A Stochastic Multiple Leader Stackelberg Model: Analysis, Computation, and Application

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Outline

1. The model and contribution
2. Equilibrium existence and uniqueness
3. Computing equilibria
4. An example
5. Conclusion

Stochastic multiple-leader multiple-follower game

- **New service technology** about to enter a market.
- M **leaders and** N **followers.**
- **Leaders:**
  - Have NO capacity installed.
  - Make early decision to allow time to install capacity—they lead.
  - Know demand function only in distribution.
- **Followers:**
  - Have capacity installed.
  - Wait and see leaders supply and realized demand function.
- **Stackelberg Nash-Cournot equilibrium:**
  - Leaders: Stackelberg players for followers but Cournot for leaders.
  - Followers are Cournot for all other players.

Stochastic multiple-leader multiple-follower game

- Leaders know demand in distribution, followers wait and see.
- Leaders are Stackelberg players for followers but Cournot for leaders.
- Followers are Cournot for all other players.

Leader 1

$$\max_{x_1 \geq 0} \quad \mathbb{E}[x_1 p(X + Y(X, \xi), \xi)] - C_1(x_1)$$

Leader 2

$$\max_{x_2 \geq 0} \quad \mathbb{E}[x_2 p(X + Y(X, \xi), \xi)] - C_2(x_2)$$

Follower 1

$$\max_{y_1 \geq 0} \quad y_1 p(X + Y, \xi) - c_1(y_1)$$

Follower 2

$$\max_{y_2 \geq 0} \quad y_2 p(X + Y, \xi) - c_2(y_2)$$

Related literature

- **Stackelberg game**—Stackelberg (1934).
  - Single leader and single follower.
  - Deterministic demand.
- **Economics literature**—Vives (1999).
  - "Leader can commit capacity."
  - "Burning the ships."
- **Management Science literature:**
  - Sherali (OR, 84)
    - Multiple leaders and followers.
    - Deterministic demand.
    - Existence and uniqueness (symmetric) + Computation.
  - DeWolf & Smeers (MS, 97)
    - Single leader but stochastic demand (finite discrete distribution).
    - Application to European gas market.

Our contribution

1. **Show existence and uniqueness of equilibrium:**
   - Multiple-leader-follower games may lack equilibrium; Ehrenmann (04).
   - Assumptions similar or weaker than Sherali's (OR, 87) for deterministic model.
   - Unlike deterministic model, leaders expected profit is smooth—we can use Rosen (Econometrica 1965).

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2. **Computational approach to find equilibrium:**
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    - Sample average approximation method converges exponentially fast.

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2. **Computational approach to find equilibrium:**
   - Continuous demand distributions allowed—unlike in DeWolf and Smeers (MS, 97).
   - Sample average approximation method converges exponentially fast.
3. **Model competition in telecommunication industry:**
   - Competition between potential providers of new service technology.
   - Leaders: consider installing capacity.
   - Followers: have capacity installed.

Outline

Existence and Uniqueness

- **Conventional wisdom:**
  - "Proving existence and uniqueness for multiple-leader multiple-follower games is hard."
  - Multiple-leader-follower games may lack equilibrium; Ehrenmann (04).
- **Why is it hard?**
  - Followers' reaction functions nonsmooth.
  - Leaders' reach equilibrium subject to equilibrium constraints.
- **Assumptions:**

A1 Linear inverse demand function:

$$p(q, \xi) = \alpha(\xi) - \beta(\xi)q.$$

A2 Bounded support set for random factor ξ.

A3 Bounded capacity limits.

A4 Smooth, convex cost functions: linear, quadratic—may be asymmetric.

Existence and uniqueness
Sketch of proof

- Leader expected profits are concave:
  - Followers equilibrium unique.
  - Followers reaction function $Y(X, \xi)$ convex in x.
- Leader expected profit smooth:
  - Piecewise smooth for deterministic case.
- Leader expected profits are strictly diagonally concave:
- Uniqueness follows from Rosen (Econometrica, 1965):
  - Under weaker conditions than Sherali (OR, 84) for deterministic case—we do not identical leaders.

Outline

Sample average approximation (SAA) method + All-KKT method

- Sample average approximation method:
  - Deal with continuous distribution for random factor ξ:
  - Approximate leader expected profit with sample average:

$$\mathbb{E}[x_i p(X + Y(X,\xi),\xi)] = \frac{1}{k}\sum_{l=1}^{k} x_i p(X + Y(X,\xi^l),\xi^l)$$

  - Sample directly from historical data or distribution if known.
  - Convergence analysis:
    - SAA equilibria converge to unique equilibrium almost sure.
    - SAA equilibria converge to approximate equilibrium with probability approaching one exponentially fast.
- All-KKT method (Hu 02, Leyffer and Munson 05, Hu and Ralph):
  - KKT conditions of leaders and followers as mixed complementarity problem.
  - Solve with commercial solver PATH.

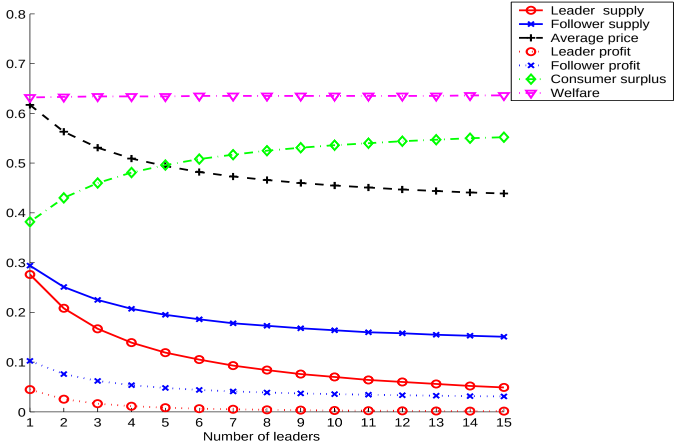
Outline

Competition in a telecommunication market

- **New service technology** in a telecommunication market:
  - 3G, bandwidth, etc.
- **Entrants (leaders). . .**
  - have NO spare capacity.
  - decide whether to install capacity.
  - know demand function only in distribution.
- **Incumbents (followers). . .**
  - have capacity installed and only re-allocate from other services.
  - have lower costs.
  - wait and see realized demand function and leaders supply.
- **Comparative statics:**
  - number of leaders and followers.
  - leader costs.
  - demand uncertainty.

Number of leaders

Figure 1: Comparative static analysis with respect to number of leaders



Number of followers

Figure 2: Comparative static analysis with respect to number of followers

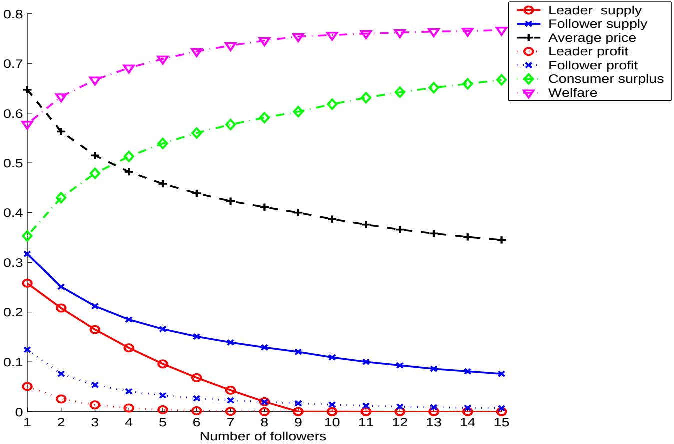
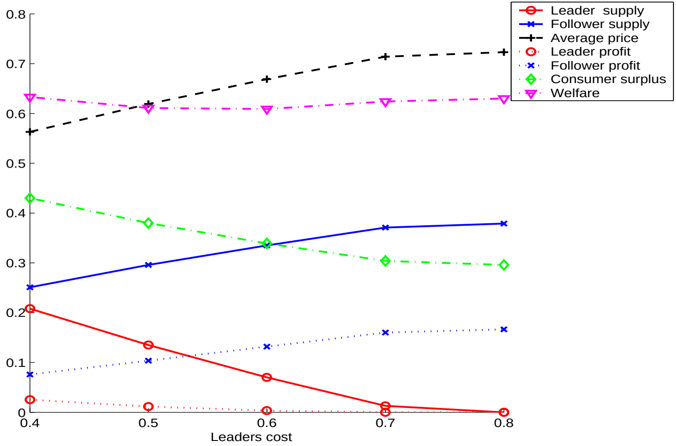
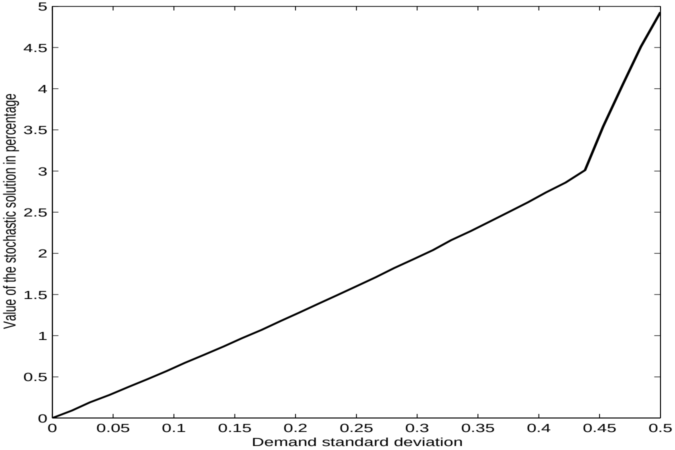


Figure 3: Comparative static analysis with respect to leaders cost



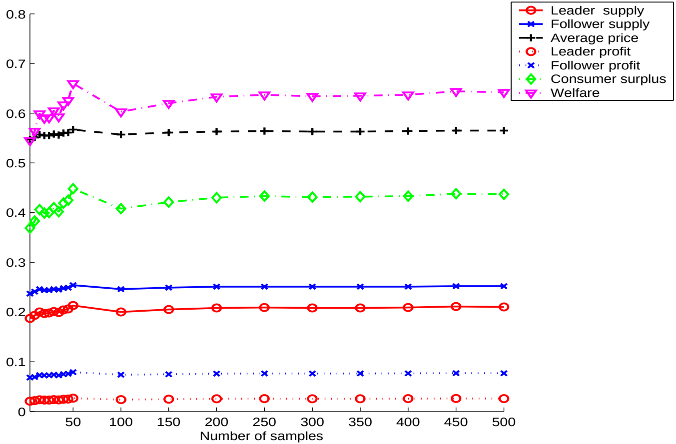
Demand function standard deviation

Figure 4: Comparative static analysis with respect to demand standard deviation



Number of samples

Figure 5: Convergence of the sample average approximation solutions



Outline

Conclusion

1. **Show existence and uniqueness of equilibrium:**
   - Assumptions similar or weaker than Sherali's (OR, 87) for deterministic model.
   - Unlike for deterministic model, leaders expected profit is a smooth function—allows us to use results in Rosen.
2. **Computational approach to find equilibrium:**
   - Sample average approximation method converges exponentially fast (continuous distributions allowed, unlike in DeWolf & Smeers (MS, 97).
   - Reformulate sample average approximation problems as mixed complementarity problem.
3. **Insights from model of competition in telecommunication industry:**
   - Followers have competitive advantage:
     - Lower costs.
     - Perfect information on demand function.
   - More followers prevent entry of leaders, more leaders do not persuade followers out of market.

Thank you!

A Stochastic Multiple Leader Stackelberg Nash Cournot Model: Analysis, Computation, and Application.

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