} \geq 0. \) To complete the proof we need to show that \( \frac{\partial S_t}{\partial N^2} \geq 0. \) Since \( N^2 = N - N^1, \) using (12), we can express \( S_t \) as follows: \( S_t = \frac{1}{2} \frac{(N^2 + (T-t)(N^2 - 2S_{t+1}) + N}{(T-t)(2N^2 + N^2 - 2S_{t+1}) + 2(N - S_{t+1})} \). Thus, we want to show that \( \frac{\partial S_t}{\partial N^2} = \frac{\partial S_{t+1}}{\partial N^2} = \frac{\partial S_{t+1}}{\partial N^2} \) holds if \( \frac{\partial S_{t+1}}{\partial N^2} \geq 0. \) Simplifying, this holds if \( \frac{\partial S_{t+1}}{\partial N^2} \geq \frac{\partial S_{t+1}}{\partial N^2} \), which certainly holds since the right hand side is negative. □