Pricing Counterparty Risk Using Good Deal Bounds

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Abstract

We develop a method for pricing counterparty risk by using good deal bounds. The method imposes a new restriction in the arbitrage free model by setting upper bounds on the Sharpe ratios of the assets. The potential prices which are eliminated represent unreasonably good deals. The constraint on the Sharpe ratio translates into a constraint on the stochastic discount factor. Thus, one can obtain tight pricing bounds. Previous literature on counterparty risk and good deal bounds involved structural models. We allow for counterparty risk to be given by intensity-based models. Also, previous literature on counterparty risk with intensity models uses pricing directly under the risk neutral measure - which is not unique. We provide a link between the objective probability measure and the range of potential risk neutral measures which has an intuitive economic meaning. Also, we study numerically the tightness of the bounds and underline the use of good deal bounds for risk management. In this context, we also study portfolio effects on the good deal bounds prices.

Keywords: incomplete markets, good deal bounds, vulnerable options, counterparty risk, coherent risk measure, over-the-counter derivatives market.

JEL classification codes: C61, G13, G19

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1 Introduction

Counterparty risk has been brought to the forefront by recent events. The current financial crisis has underlined the importance of good pricing and risk management tools for counterparty risk. This paper approaches the issue by developing tools which address the market incompleteness due to the counterparty risk.

In the context of derivatives, the source for counterparty risk is the fact that the products are traded over-the-counter (OTC). According to the Bank of International Settlements, in December 2007, the OTC notional amounts outstanding were 417 trillion US dollars. By comparison, at the end of the same period, the notional amounts outstanding in exchange traded futures were 28 trillion US dollars and the notional amounts outstanding in exchange traded option were 52.5 trillion. Since the market for OTC derivatives is big, managing counterparty risk for OTC derivatives is essential\footnote{source: BIS report \textit{Statistical Annex to Quarterly review Sep 08}}. If traded on an organized exchange, the counterparty risk associated with the derivatives disappears due to the presence of the market maker. The market incompleteness comes from not having liquidly exchange-traded financial products (credit derivatives) that would help pin down the market price of risk for the counterparty’s default. This is a classic case of market incompleteness.

As a way of solving the pricing issues raised by the market incompleteness, I propose the good deal bounds method. The method imposes a new restriction in the arbitrage free model by setting upper bounds on the Sharpe ratios of the assets. The potential prices which are eliminated represent unreasonably good deals. The constraint on the Sharpe ratio translates into a constraint on the stochastic discount factor. Thus, one can obtain tight pricing bounds. One has to note that by eliminating unusually good deals, we do not eliminate extreme market outcomes, but extreme attitudes toward risk (i.e. investors asking for extreme compensation for the risks taken).

To put good deal bounds in a general context, we remember that one of the consequences of having an incomplete market setup is the fact that we no longer have a unique stochastic discount factor or a unique equivalent martingale measure, and consequently not a unique price. One could simply calculate the bounds of the prices, generated by the interval of all possible risk-neutral measures (or all possible stochastic discount factors). These bounds are known as the no-arbitrage bounds. However, they are too large to be of any practical use.

Another alternative would be to pick one of the possible equivalent martingale measures, according to some criterium, chosen by the researcher/implementer of the model. The literature adopting this path is vast. For further reference to different strands of literature
dealing with this approach see Schweizer (2001), Henderson and Hobson (2004), Barrieu and Karoui (2005). However, there is no clear cut way of choosing between different criteria and some of them are somewhat ad-hoc, in the sense that they do not have a clear economic interpretation.

In contrast to this, Cochrane and Saa-Raquejo (2000) proposed the method of good deal bounds. The good deal approach aims at obtaining an interval of “reasonable” prices in incomplete markets, rather than concentrating at obtaining a unique price. Since the no-arbitrage bounds are too large to be used, Cochrane and Saa-Raquejo (2000) suggested to rule out not only arbitrage opportunities, but also trade opportunities which are too favorable to be observed on a real market. These unrealistically-favorable deals are considered “too good to be true”, hence the name of “good deal bounds” (GDB). One possible measure for the “goodness” of a deal is its Sharpe Ratio (SR), and thus, trades/portfolios which have a SR above a certain threshold are eliminated. Since the SR links the return of financial assets to the risk undertaken, it is not extreme events which are eliminated from the set, but extreme compensation for the risk undertaken. The SR is chosen as a measure for the “goodness of the deal” because of its intuitive meaning, but also due to a large empirical literature which can tell us the range of the Sharpe Ratios observed on the market. Thus, the bound on the SR will not be arbitrary. The procedure reduces the set of possible prices for the claims traded. Hence, the good-deal bounds methodology leads to a much tighter interval of possible prices than the bounds obtained by no-arbitrage.

The next step in developing a theory for “good deal bounds” was done by Björk and Slinko (2005). They proposed a new frame for solving the optimization problem defined by Cochrane and Saa-Raquejo (2000) while at the same time allowing for more complex dynamics for the underlying assets, such as jump-diffusion processes, to be taken into account. This formulation of the good deal bounds will be used in the current project.

Previous literature on counterparty risk and good deal bounds involved structural models (e.g. Hung and Liu 2005). We allow for counterparty risk to be given by intensity-based models, which is a standard tool in credit-risk pricing and management. Also, previous literature on counterparty risk with intensity models uses pricing directly under the risk neutral measure - which is not unique (e.g. Brigo and Masetti 2005, Brigo and Pallavicini 2008). I provide a link between the objective probability measure and the range of potential risk neutral measures which has an intuitive economic meaning. Furthermore, I study how the interval of prices induced by the good deal bounds changes with different important parameters in the model: i.e. the current intensity of default, the parameters of the intensity process, the good deal bound constant chosen by the modeler, the recovery rate. Results show
that the current intensity of default and the recovery rate impact the GDB price interval more than the chosen GDB constant.

Besides the theoretical interest in the link between the risk neutral and objective probability measure, calculating good deal bounds can be useful from a risk management perspective. The good deal bound pricing problem can be reformulated as follows: we are trying to find the highest and the lowest arbitrage free pricing processes, subject to an upper bound on the norm of the market price for risk present on the market. This means that one can use the lower good deal bound as a measure for how low one can expect for the price of a derivative or a portfolio of derivatives to fall, when counterparty risk is taken into account, provided that there are no other changes in the underlying financial product. I prove that this bound is a coherent risk measure according to Artzner P. and Heath (1999).².

I study how stable are the pricing measures induced by good deal bounds in the context of introducing new financial products in a portfolio. Alternatively, keeping the set of financial products traded fixed, I study the quantitative effect of a new counterparty for the pricing measure of the lower good deal bound prices. These investigations are necessary in order to assess how useful is the lower GDB price as a measure of counterparty risk in a more complex setting. The measure is stable with respect to the introduction of new assets traded with existing counterparties. Since we do not have a good model for the correlation of defaults, the lower good deal bounds price is also sensitive to this general drawback of the credit derivatives literature.

The paper is organized as follows. First, I present the GDB methodology. Then, I use vulnerable options as an example for implementing GDB and analyze numerically the results. As a next step, I analyze good deal bounds in the context of portfolio management and introduce the lower good deal bound as a coherent risk measure for portfolio management. Then, I conclude.

2 Good Deal Bounds

One of the main limitations when pricing counterparty risk is the assumption that either the assets of our counterparty or a credit derivative (e.g. a credit default swap, CDS) on our counterparty are liquidly traded on an exchange. In practice, CDS-es are traded OTC and thus bear counterparty risk themselves. This means that it is difficult to pin down whether

²The link between GDB and coherent risk measures was first noticed by Jaschke and Küchler (2001). However, he excludes SR based good deal bounds. We show that under the framework of Björk and Slinko (2005), SR based good deal bounds are coherent risk measures as well
a change in the CDS spread is due to a change in the risk of default of the CDS name or a change in the risk of default of the CDS counterparty. Figure 1 represents a comparison between a real world measure probability of default like KMV Moody’s EDF and the risk-neutral probability of default calibrated from CDS prices (without taking into account CDS counterparty risk). As we see, the risk neutral probability of default varies much more than the objective one and we cannot clearly separate the cause (changing measure effect or additional risk undertaken through the CDS trade). However, since we have one asset (the CDS) and two sources of randomness, we are still in an incomplete market setup.

In order to deal with the market incompleteness, we are going to employ good deal bounds. We eliminate trade opportunities which are considered too favorable to be observed in the real markets. The elimination of the unrealistically good trades is done as follows. From the extended Hansen-Jaganathan bounds, we know that a constraint on the generalized Sharpe Ratio translates into a constraint on the market price of risk (or the Girsanov kernel for the equivalent martingale measure) - for a detailed explanation see Björk and Slinko (2005). We are going to show that these bounds are quite tight and investigate numerically how sensitive to different specific factors the bounds are.

A major difference with the good deal bounds (GDB) approach is the fact that the model is specified under $P$ - the objective probability measure. Most derivative pricing models are specified directly under $Q$ - the risk-neutral probability measure. By doing so, we do not run into the difficulty of separating the probability of default of the name of the CDS from the probability of the counterparty of the CDS, implied by the series of prices. We need, however, a good measure of the real world probability of default. One such measure is KMV Moody’s EDF measure. Among the advantages of such a measure is the fact that it is a continuous measure which does not cluster heterogeneous companies together as ratings usually do.

In the next section, we will demonstrate how to price counterparty risk in the context of vulnerable options on equity. Although interest rate derivatives are more widely traded on the OTC markets, they are also more complex products which require much more sophisticated modeling. Also, the most traded fixed income derivatives are swaps, which are two-sided deals. If we take counterparty risk into account, in the case of no recovery, the value of the swap rate for a swap with maturity $T_N$ is given by:

$$\sum_{i=1}^{N} K_p(t, T_i) I[Y_1(T_i) = 0] = \sum_{i=1}^{N} L(t, T_{i-1}, T, i)p(t, T_i)I[Y_2(T_i) = 0]$$
where $K$ is the swap rate, $p(t, T_i)$ is the price of a zero-coupon bond with maturity $T_i$, $L(t, T_{i-1}, T, i)$ is the forward LIBOR rate with maturity $T_i$, $Y_j(T_i)$ is the probability of survival up to time $T_i$ for the counterparty $j$, $j = 1, 2$. Thus, we need to take into account the probability of default of 2 counterparties and potentially the 2 different recovery rates for both participants in the transaction. By comparison, pricing a vulnerable option requires taking into account only the default risk for the writer of the option. Thus, we proceed with computing the good deal bound prices for a vulnerable option.

## 3 Example on Vulnerable Options

In this section, we are going to show how to implement the good deal bounds for vulnerable options - i.e. options where the counterparty may default. The OTC equity marked options gross market value in December 2007 was 6.2 trillion US dollars. Although a small proportion from the total derivatives transactions in the OTC markets, it is almost a fifth of the exchange traded futures market.\(^3\)

The underlying stock for our chosen derivative is traded and we choose to model the stock as a geometric Brownian motion in order to isolate counterparty risk. Jumps and stochastic volatility extensions are straightforward. However, they would add to the market incompleteness generated by counterparty risk and it would make it harder to separate the impact

\(^3\)source: BIS report *Statistical Annex to Quarterly review Sep 08*
of counterparty risk on the prices.

Our model is defined under the measure $P$. The market is formed by the stock and a risk free bank account. We also have a non-traded default indicator $Y$, which is modeled as a point process with intensity of default $\lambda$. Default occurs at the first jump of the process $Y$. In the main part of the paper, we model $\lambda$ as an affine process. Appendix A presents computations for the good deal bound problem when $\lambda$ is constant.

The assumptions we make are summarized as follows:

**Assumption 3.1**

1. Let the filtration space $(\Omega, \mathcal{F}, P, \mathbf{F})$ be given, where $\mathcal{F}$ is the internal filtration generated by the processes $W^P, \tilde{W}^P$ and $N$, defined below.

2. $W^P$ and $\tilde{W}^P$ are $P$-Wiener processes and $dW^P d\tilde{W}^P = \rho dt$. $N$ is a Cox process with predictable intensity $\lambda_t$.

3. We assume the intensity of the Cox process $\lambda_t$ to follow the dynamics
   \[
   d\lambda_t = \kappa(\theta - \lambda_t)dt + \sigma \sqrt{\lambda_t} dW^P_t
   \]

4. The market model under the objective probability measure $P$ is given by the following dynamics:
   \[
   dS_t = S_t \alpha_t dt + S_t \gamma_t d\tilde{W}^P_t
   \]
   \[
   dB_t = r B_t dt
   \]
   where $S_t$ denotes a traded stock and $B_t$ the money bank account.

5. $\alpha_t$ and $\gamma_t$ are scalar deterministic functions of time.

6. We assume a European call option is written on the stock. Default of the counterparty/writer of the option is described by the process $Y_t = N_t$. Default occurs at the first jump of the Cox process $N_t$. 


3.1 The payoff function

The payoff for a vulnerable European derivative is given by

\[
X = \begin{cases} 
\Phi(S_T), & \text{if } Y_T = 0 \\
R, & \text{if } Y_\tau > 0 \text{ for some } 0 < \tau \leq T 
\end{cases}
\]

where \( R \) denotes the recovery payoff.

We will compute the good deal bound price for the specific example of a vulnerable European call, \( \Phi(S_T) = \max[S_T - K, 0] \). However, since the reasoning carries through for more derivatives, we prefer solving for the general case as far as possible and taking the European call as an example only in the last step.

We model the recovery payoff as recovery to market value (RMV). For this type of recovery specification, the payment of the recovery is done immediately after default. Let \( \tau \) be the time of default. Define the stochastic variable \( V_t \) as the market value of the vulnerable option, conditional on no default up to time \( t \):

\[
V_t = E^Q \left[ e^{-r(T-t)} \Phi(S_T, Y_T) \bigg| \mathcal{F}_t, Y_t = 0 \right]
\]

where \( Q \) is the equivalent martingale measure under which the pricing is done, as defined in the next section. If default occurs at \( \tau \), the recovery process is equal to:

\[
R = (1 - q) V_\tau - \text{ where } 0 < q < 1 \text{ and } V_\tau = \lim_{s \uparrow \tau} V_s
\]

It was proven that, for this recovery specification, the price of a derivative with counterparty risk is:

\[
\Pi = E^Q \left[ e^{-r(T-t)} (\Phi(S_T) \mathbb{I}\{Y_T = 0\} + R \mathbb{I}\{Y_T > 0\}) | \mathcal{F}_t \right]
\]

\[
= E^Q \left[ e^{\int_t^T -(\rho_u + q\lambda_u)du} \Phi(S_T) | \mathcal{F}_t \right]
\]

Besides the mathematical convenience of RMV, the specification is generally preferred for modeling OTC derivatives counterparty risk, according to Schönbucher (2003).

3.2 Q dynamics

Any intensity-based credit risk model assumes an incomplete market setup, since we have two sources of risk and only one traded asset. Hence, we do not have a unique equivalent
martingale measure (EMM), but a whole class of potential EMM. For any potential EMM $Q \sim P$ we define $L$ by:

$$L_t = \frac{dQ}{dP} \text{ on } \mathcal{F}_T$$

The fact that $\mathcal{F}$ is the internal filtration implies that $L_t$ must have dynamics of the form:

$$dL_t = L_t h_t d\tilde{W}_t^P + L_t g_t \sqrt{\lambda} dW_t^P + L_t \varphi_t (dN_t - \lambda_t dt)$$

$$L_0 = 1$$

where $h_t$ and $g_t$ are adapted processes and $\varphi_t$ is a predictable stochastic process. We have chosen to model the Girsanov kernel corresponding to $W^P$ as $g_t \sqrt{\lambda}$ in order to preserve the affine character of $\lambda$ under the risk neutral measure.

From an economic point of view, $-h_t$ corresponds to the market price of risk for the stock, $S_t$; $\varphi$ compensates for the default event itself, while $-g_t \sqrt{\lambda}$ corresponds to the market compensation for the uncertainty over the probability of default.

From Girsanov’s theorem, it follows that:

$$d\tilde{W}_t^P = h_t dt + d\tilde{W}_t$$

$$dW_t^P = g_t \sqrt{\lambda_t} dt + dW_t$$

where $W_t$ and $\tilde{W}_t$ are Q-Wiener processes.

Also, the intensity of the Cox process becomes $\lambda_t^Q = (1 + \varphi_t) \lambda_t$. This leads to a positivity constraint on $\varphi_t$:

$$\varphi_t \geq -1$$

$S_t$ is a traded asset and, from the definition of an EMM, the drift of any traded asset under the EMM must equal the risk free interest rate. Thus, $h_t$ must satisfy the martingale condition:

$$r = \alpha_t + \gamma_t h_t$$

The class of equivalent martingale measures is defined as the class of measures obtained by (2), (3) and (4) and satisfying the conditions (5) and (6).
3.3 Optimization Problem

As mentioned in the introduction, we are trying to find the highest and the lowest arbitrage
free pricing processes, subject to an upper bound on the norm of the market price for risk, or
equivalently, a bound on the Girsanov kernel. Dealing with the market price of risk translates
to dealing with the Girsanov kernel of the equivalent martingale measures. Thus, we define
the good deal bounds as follows

**Definition 3.1** The lower good deal bound price process for a vulnerable option is defined
as the optimal value process for the following optimal control problem:

\[
\min_{h_t, g_t, \varphi_t} E_Q \left[ e^{\int_t^T - (r_u + q \lambda_u) du} \Phi(S_T) \right] | \mathcal{F}_t
\]

\[
dS_t = rS_t dt + S_t \gamma_t \tilde{W}_t
\]

\[
d\lambda_t = [\kappa (\theta - \lambda_t) + g_t \sigma \lambda_t] dt + \sigma \sqrt{\lambda_t} dW_t
\]

\[
\lambda_t^Q = \lambda_t (1 + \varphi_t)
\]

\[
\alpha_t + \gamma_t h_t = r
\]

\[
\varphi_t \geq -1
\]

\[
h_t^2 + g_t^2 \lambda_t + \varphi_t^2 \lambda_t \leq C^2
\]

The upper good deal bound process is the optimal value process for a similar optimal control
problem, with the only difference that we maximize the expression, subject to the same
constraints.

*We denote the optimal value process by* \( V(t, S_t, Y_T) \), *where* \( V \) *is the optimal value function.*

Before proceeding, let us comment on the structure of the optimization problem. The
objective function is the arbitrage-free price for the payoff function, where the expectation is
computed under the risk neutral measure generated by \( h_t, g_t \), and \( \varphi \). Since we have to select
this measure from a continuum of eligible EMM, we maximize with respect to the Girsanov
kernels.

The optimization is subject to the dynamics of the assets on the market, under the appro-
priate probability measure. The first five constraints are the usual constraints necessary for
changing the measure and establishing it as a probability measure, in general (8), and a
risk-neutral measure (7).

If all the Girsanov kernel elements could be identified from these constraints, we would be
in a complete market setup and would be able to find a unique price. Since the number of traded assets is smaller than the number of risk sources, we cannot price all the risk factors and need the last inequality in order to tighten the no arbitrage price bounds. We will refer to this inequality:

\[ h_t^2 + g_t^2 \lambda_t + \varphi_t^2 \lambda_t \leq C^2, \quad 0 \leq t \leq T \]

as the good deal bounds condition.

Notice that \( \varphi_t \) does not appear in equation (7), the condition for the drift of the stock under the martingale measure, but in equations (8) and (9). This separation of the two components of the Girsanov kernel allows us to obtain more elegant closed form solutions.

Classical control theory allows us to solve for the lower good deal bound by solving the **Hamilton Jacobi Bellman equation**, given by the following PDE:

\[
\frac{\partial V}{\partial t} + \inf_{h, g, \varphi} A^{h, g, \varphi} V - rV = 0
\]

\[
V(T, s, y, \lambda) = \Phi(S_T)
\]

where \( A \) is the infinitesimal operator of \((W, \tilde{W}, N)\):

\[
AV = V_s r + V_\lambda [\kappa (\theta - \lambda(t)) + g_t \sigma \lambda(t)]
\]

\[
+ \Delta V_{\lambda} + \frac{1}{2} \sigma^2 s^2 V_{ss}
\]

\[
+ \frac{1}{2} \sigma^2 \lambda V_{\lambda\lambda} + \gamma \sigma s \sqrt{\lambda} V_{s\lambda}
\]

where \( \Delta V = V(t, s, 1, \lambda) - V(t, s, 0, \lambda) = -qV \).

The problem for the upper bound is reduced to an similar PDE, but the inf-problem is replaced by \( \sup_{h_t, g_t, \varphi_t} A^{h, g, \varphi} V \).

The HJB equation is solved in 2 steps:

- solving for each \( t, s, \lambda \) the embedded static problem, in order to obtain the Girsanov kernel;

- solving the PDE, in order to obtain the price of the vulnerable option

Solving the static problem from the HJB equation reduces to solving the following simple problem:

\[
\min_{h, g, \varphi} \quad -qV \lambda \varphi + \sigma \lambda V_\lambda g
\]
\[ \alpha + \gamma h = r \]
\[ \varphi \geq -1 \]
\[ h^2 + g^2 + \varphi^2 \lambda \leq B^2 \]

This is solved by standard Karoush-Kuhn-Tucker and we find that the lower bound Girsanov kernel is given by:

- \( \hat{h}_t = \frac{r - \alpha t}{\gamma t} \)
- \( \hat{\varphi}_t = -qV \sqrt{\frac{C^2 - h^2}{\lambda [(qV)^2 + (\sigma V \lambda)^2]}} \)
- \( \hat{g}_t = \sigma V \lambda \sqrt{\frac{C^2 - h^2}{\lambda [(qV)^2 + (\sigma V \lambda)^2]}} \).

We notice the Girsanov kernel depends on the optimal value function \( V \). In a similar way, we can compute the upper GDB Girsanov kernel.

**Proposition 3.1** Under assumptions 3.1, the Girsanov kernel for the lower good deal bound EMM as defined in definition 3.1 is given by:

- \( \hat{h}_t = \frac{r - \alpha t}{\gamma t} \)
- \( \hat{\varphi}_t = -qV \sqrt{\frac{C^2 - h^2}{\lambda [(qV)^2 + (\sigma V \lambda)^2]}} \)
- \( \hat{g}_t = \sigma V \lambda \sqrt{\frac{C^2 - h^2}{\lambda [(qV)^2 + (\sigma V \lambda)^2]}} \).

The Girsanov kernel corresponding to the upper good deal bound EMM is given by:

- \( \hat{h}_t = \frac{r - \alpha t}{\gamma t} \)
- \( \hat{\varphi}_t = \max \left[ \frac{\Delta V \sqrt{\frac{B^2 - h^2}{\lambda (\Delta V)^2 + (\sigma V \lambda)^2}}} {L \left( \frac{-1}{R} \right)} \right] \)
- \( \hat{g}_t = \begin{cases} \sigma V \lambda \sqrt{\frac{B^2 - h^2}{\lambda (\Delta V)^2 + (\sigma V \lambda)^2}}, & \text{if } \hat{\varphi}_t = L \\ -\sqrt{B^2 - h^2 - \lambda} & \text{if } \hat{\varphi}_t = R \end{cases} \)
Now, we should plug in the above solution in the HJB equation and solve the PDE. The PDE proves to be unmanageable, and we need to employ some different techniques. We will use a first order Taylor expansion in order to approximate the solution of the HJB equation. We would like to have an approximation that incorporates the tightness of our good deal bounds constraint. Approximating it around $C$ yields explosive solutions. It turns out that the proper variable for this is $y = \sqrt{C^2 - h^2}$, where $h$ was determined by the martingale constraint.

We will do the approximation around the **minimal martingale measure** result given by:

$$\min_{h,g,\varphi} \ h^2 + g^2 + \varphi^2$$

$$\alpha + \sigma h = r$$

Formally, we define the minimal martingale measure as follows:

**Definition 3.2** Let $Q^{MM} \sim P$, we define $L$ by:

$$L_t = \frac{dQ^{MM}}{dP} \text{ on } \mathcal{F}_T$$

(10)

with dynamics of the form:

$$dL_t = L_t h_t d\tilde{W}_t^P$$

(11)

$$L_0 = 1$$

(12)

where $h_t$ is given by the equation

$$r = \alpha_t + \gamma_t h_t$$

(13)

The minimal martingale measure $Q^{MM}$ is defined as the measure obtained by (10), (11) and (12) and satisfying the condition (13).

**Remark 3.1** For a thorough analysis on the minimal martingale measure (MMM) and its properties, we refer to Schweizer (1995). Approximations of the good deal bounds solutions around the MMM were first used by Björk and Slinko (2008). In their paper, approximations of GDB are used to price derivatives on underlying with jump-diffusion dynamics and stochastic volatility. The approximations seem to perform well.
For us, the minimal martingale measure Girsanov kernels are trivially 0. This means that \( \lambda^Q = \lambda \) and the dynamics of \( \lambda \) under the minimal martingale measure do not change.

The intuition behind these approximations is as follows: we can re-write the HJB equation by incorporating the Karush-Kuhn-Tucker constraints:

\[
\begin{align*}
\frac{\partial V}{\partial t}(t,s,y,\lambda) + V_s sr + V_\lambda [\kappa (\theta - \lambda t) + g_t \sigma \lambda t] - qV \lambda (1 + \varphi_t) + \frac{1}{2} \gamma^2 s^2 V_{ss} + \frac{1}{2} \sigma^2 \lambda V_{\lambda \lambda} + \gamma \sigma s \sqrt{\lambda} V_{\lambda s} - rV(t,s,y,\lambda) + \nu [g_t^2 \lambda + \varphi_t^2 \lambda + h_t^2 - C^2] &= 0 \\
\end{align*}
\]

where \( y \) is defined as above. If we replace \( g, h \) and \( \varphi \) by \( \hat{g}, \hat{h} \) and \( \hat{\varphi} \) as computed above, the solution of (14) is the lower good deal bound price.

When we compute the lower good deal bound price by approximating around the minimal martingale solution, we obtain:

\[
V_{LGB} = V_{MM} + (y_{MM} - y) \frac{\partial V}{\partial y}(y_{MM})
\]

where \( V_{LGB} \) denotes the lower GDB price, \( V_{MM} \) the minimal martingale price and \( \frac{\partial V}{\partial y} \) represents the sensitivity of \( V \), the solution of PDE (14), with respect to the variable \( y \). The variable \( y_{MM} \) is taken to be zero. This means that the above equation translates into

\[
V_{LGB} = V_{MM} - \sqrt{C^2 - h^2} \frac{\partial V}{\partial y}(0)
\]

Hence, we need to compute two objects: \( V_{MM} \) and \( \frac{\partial V}{\partial y}(0) \). The price under the minimal martingale measure is given by:

\[
V_{MM} = E_t^{MM} \left[ \exp \left\{ - \int_t^T (r_u + q \lambda_u) du \right\} \Phi(S_T) \right]
\]

where \( E_t^{MM}[\bullet] \) denotes the expectations under the minimal martingale measure. If the intensity of default \( \lambda \) and the stock price \( S \) are independent, we have a closed-form solution

\[
V_{MM} = E_t^{MM} \left[ \exp \left\{ - \int_t^T (r + q \lambda_u) du \right\} \right] E_t^{MM} [\Phi(S_T)]
\]

where

\[
dS_t = rS_t dt + S_t \gamma_t d\tilde{W}_t
\]
\[ d\lambda_t = \kappa (\theta - \lambda_t) dt + \sigma \sqrt{\lambda_t} dW_t \]

which yields

\[ V_{MM} = \exp \{-r(T - t) + [A(t, T, q) + B(t, T, q)q\lambda_t] \} E_t^{MM} [\Phi(S_T)] \]

The terms \( A(t, T, q) \) and \( B(t, T, q) \) can be computed by employing the classical machinery of affine processes. For \( \kappa, \theta, \sigma \) are constants, \( \lambda \) has CIR dynamics and

\[
\begin{align*}
B(t, T, q) &= \frac{2 \left( e^{\delta(T-t)} - 1 \right)}{(\delta + q\kappa)(e^{\delta(T-t)} - 1) + 2\delta} \gamma^2 r^2 \\
A(t, T, q) &= \left[ \frac{2\delta e^{(q\kappa+\delta)(T-t)/2}}{(\delta + q\kappa)(e^{\delta(T-t)} - 1) + 2\delta} \right]^{2q\kappa/\sigma^2}
\end{align*}
\]

where \( \delta = \sqrt{(q\kappa)^2 + 2q\sigma^2} \). If our claim is a European call, then \( \Phi(S_T) = \max[S_T - K] \) and \( E_t^{MM} [\Phi(S_T)] \) is the Black Scholes price.

If the underlying stock for the derivative and the intensity of default are not independent and \( \rho \neq 0 \), we cannot obtain the \( V_{MM} \) in closed form and need to use Monte Carlo simulations. Monte Carlo methods for affine processes and geometric Brownian motion are well known and developed.

Now, we need to compute the sensitivity factor \( \frac{\partial V}{\partial y}(0) \). First, we are going to present results for \( \rho = 0 \).

We denote \( \frac{\partial V}{\partial y}(\bullet) \) by \( Z(t, \bullet) \). We re-write HJB equation as (14) and take the first derivative with respect to \( y \). By applying the envelope theorem and some computations detailed in Appendix B, we obtain:

\[
\begin{align*}
\frac{\partial Z}{\partial t} + Z_s r + Z_{\lambda} [\kappa (\theta - \lambda_t)] + \frac{1}{2} \gamma^2 s^2 Z_{ss} + \frac{1}{2} \sigma^2 \lambda Z_{\lambda\lambda} \\
+ \rho \gamma s \sqrt{\lambda} Z_{s\lambda} - [q\lambda_t + r] Z + \sqrt{\lambda} \sqrt{(qV_{MM})^2 + \left( \frac{\partial V_{MM}}{\partial \lambda} \sigma \right)^2} = 0 \\
Z(T, s, \lambda) = 0
\end{align*}
\]
We can solve the above problem by applying Feinman-Kac. The general solution is:

\[
Z(t, s, \lambda) = \int_t^T E_{t,s}^{MM} \left[ \exp \left\{ - \int_t^u r(\tau) + q\lambda(\tau)d\tau \right\} \sqrt{\lambda_u M(u)} \right] d\tau
\]

where \(M(u) = \sqrt{(qV_{MM})^2(u, S_u, \lambda_u) + \left(\frac{\partial V_{MM}}{\partial \lambda}(u, S_u, \lambda_u)\sigma\right)^2}\).

If we have obtained \(V_{MM}\) by Monte Carlo simulation (in the general case, when \(\rho\) is different from zero), we will need to compute also the sensitivity of the solution with respect to \(\lambda\). We can use the likelihood ratio method in order to do so, as explained by Glasserman (2003).

Appendix C presents graphs that show the impact of \(\rho\) on the lower GDB price.

However, if \(\rho = 0\), then we know

\[
\frac{\partial V_{MM}}{\partial \lambda}(t) = qB(t, T, q)V_{MM}(t)
\]

and, we obtain

\[
Z(t, s, \lambda) = \int_t^T E_{t,s}^{MM} \left[ \exp \left\{ - \int_t^u r(\tau) + q\lambda(\tau)d\tau \right\} qV_{MM}(u)\sqrt{\lambda_u \sqrt{1 + B^2(u, T, q)\sigma^2}} \right] du
\]

After replacing \(V_{MM}\) in the above, we obtain:

\[
Z(t, s, \lambda) = qV_{MM}(t) \int_t^T E_{t}^{\hat{Q}} \left[ \sqrt{\lambda_u} \sqrt{1 + \sigma^2 B^2(u, T, q)\sigma^2} \right] du
\]

where \(\hat{Q}\) is as defined in Appendix B. It is a well known fact (see Glasserman 2003, Schönbucher 2003) that \(\lambda\) is non-central chi-square distributed with weighting factor

\[
\eta = \frac{q\sigma^2}{4} B(t, T, q), \quad (16)
\]

degrees of freedom

\[
\nu = \frac{\kappa\theta}{\sigma^2}, \quad (17)
\]

and non-centrality factor\(^4\)

\[
\Lambda = \frac{4}{\sigma^2} \frac{\partial}{\partial T} B(t, T, q) \lambda(t). \quad (18)
\]

\(^4\)For detailed derivations of the parameters of the non-central chi-square distribution under the \(\hat{Q}\) measure, we refer to chapter 7 from Schönbucher (2003)
This means that $\sqrt{\lambda_t}$ is non-central chi distributed with non-centrality factor $\Lambda$ and the formula for the mean is given by:

$$E^Q[\sqrt{\lambda}] = \sqrt{\frac{\pi}{2}} \eta L_{1/2}^{(\nu/2-1)} \left( -\frac{\Lambda^2}{2} \right)$$

where $L_i^{(a)}(x)$ is the generalized Laguerre polynomial and $\eta$, $\nu$ and $\Lambda$ are given by (27), (28) and (29). We can summarize our results about $Z$ in the following proposition:

**Proposition 3.2** Let assumptions 3.1 hold. Let $V$ be the optimal value function that solves the lower good deal bound problem, as defined by Definition 3.1. Let $V_{MM}$ be the price of a vulnerable option computed under the minimal martingale measure defined by Definition 3.2.

- The derivative of the optimal value function $V$ with respect to the variable $y = \sqrt{C^2 - h^2}$ and evaluated at $y_{MM} = 0$, $Z(t,s,\lambda)$, is given by the following PDE:

$$\begin{align*}
\frac{\partial Z}{\partial t} + Z_s s r + Z_\lambda [\kappa (\theta - \lambda t)] + \frac{1}{2} \gamma s^2 Z_{ss} + \frac{1}{2} \sigma^2 \lambda Z_{\lambda\lambda} \\
+ \rho \gamma s \sqrt{\lambda} Z_{s\lambda} - [q \lambda_t + r] Z - \sqrt{\lambda} \left( q V_{MM} \right)^2 + \left( \frac{\partial V_{MM}}{\partial \lambda} \sigma \right)^2 = 0
\end{align*}$$

$Z(T,s,\lambda) = 0$

- The derivative can also be found as:

$$Z(t,s,\lambda) = \int_t^T E^\hat{Q}_{t,s,\lambda} \left[ \exp \left\{ - \int_t^u r(\tau) + q \lambda(\tau) d\tau \right\} \sqrt{\lambda_u} M(u) \right] d\tau$$

where $M(u) = \sqrt{(q V_{MM}(u,S_u,\lambda_u))^2 + \left( \frac{\partial V_{MM}}{\partial \lambda} (u,S_u,\lambda_u) \sigma \right)^2}$.

- For the case when the underlying stock and the probability of default are uncorrelated ($\rho = 0$), $Z(t,s,\lambda)$ is given by:

$$Z(t,s,\lambda) = q V_{MM}(t) \int_t^T E^\hat{Q}_t \left[ \sqrt{\lambda_u} \right] \sqrt{1 + \sigma^2 B^2(u,T,q)} du$$

with $E^\hat{Q}_t \left[ \sqrt{\lambda_u} \right]$ given by equation (30).
Hence, when $\rho = 0$, equations (15) and (20) imply that the lower bound for the price of a derivative with counterparty risk is

$$V_{LGB} = V_{MM} \left[ 1 - q\sqrt{C^2 - h^2} \int_t^T E_t^Q \left[ \sqrt{\lambda_u} \right] \sqrt{1 + \sigma^2 B^2(u, T, q)} du \right]$$

We summarize results about the good deal bound prices in the proposition below.

**Proposition 3.3** Let assumptions 3.1 hold. Let $V_{MM}$ be the price of a vulnerable option computed under the minimal martingale measure defined by Definition 3.2. Let $Z$ be the derivative of the optimal value function $V$ with respect to the variable $y = \sqrt{C^2 - h^2}$ and evaluated at $y_{MM} = 0$, as in Proposition 3.2. The upper/lower good deal bound price for a vulnerable option is given by:

$$V_{U/LGB} = V_{MM} \pm \sqrt{C^2 - h^2} Z$$

where

$$V_{MM} = E_t^{MM} \left[ \exp \left\{ - \int_t^T (r_u + q\lambda_u) du \right\} \Phi(S_T) \right]$$

and

$$Z(t, s, \lambda) = \int_t^T E_{t,s,\lambda}^{MM} \left[ \exp \left\{ - \int_t^u r(\tau) + q\lambda(\tau) d\tau \right\} \sqrt{\lambda_u} M(u) \right] d\tau.$$

where $M(u) = \sqrt{(qV_{MM}(u, S_u, \lambda_u))^2 + \left( \frac{\partial V_{MM}}{\partial \lambda}(u, S_u, \lambda_u) \sigma \right)^2}$.

For the special case when $\rho = 0$ and $\kappa$, $\theta$, $\sigma$ are constants, the upper/lower good deal bound price is given by:

$$V_{U/LGB} = V_{MM} \left[ 1 \pm q\sqrt{C^2 - h^2} \int_t^T E_t^Q \left[ \sqrt{\lambda_u} \right] \sqrt{1 + \sigma^2 B^2(u, T, q)} du \right]$$

with $E_t^Q \left[ \sqrt{\lambda_u} \right]$ given by equation (30) and

$$V_{MM} = \exp \left\{ -r(T - t) + [A(t, T, q) + B(t, T, q)q\lambda_t] \right\} E_t^{MM} [\Phi(S_T)]$$

where $E_t^{MM} [\Phi(S_T)]$ is the Black Scholes price and

$$B(t, T, q) = \frac{2 \left( e^{\delta(T-t)} - 1 \right)}{(\delta + q\kappa) (e^{\delta(T-t)} - 1) + 2\delta}$$
\[
A(t, T, q) = \left[ \frac{2\delta e^{(q\kappa+\delta)(T-t)/2}}{(\delta + q\kappa)(e^{\delta(T-t)} - 1) + 2\delta} \right]^{2q\kappa\theta/\sigma^2}
\]

with \( \delta = \sqrt{(q\kappa)^2 + 2q\sigma^2} \).

### 3.4 Variation of the GDB interval for different model specifications

In this section, we are analyzing how the low good deal bound price varies with respect to several model parameters. The upper good deal bound price is either very close or identical with the Black-Scholes price. The intuition behind this fact is that the higher price we can get for an asset with counterparty risk is the one that we obtain when that particular risk is ignored.

When we analyze the sensitivity of the lower GDB price to different factors, we notice that we can group them in 2 categories: parameters specific to each transaction and parameters specific to the market environment. In the first class, we mention the initial level of the probability of default for our counterparty \( \lambda_0 \), the volatility of the intensity of default \( \sigma \), the long term level of the intensity of default \( \theta \). In the second class, one can include the size of the GDB constraint - the parameter \( C \) and, to a certain extent, the loss to default parameter \( q \).

The GDB constraint parameter \( C \) is chosen by the modeler as the bound of the Sharpe ratios for the all transactions on the market. We remember that we place an upper bound on the SR of all the portfolios that can be formed on the market consisting of the underlying assets, the derivative claim and the money account; binding the Sharpe Ratio of all possible portfolios is equivalent to using the Hansen-Jagannathan bounds, which state that the SR of all portfolios formed on the market are less or equal to the market price of risk. Thus, the choice of the GDB parameter \( C \) should be dictated in part by the characteristics of the market on which we are performing the transaction. Empirical evidence suggests that, for mature markets, a Sharpe Ratio above 2 is rare. Thus, even if \( C \) is chosen by the modeler, its choice should reflect general characteristics about the market on which we deal.

The loss to default parameter \( q \) reflects both characteristics specific to the counterparty and to the market environment - the dead weight loss due to bankruptcy procedures.

Graphs and tables for this section are presented in the Appendix C. The stock price varies between 1 and 60, the strike price is 30. We present results for the stock being 20, 30 and 50 in order to capture the effect of the moneyness of the option. In the baseline case, \( \lambda \) is 0.03,
\( \sigma \) is 0.1 and the long term intensity of default level \( \theta \) is 0.2. The good deal bound constraint \( C \) is 2.5 and the loss to default parameter \( q \) is 0.4.

First, we present the impact of changing the size of the parameter \( C \) on the lower GDB prices. As one might expect, the size of the good deal bound interval increases with the size of the parameter \( C \). This happens because, by relaxing the good deal bound constraint, we simply increase the set of the admissible equivalent martingale measures and hence the set of possible prices. Result for \( C \) equal to 2, 2.5, 3, 4 are tabulated in the appendix. For an option in the money, this translates into a change in price from 18.17 to 15.19. This is a considerable price impact.

However, when we compute the impact of the loss to default parameter, we notice it has a strong influence as well. For a loss to default 0.2, the LGDB price is 19.73 only to fall at 13.90 for a loss to default of 0.8. The higher is the loss to default, the more significant becomes the impact of the counterparty risk on the price of the derivatives.

The intensity of default when the contract is concluded, \( \lambda_0 \) plays a significant role as well. We compute the price for intensities of 0.01, 0.03, 0.1 and 0.3. The lower good deal price changes from 19.37 to 7.23. Thus, we see the choice of counterparty is crucial for the value of a derivative traded OTC. This is even more striking when we see that the parameters for the dynamics of \( \lambda \) do not have a big impact on the lower good deal bound price. The main explanation for this is the fact that default is a 1-0 phenomenon - we do not care if the intensity of default would revert in the future to a lower probability of default, as much as we care what is the probability of default in the next time interval. Once the default is realized, so are the losses and the notion of intensity of default \( \lambda \) is not meaningful anymore. Thus, the impact of \( \sigma \) and \( \theta \) on the lower good deal bound price is only of the order of decimals.

From the above exercise, we notice that the choice of GDB constraint is not the most important factor in determining the lower good deal bound price. The "amount of counterparty risk" undertaken, as reflected by the loss to default parameter and the current intensity of default, has a bigger impact on the price then the constraint parameter. The high impact of the price of \( \lambda_0 \) - the current level of the probability of default - also points out toward an "old time wisdom" - i.e. the most important part of risk management is to choose your counterparty and monitor it carefully.
4 Counterparty risk for a portfolio

In the previous sections, we have studied the effect of counterparty risk on one OTC financial derivative at a time. However, for risk management purposes, we are interested more on how good deal bounds are computed and “behave” in a portfolio framework. Before this, however, we have to check if good deal bounds meet the the requirements to be a good risk measure. Such requirements were put forward in Artzner P. and Heath (1999), and the resulting risk management instrument carries the name of coherent risk measures. They are defined as follows:

**Definition 4.1** A risk measure \( \rho : \mathcal{G} \rightarrow \mathbb{R} \) satisfying the four axioms of

a) **translation invariance:**

\[
\rho(X + \alpha r) = \rho(X) - \alpha,
\]

where \( X \) is a risky portfolio, \( \alpha \) a real number and \( r \) is a reference risk free investment.

b) **subadditivity:** for all risky portfolios \( X_1 \) and \( X_2 \in \mathcal{G} \),

\[
\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2).
\]

c) **positive homogeneity:** for all \( \lambda > 0 \) and all \( X \in \mathcal{G} \),

\[
\rho(\lambda X) = \lambda \rho(X).
\]

d) **monotonicity:** for all \( X \) and \( Y \in \mathcal{G} \) with \( X \leq Y \); we have \( \rho(Y) \leq \rho(X) \).

is called coherent.

Note that, although abstract, the requirements for coherent risk measures actually have intuitive economic meaning: translation invariance, for example, basically asks that by adding capital to our position we reduce the amount of riskiness of the position; the subadditivity property captures the beneficial effects of diversification.

Artzner P. and Heath (1999) prove that given the total return \( R \) on a reference investment, a risk measure \( \rho \) is coherent if and only if there exists a family \( \mathcal{P} \) of probability measures on the set of states of nature, such that

\[
\rho(X) = \sup \left\{ E^{P} \left[ \frac{-X}{R} \right] | P \in \mathcal{P} \right\} \tag{21}
\]
The first to notice the link between good deal bounds and coherent risk measures were Jaschke and Küchler (2001). However, they dismiss the good deal bounds on the Sharpe Ratio a la Cochrane as not satisfying the monotonicity requirement. Under the new reformulation of the GDB based on the SR done by Björk-Slinko, one can notice that the lower GDB trivially satisfies (21) and hence, it is a coherent risk measure.

In the rest of the paper, we are going to study the effect of adding more assets traded with the same counterparty to our portfolio and see how the GDB behave in this context. Then, we are going to check how adding a new counterparty is going to affect the lower good deal bound.

4.1 Good deal bounds for a portfolio with several assets against one counterparty

As mentioned above, when we deal with counterparty risk, we are more interested in the impact of the risk on the value of a portfolio rather than the impact on the price of each asset. Usually, the two notions coincide since, once the risk-neutral pricing measure is fixed, the price of a vulnerable derivative $\Pi^V(X)$ is given by

$$\Pi^V(X) = p_c \Pi(X)$$

where $\Pi(X)$ is the price of the derivative in non-vulnerable form, and $p_c$ is the additional discounting we need to do to account for counterparty risk. However, when using good deal bounds, the choice of a particular set of risk-neutral measures depends on the risk factors on our market, and introducing a new asset traded with the same counterparty changes the choice of EMM used for the pricing of the lower good deal bound.

In order to address this issue, we are going to examine what happens to the prices when we take into account more traded assets in order to fix the lower bound measure. We remember that, when we have included only the underlying of our derivative, the lower good deal bound price is given by:

$$V_{LGB} = V_{MM} - \sqrt{C^2 - h^2} Z(t)$$

(22)

where $Z_t$ is the first derivative of the pricing function $V$ with respect to $y = \sqrt{C^2 - h^2}$ and $h^2$ is the norm of the market price of risk for the traded asset. We notice that neither $V_{MM}$ nor $Z(t)$ do not depend on the value of $y$. They depend only on the traded underlying and the dynamics of $\lambda$ under $P$. Thus, it is straightforward to generalize the above expression to the case when we have $n$ traded assets. Consider a market model with $n$ stocks, a risk-free
bank account. We trade $n$ OTC derivatives against the same counterparty. Each derivative is written on one asset.

Assumption 4.1

1. Let the filtration space $(\Omega, \mathcal{F}, P, \mathbf{F})$ be given, where $\mathbf{F}$ is the internal filtration generated by the processes $W^P, \tilde{W}_i^P$, with $i = 1, \ldots, n$ and $N$, defined below.

2. $W^P$ and $\tilde{W}_i^P$ are $P$-Wiener processes and $dW^P d\tilde{W}_i^P = \rho_i dt$. $N$ is a Cox process with predictable intensity $\lambda_t$.

3. $\tilde{W}_i^P, \tilde{W}_j^P$ are independent $P$-Wiener processes; $(\tilde{W}_i^P)_{i=1,\ldots,n} = \tilde{W}^P$

4. We assume the intensity of the Cox process $\lambda_t$ to follow the dynamics

\[ d\lambda_t = \kappa(\theta - \lambda_t) dt + \sigma \sqrt{\lambda_t} dW^P_t \]

5. The market model under the objective probability measure $P$ is given by the following dynamics:

\[
\begin{align*}
    dS_t^i &= S_t^i \alpha^i_t dt + S_t^i \gamma^i_t d\tilde{W}_t^i & i = 1, \ldots, n \\
    dB_t &= rB_t dt
\end{align*}
\]

where $S_t^i$ denotes the traded stock $i$, for $i = 1, \ldots, n$ and $B_t$ the money bank account.

6. $\alpha^i_t$ are scalar deterministic functions of time and $\gamma^i_t$ are $(1,n)$ deterministic vector functions of time.

7. We assume a European call option is written on each stock. Default of the counterparty/writer of the option is described by the process $Y_t = N_t$. Default occurs at the first jump of the Cox process $N_t$.

We denote the payoff function of each European call on stock $S_T^i$ as $\Phi(S_T^i)$ and write the lower good deal problem as

\[
\min_{h,g,\varphi} E^Q \left[ e^{\int_T^{T^-} - (r_u + q\lambda_u) du} \sum_{i=1}^n \Phi(S_T^i) | \mathcal{F}_t \right]
\]

\[
\begin{align*}
    dS_t^i &= rS_t^i dt + S_t^i \gamma^i_t d\tilde{W}_t \\
    d\lambda_t &= [\kappa(\theta - \lambda_t) + g_t \sigma \lambda_t] dt + \sigma \sqrt{\lambda_t} dW_t
\end{align*}
\]
\[ \lambda_t^Q = \lambda_t (1 + \varphi_t) \]
\[ \bar{\alpha}_t + \bar{\gamma}_t \bar{h}_t = r \]
\[ \varphi_t \geq -1 \]
\[ \sum_{i=1}^{n} h_i^2 + g^2 \lambda + \varphi^2 \lambda_t \leq C^2 \]

where \( \bar{\alpha} = (\alpha^i)_{i=1\ldots n} \) is a \((n,1)\)-vector, \( \bar{\gamma} = (\gamma^i)_{i=1\ldots n} \) is a \((n,n)\) matrix and \( \bar{h} = (h^i)_{i=1\ldots n} \), \( g \) and \( \varphi \) are the Girsanov kernels for the EMM measure change:

\[ L_t = \frac{dQ}{dP} \quad \text{on } \mathcal{F}_T \]
\[ dL_t = L_t \sum_{i=1}^{n} h_i(t) d\bar{W}_i^P(t) + L_t g_t \sqrt{\lambda} dW_P(t) + L_t \varphi_t (dN_t - \lambda_t dt) \]
\[ L_0 = 1 \]

The Girsanov kernels \( h_i \) are computed as the solution to the linear equation system given by:

\[ \bar{\alpha}_t + \bar{\gamma}_t \bar{h}_t = r \]

As before, the minimal martingale measure \( \bar{h} \) does not change, but the minimal martingale measure \( g \) and \( \varphi \) are 0. We solve the problem through methods similar to the previous sections and the lower good deal bound portfolio value \( V_{LGB}^n \) becomes:

\[ V_{LGB}^n = V_{MM} - \sqrt{C^2 - \sum_{i=1}^{n} h_i^2 Z(t)} \]

where \( V_{MM} \) is the sum of the individual \( V_{MM}^i \) computed as in Proposition 3.3 and \( Z(t) \) computed as in Proposition 3.3 for the new \( V_{MM} \).

**Proposition 4.1** Let assumptions 4.1 hold. Let \( V_{MM} \) be the price of a vulnerable portfolio of options computed under the minimal martingale measure. Let \( Z \) be the derivative of the optimal value function \( V \) with respect to the variable \( y = \sqrt{C^2 - \sum_{i=1}^{n} h_i^2} \) and evaluated at \( y_{MM} = 0 \). The lower good deal bound price for a vulnerable option is given by:

\[ V_{LGB} = V_{MM} - \sqrt{C^2 - \sum_{i=1}^{n} h_i^2 Z} \]

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where

\[ V_{MM} = E_t^{MM} \left[ \exp \left\{ -\int_t^T (r_u + q\lambda_u) du \right\} \sum_{i=1}^n \Phi(S^i_T) \right] \]

and

\[ Z(t, s, \lambda) = \int_t^T E_{t,s,\lambda}^{MM} \left[ \exp \left\{ -\int_t^u r(\tau) + q\lambda(\tau) d\tau \right\} \sqrt{\lambda_u M(u)} \right] d\tau. \]

where \( M(u) = \sqrt{(qV_{MM})^2(u, (S^i_u)_{i=1}^n, \lambda_u) + \left( \frac{\partial V_{MM}}{\partial \lambda}(u, (S^i_u)_{i=1}^n, \lambda_u) \sigma \right)^2}. \)

For the special case when all \( \rho_i = 0, i = 1, ..., n \) and \( \kappa, \theta, \sigma \) are constants, the upper/lower good deal bound price is given by:

\[ V_{LGB} = V_{MM} \left[ 1 - q \sqrt{C^2 - \sum_{i=1}^n h_i^2 \int_t^T E_t^Q[\sqrt{\lambda_u}] \sqrt{1 + \sigma^2 B^2(u, T, q)} du} \right] \]

with \( E_t^Q[\sqrt{\lambda_u}] \) given by equation (30) and

\[ V_{MM} = \exp \left\{ -r(T-t) + [A(t, T, q) + B(t, T, q) q \lambda_1] \sum_{i=1}^n E_t^{MM} [\Phi(S^i_T)] \right\} \]

where \( E_t^{MM} [\Phi(S^i_T)] \) is the Black Scholes price for the option written on the stock \( S^i \)

\[ B(t, T, q) = \frac{2 \left( e^{\delta(T-t)} - 1 \right)}{(\delta + q\kappa) (e^{\delta(T-t)} - 1) + 2\delta} \]

\[ A(t, T, q) = \left[ \frac{2\delta e^{(q\kappa+\delta)(T-t)/2}}{(\delta + q\kappa) (e^{\delta(T-t)} - 1) + 2\delta} \right]^{2q\kappa/\sigma^2} \]

with \( \delta = \sqrt{(q\kappa)^2 + 2q\sigma^2}. \)

The individual good deal bound prices for each stock \( S^j \) from the option portfolio are given by:

\[ V_{LGB}^j = V_{MM}^j - \sqrt{C^2 - \sum_{i=1}^n h_i^2 Z^j} \]

where \( V_{MM}(S^j) \) and \( Z(S^j) \) are computed as in Proposition 3.3.

**Proof.** It is straightforward, by following exactly the same steps as in Proposition 3.3. \( \blacksquare \)

This means it is easy to use the same setup as before in order to price the value of a portfolio of derivatives traded with one counterparty. Appendix C shows that lower good deal bound is fairly stable to the introduction of new assets. For options deep in the money,
the prices changes from 17.7698 to 17.8066 when we introduce 9 more assets. There are two main reasons for this phenomenon. First, the value of \( Z(t) \) is close to zero. Second, as shown in the last graph, the variable \( y = \sqrt{C^2 - \sum_{i=1}^n h_i^2} \) changes very little with the introduction of a new asset - by the introduction of 20 new assets, we obtain a change of 0.8.

4.2 Good deal bounds for several counterparties

In this section, we are going to see how the introduction of a new counterparty affects the good deal bound price of an asset. Our vulnerable option will be written on the stock \( S \) with dynamics given by:

\[
dS_t = \alpha S_t dt + \gamma S_t d\tilde{W}_t
\]

We can trade the vulnerable option either with counterparty 1 or with counterparty 2. The default indicator of counterparty \( i \) is given by a point process \( N_i \) with intensity \( \lambda_i \), which is modeled as an affine process:

\[
d\lambda_i^i = \kappa^i(\theta^i - \lambda_i^i)dt + \sigma^i \sqrt{\lambda_i^i} dW_i^P
\]

where \( W_i^P, i = 1, 2 \) are two Wiener processes, with \( dW_i^P dW_j^P = \rho dt \). We denote the correlation between the two counterparties intensities of default as \( \rho \).

Assumption 4.2

1. Let the filtration space \((\Omega, F, P, \mathcal{F})\) be given, where \( \mathcal{F} \) is the internal filtration generated by the processes \( W^P, \bar{W}^P \) and \( N \), defined below.

2. \( W_i^P, i = 1, 2 \) and \( \bar{W}^P \) are \( P \)-Wiener processes and \( dW_i^P d\bar{W}^P = \rho_i dt, i = 1, 2 \) and \( dW_1^P dW_2^P = \rho dt \). \( N^i \) is a Cox process with predictable intensity \( \lambda_i^i \).

3. We assume the intensity of the Cox process \( \lambda_i^i \) to follow the dynamics

\[
d\lambda_i^i = \kappa^i(\theta^i - \lambda_i^i)dt + \sigma^i \sqrt{\lambda_i^i} dW_i^P
\]

4. \( \kappa^i, \theta^i, \sigma^i \) are scalar deterministic functions of time, \( i = 1, 2 \).

5. The market model under the objective probability measure \( P \) is given by the following dynamics:

\[
dS_t = S_t \alpha_t dt + S_t \gamma_t d\bar{W}_t^P
\]
\[ dB_t = r B_t dt \]

where \( S_t \) denotes a traded stock and \( B_t \) the money bank account.

6. \( \alpha_t \) and \( \gamma_t \) are scalar deterministic functions of time.

7. We assume a European call option is written on the stock. Default of the counter-party/writer of the option is described by the process \( Y^i_t = N^i_t \). Default of counterparty \( i \), \( i = 1, 2 \) occurs at the first jump of the Cox process \( N^i_t \).

I am trying to compute the lower good deal bound price for a derivative on \( S \) with counterparty 1. This is formulated as follows:

\[
\max_{h, g_1, g_2, \varphi_1, \varphi_2} E^Q \left[ \exp \left\{ - \int_t^T (r_u + q^1 \lambda^1_u) du \right\} \Phi(S_T) \right] | \mathcal{F}_t
\]

\[
dS_t = r S_t dt + S_t \gamma_t d\tilde{W}^Q_t
\]

\[
\lambda^Q_t = \lambda^1 (1 + \varphi_1)
\]

\[
\lambda^Q_2 = \lambda^2 (1 + \varphi_2)
\]

\[
d\lambda^1_t = \left( \kappa^1 (\theta^1 - \lambda^1_t) + g_1 \sigma^1 \sqrt{\lambda^1_t} \right) dt + \sigma^1 \sqrt{\lambda^1_t} dW^Q_1
\]

\[
d\lambda^2_t = \left( \kappa^2 (\theta^2 - \lambda^2_t) + g_2 \sigma^2 \sqrt{\lambda^2_t} \right) dt + \sigma^2 \sqrt{\lambda^2_t} dW^Q_2
\]

\[
\alpha_t + \gamma_t h_t = r
\]

\[
\varphi_1 \geq -1
\]

\[
\varphi_2 \geq -1
\]

\[
h^2_t + \varphi^2_1 \lambda^1 + \varphi^2_2 \lambda^2 + g^2_1 \lambda^1 + g^2_2 \lambda^2 \leq C^2
\]

Please note that \( \lambda^2 \) does not affect the payoff function and, in case of default of the counterparty 2, we do not have a jump in the price process for the asset traded with counterparty 1. However, the coefficients of the process \( \lambda^Q_1 \) will change, since \( dW^P_1 dW^P_2 = \rho dt \) - the two Wiener processes are correlated. However, this will have a small impact on the price of our option. If you remember the numerical results from section 3.4, a change in the parameters of the intensity process has only a very small effect on the price of a vulnerable option. Basically, good deal bounds shares the same flaw as the generic credit risk framework of today, in not capturing contagion effects properly.
5 Conclusion

We have developed a method for pricing counterparty risk by using good deal bounds. The method imposes a new restriction in the arbitrage free model by setting upper bounds on the Sharpe ratios of the assets. The potential prices which are eliminated represent unreasonably good deals. The constraint on the Sharpe ratio translates into a constraint on the stochastic discount factor. Thus, one can obtain tight pricing bounds. Previous literature on counterparty risk and good-deal bounds involved structural models. We allow for counterparty risk to be given by intensity-based models. Also, previous literature on counterparty risk with intensity models uses pricing directly under the risk neutral measure - which is not unique. We provide a link between the objective probability measure and the range of potential risk neutral measures which has an intuitive economic meaning.

Also, we have studied numerically the tightness of the bounds and the change in the GDB interval due to various factors. Thus, we have noticed that the choice of GDB constraint (which is left at the discretion of the modeler) is not the most important factor in determining the lower good deal bound price. The “amount of counterparty risk” undertaken, as reflected by the loss to default parameter and the current intensity of default, has a bigger impact on the price than the constraint parameter. The high impact of the price of \( \lambda_0 \) - the current level of the probability of default - also points out toward an “old time wisdom”- i.e. the most important part of risk management is to choose your counterparty and monitor it carefully.

Finally, we underline the use of good deal bounds for risk management. We prove the link between the lower good deal bound price and the coherent risk measures. In this context, we also study portfolio effects on the good deal bounds prices. We notice that the GDB are robust to the introduction of new assets in the portfolio traded with our counterparty. Also, the techniques and results used in computing the price of an asset are easily transferable to the risk management framework for a portfolio. However, when it comes to aggregating the effect of different counterparties, good deal bounds are not good at capturing the effect of contagion, a feature transmitted through the generic form of modeling default risk employed today.
A The Simple Poisson Process Case

This appendix will deal with the special case when the Cox process has a constant intensity, being reduced to the Poisson process. In this case, we can obtain closed form solutions for the price of the vulnerable options. Results can be easily generalized for the case of Poison processes with piecewise constant intensity.

In the case of deterministic intensity, the upper good deal bound problem becomes:

$$\max_{h, g, \varphi} \quad E_t^Q \left[ \exp \left\{ - \int_t^T (r_s + q \lambda_s^Q) ds \right\} \Phi(S_T) \right]$$

$$dS_t = r S_t dt + S_t \gamma_t d\tilde{W}_t$$

$$\lambda_t^Q = \lambda_t (1 + \varphi_t)$$

$$\alpha_t + \gamma_t h_t = r$$

$$\varphi_t \geq 1$$

$$h_t^2 + \varphi_t^2 \lambda \leq C^2$$

By solving the embedded static problem, we obtain the following optimal Girsanov kernel

- $\hat{h}_t = \frac{r - \alpha_t}{\gamma_t}$, $\hat{\varphi}_t = \max \left[ -\sqrt{\frac{C^2 - h^2}{\lambda}}, -1 \right]$.  

The case $\hat{\varphi} = -1$ reduces the jump intensity under $Q$ to zero. Hence forward, we will analyze only the case $\hat{\varphi}_{u/l} = \pm \sqrt{\frac{C^2 - (\alpha - r)^2}{\lambda}}$.

We have the following valuation formula: Provided no default occurred until the time of the pricing $t$, the general pricing formula for a defaultable claim with payoff $X$ and recovery to market value, as defined before is:

$$\Pi_t = E_t^Q \left[ \exp \left\{ - \int_t^T (r_s + q \lambda_s^Q) ds \right\} X \right]$$

In our case, since both the interest rate and the jump intensity of default are constant, we have:

$$\Pi = \exp \left\{ -(r + q \lambda^Q)(T - t) \right\} E_t^Q [\max[S_T - K, 0]]$$

We obtain the expectation $E_t^Q [\max[S_T - K, 0]]$ as $se^{r(T-t)}N[d_1(t,s)] - KN[d_2(t,s)]$. Hence, the upper/lower good deal bound price for a vulnerable option, in the case of constant intensity is given by:

$$\Pi_{u/l} = e^{-(r + q \alpha_{u/l})(T-t)} \left\{ se^{r(T-t)}N[d_1(t,s)] - KN[d_2(t,s)] \right\}$$

(24)
where:
\[
\begin{align*}
  d_1(t, s) &= \frac{1}{\sigma \sqrt{T - t}} \left\{ \ln \left( \frac{s}{K} \right) + \left( r + \frac{1}{2} \sigma^2 \right) (T - t) \right\}, \\
  d_2(t, s) &= d_1(t, s) - \sigma \sqrt{T - t}, \\
  \lambda^Q_{u/l} &= \lambda \left( 1 \mp \sqrt{\frac{C^2 - \frac{(\alpha - r)^2}{\gamma}}{\lambda}} \right)
\end{align*}
\]

B Computations for \( \frac{\partial V}{\partial y}(y_{MM}) \) - the sensitivity of \( V \) with respect to the GDB constraint

In this section, we detail the computations needed in order to obtain the sensitivity of the lower good deal bounds pricing equation with respect to the variable \( y = \sqrt{C^2 - h^2} \). We re-write the HJB as
\[
\frac{\partial V}{\partial t}(t, s, y, \lambda) + V_s r + V_\lambda [\kappa (\theta - \lambda t) + g_t \sigma \lambda_t] - qV \lambda_t (1 + \varphi_t) + \frac{1}{2} \gamma^2 s^2 V_{ss} + \frac{1}{2} \sigma^2 \lambda V_{\lambda\lambda} + \rho \gamma \sigma s \sqrt{\lambda} V_{\lambda s} - rV(t, s, y, \lambda) + \nu \left[ g_t^2 \lambda + \varphi_t^2 \lambda + h_t^2 - C^2 \right] = 0
\]
where \( \nu \) is the Langrange coefficient of the good deal bound constraint. We apply the envelope theorem which allows us to derive the above PDE by \( y = \sqrt{C^2 - h^2} \) and denote \( \frac{\partial V}{\partial y} \) by \( Z \). As it turns out, we obtain a different PDE for \( Z \):
\[
\frac{\partial Z}{\partial t} + Z_s r + Z_\lambda [\kappa (\theta - \lambda t) + g_t \sigma \lambda_t] - qZ \lambda_t (1 + \varphi_t)
+ \frac{1}{2} \gamma^2 s^2 Z_{ss} + \frac{1}{2} \sigma^2 \lambda Z_{\lambda\lambda} + \rho \gamma \sigma s \sqrt{\lambda} Z_{\lambda s} - rZ(t, s, y, \lambda) - \nu 2 \sqrt{C^2 - h^2} = 0
\]
From the previous computations, we have obtained the Lagrange coefficient
\[
\nu = -\frac{1}{2} \frac{\sqrt{\lambda} \sqrt{(qV)_t^2 + (\sigma V_\lambda)_t^2}}{\sqrt{C^2 - h^2}}
\]
and replace \( \nu \) in the PDE for \( Z \). Note that both the values for \( g \) and \( \varphi \) and the terms \( qV \) and \( V_\lambda \) should be evaluated at the minimal martingale value. The values for \( g_{MM} \) and \( \varphi_{MM} \) are 0. Our PDE becomes
\[
\frac{\partial Z}{\partial t} + Z_s r + Z_\lambda [\kappa (\theta - \lambda t) + g_t \sigma \lambda_t] - qZ \lambda_t (1 + \varphi_t)
\]
\[ + \frac{1}{2} \gamma^2 s^2 Z_{ss} + \frac{1}{2} \sigma^2 \lambda Z_{\lambda \lambda} + \rho \gamma \sigma s \sqrt{\lambda} Z_{s \lambda} - r Z(t, s, y, \lambda) + \sqrt{\lambda} \sqrt{(qV^{MM})^2 + (\sigma V^{MM})^2} = 0 \]

We solve this by Feynman-Kac:

\[
Z = \int_t^T E_{t, s, \lambda}^{MM} \left[ \exp \left\{ - \int_t^u (r_{\tau} + q_{\lambda_{\tau}}) d\tau \right\} \sqrt{\lambda_u \sqrt{(qV^{MM})^2 + (\sigma V^{MM})^2}} \right] du
\]

We can distinguish 2 situations:

- for \( \rho \neq 0 \), we need to insert MC simulation results and solve the PDE numerically.
  In order to compute \( V^{MM}_\lambda \), we use the maximum likelihood method as explained in Glasserman (2003)

- for \( \rho = 0 \), we can go further:
  from \( V^{MM}_\lambda = -V^{MM}_t qB(t, T, q) \), we get that:

\[
Z = \int_t^T E_{t, s, \lambda}^{MM} \left[ \exp \left\{ - \int_t^u (r_{\tau} + q_{\lambda_{\tau}}) d\tau \right\} \sqrt{\lambda_u V^{MM} \sqrt{q^2 + (\sigma q B(u, T, q))^2}} \right] du
\]

\[
= \int_t^T q \sqrt{1 + (\sigma q B(u, T, q))^2} E_{t, \lambda}^{MM} \left[ \exp \left\{ - \int_t^T (r_{\tau} + q_{\lambda_{\tau}}) d\tau \right\} \sqrt{\lambda_u} E_{t, s}^{MM} [\Phi (S_T)] \right] du
\]

We need to compute

\[
A = E_{t, \lambda}^{MM} \left[ \exp \left\{ - \int_t^T (r_{\tau} + q_{\lambda_{\tau}}) d\tau \right\} \sqrt{\lambda_u} \right] = E_{t, \lambda}^{MM} [M_T X] = E_{t, \lambda}^{MM} [m_T R_T X] \quad (25)
\]

where \( m_T = \frac{E^{MM}[\exp \{ \int_0^T q_{\lambda_{\tau}} d\tau \}]}{E^{MM}[\exp \{ \int_0^T q_{\lambda_{\tau}} d\tau \}]} \) and \( R_T = \frac{\exp \{ - \int_t^T (r_{\tau} + q_{\lambda_{\tau}}) d\tau \}}{m_T} \).

We have \( \exp \{ - \int_t^T (r_{\tau} + q_{\lambda_{\tau}}) d\tau \} \geq 0 \) by definition. Also, we note that \( E^{MM}[R_T] = 1 \).

These facts allow us to use \( R_T \) as a Radon-Nycodim derivative in a change of measure and define a measure \( \hat{Q} \) by:

\[
d\hat{Q} = R_T dQ^{MM} \text{ on } \mathcal{F}_T
\]

(26)
Using Bayes’ Theorem, we can re-write (25) as:

$$A = m_T E^{MM} [R_T | \mathcal{F}_t] E^{\hat{Q}} [Z | \mathcal{F}_t]$$

If we define the likelihood process $L_t$, $0 \leq t \leq T$, by:

$$d\hat{Q} = L_t dQ^{MM} \text{ on } \mathcal{F}_t$$

by standard theory, we have:

$$L_t = E^{MM} [L_T | \mathcal{F}_t] = E^{MM} [R_T | \mathcal{F}_t]$$

Note that even if $L_T = R_T$, we cannot draw the conclusion $L_t = R_t$ for $t < T$. This is a consequence of the fact that $\exp \left\{ - \int_t^T (r_{\tau} + q\lambda_{\tau}) d\tau \right\}$ is not a traded asset.

We notice that

$$m_T E^{MM} [R_T | \mathcal{F}_t] = E^{MM} \left[ \exp \left\{ - \int_t^T (r_{\tau} + q\lambda_{\tau}) d\tau \right\} \bigg| \mathcal{F}_t \right]$$

Thus, in order to proceed, we need to calculate the following:

- $E^{MM} \left[ \exp \left\{ - \int_t^T (r_{\tau} + q\lambda_{\tau}) d\tau \right\} \bigg| \mathcal{F}_t \right]$,
- the dynamics for $L_t$ in order to identify the Girsanov transformation $Q^{MM} \rightarrow \hat{Q}$,
- $E^{\hat{Q}} [X | \mathcal{F}_t]$

We know from the computations linked to $V^{MM}$ that

$$E^{MM} \left[ \exp \left\{ - \int_t^T (r_{\tau} + q\lambda_{\tau}) d\tau \right\} \bigg| \mathcal{F}_t \right] = \exp \{- [r\Delta T + A(t, T, q) + B(t, T, q)q\lambda_t] \}$$

From the definition of $L_t$, one can easily derive the dynamics of $L_t$:

$$dL_t = -\sigma q \sqrt{\lambda} B(t, T, q) L_t dW_t$$

and then derive the dynamics of $\lambda$ under $\hat{Q}$ are given by:

$$d\lambda_t = [\kappa(\theta - \lambda) - \sigma^2 q \lambda B(t, T, q)] dt + \sigma \sqrt{\lambda} d\hat{W}_t$$

Finally, we need to compute $E^{\hat{Q}} \left[ \sqrt{\lambda} | \mathcal{F}_t \right]$. For a detailed proof that $\lambda$ is non-central
chi-square distributed with weighting factor

$$\eta = \frac{q\sigma^2}{4} B(t, T, q), \quad (27)$$

degrees of freedom

$$\nu = \frac{\kappa \theta}{\sigma^2}, \quad (28)$$

and non-centrality factor

$$\Lambda = \frac{4}{\sigma^2} \frac{\partial}{\partial T} B(t, T, q) \lambda(t) \quad (29)$$

under the $\hat{Q}$ measure, we refer to chapter 7 from Schönbucher (2003).

This means that $\sqrt{\lambda}$ is non-central chi distributed with non-centrality factor $\Lambda$ and the formula for the mean is given by:

$$E[\sqrt{\lambda}] = \sqrt{\frac{\pi}{2}} \eta L^{(\nu/2-1)}_{1/2} \left( -\frac{\Lambda^2}{2} \right) \quad (30)$$

where $L^{(\alpha)}_i(x)$ is the generalized Laguerre polynomial and $\eta, \nu$ and $\Lambda$ are given by (27), (28) and (29).
C  Graphs and Tables

The variable C (or the size of the GDB constraint)

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The rate of recovery

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The initial intensity of default

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The volatility of the intensity of default process

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For different values of $\sigma$, the graphs show the stock prices and option prices for $S_t=20, K=30$, $S_t=30, K=30$, and $S_t=50, K=30$. The plots illustrate the behavior of the minimal martingale sensitivity and the lower bound as $\sigma$ varies from 0.05 to 0.4.
The long term intensity of default

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<td>5.2136</td>
<td>4.3077</td>
</tr>
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<td>MM</td>
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<td>5.2173</td>
<td>4.297</td>
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<td>0.9192</td>
<td>0.7571</td>
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<td>BS: 0.759, MM: 0.7571, $V_{LGB}$: 0.7545</td>
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</table>

| | 0.9291 | 0.9201 | 0.7545 | BS: 0.9291, MM: 0.9201, $V_{LGB}$: 0.7545 |
| | 0.9186 | 0.9192 | 0.7571 | BS: 0.9186, MM: 0.9192, $V_{LGB}$: 0.7571 |
| | 0.759 | 0.7571 | 0.7545 | BS: 0.759, MM: 0.7571, $V_{LGB}$: 0.7545 |
Introducing new assets into the portfolio

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<td>2.4546</td>
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<tr>
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<td>0.0647</td>
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<tr>
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<td>0.7605</td>
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<td>1.5137</td>
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<tr>
<td>$V_{LGB}$</td>
<td>17.7698</td>
<td>17.8066</td>
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Introducing correlation between $S_t$ and $\lambda_t$ ($\sigma$ is 0.2 is this table, hence the difference in numbers. We need this in order to be able to generate a higher $\rho$)

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<th>$\rho$</th>
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<td>$V_{LGB}$</td>
<td>16.3751</td>
<td>16.3743</td>
</tr>
</tbody>
</table>

![Graphs showing correlation between stock prices and option prices for $\rho = 0$ and $\rho = 0.35$.](image)
References


Schönbucher, P. (2003). Credit derivatives pricing models - models, pricing and implementation. JWS.
