CHOICE BY SEQUENTIAL PROCEDURES*

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Abstract. Despite the growing array of boundedly rational choice models available from the literature, little is known about the formal relationships between them. Understanding the implications between the relevant models of choice is crucial for a proper assessment of the significance of the theoretical findings. In this paper, we take some first steps forward, and establish the relationships between three recent and influential boundedly rational choice models. We show that choice with a status-quo bias is a refinement of rationalizability by game trees, which, in turn, is also a refinement of sequential rationalizability. Thus, we provide a sharp taxonomy of these choice models, and show that they can be understood as sequential choice procedures.

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1. Introduction

In the classical choice model the decision-maker (DM) maximizes a single preference relation, which she uses to select the preferred element. Nevertheless, a large literature in psychology and economics has demonstrated that behavior often deviates from the classical model. Not surprisingly, these inconsistencies between the formal model and empirical studies have allowed a number of alternative boundedly rational choice models to flourish in the literature. Some recent prominent examples are:

- Status-quo bias (Masatlioglu and Ok, 2005, 2010): the DM typically values an alternative more highly when it is regarded as the status-quo, than she would otherwise.
- Rationalizability by game trees (Xu and Zhou, 2007): the choices of the DM are the equilibrium outcome of the conflict between different criteria, modeled as a game.
- Sequential rationalizability (Manzini and Mariotti, 2007): the DM sequentially applies a collection of criteria in a fixed order of priority.

Although the various choice models attempt to account for different behavioral patterns, the question arises as to the formal relationships between them. This is crucial to the understanding of the literature, the applicability of the models, and the development of the field. In this paper, we take some first steps forward in this direction. We show what

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may be regarded as a priori unexpected implications between the three influential models of choice mentioned above. In particular, we show that the concept of status-quo bias is a refinement of rationalizability by game trees, which, in turn, is also a refinement of sequential rationalizability. That is, every choice pattern that is status-quo biased is also rationalizable by a game tree. Similarly, every choice pattern that is rationalizable by a game tree is also sequentially rationalizable. The converse statements are not necessarily true. Thus, we provide a sharp taxonomy of these choice models, and show that they all can be understood as special cases of sequential choice procedures.

The first model we study responds to the large empirical literature supporting the view that DMs typically value an alternative more highly when it is regarded as the status-quo, than they would otherwise. This is the so-called status-quo bias (see, e.g., Thaler, 1980 and Kahneman, Knetsch, and Thaler, 1991). Masatlioglu and Ok (2005, 2010) provide a theoretical treatment of choice behavior that allows for the presence of a status-quo bias. Masatlioglu and Ok introduce a set of properties for choice behavior that are equivalent to the following status-quo biased choice model. The DM evaluates the alternatives by means of a vector-valued, multicriteria-type utility function, u. If the DM confronts a choice problem without a status-quo, she simply maximizes an aggregation of these criteria. If there is a status-quo, then the DM compares the status-quo with all the alternatives in the set, using all the criteria. She will stay with the status-quo unless there is an alternative that dominates it in terms of *all* the decision criteria. This represents a marked status-quo bias. If there are alternatives that dominate the status-quo by all the criteria, then the DM makes her choice using the same aggregator as above.

The second model analyzed in this paper perceives the decision-maker as a collection of selves in internal conflict (see, e.g., Ainslie, 1992 and Elster, 1989). Xu and Zhou (2007) suggest a model where multiple choice criteria compete in a game structure in order to reach a decision. In this vein, the DM's choices are the equilibrium outcome of an extensive game with perfect information. The alternatives are the terminal nodes of the game and the criteria take the form of linear orders at the non-terminal nodes. The outcome, therefore, is the subgame perfect Nash equilibrium of the game.

Finally, we examine a third model that perceives the DM as following a sequential procedure that guides her through the complex choice process. For this reason, it has attracted a great deal of attention from the psychology, marketing, and management literatures (see, for example, Tversky 1972, Manrai and Sinha 1989, and Hogarth and Karelaia 2005). Manzini and Mariotti (2007) suggest a model of sequential rationalizability where the DM faced with a choice problem, applies a number of criteria in a fixed order of priority. The criteria take the form of asymmetric relations that the DM sequentially maximizes to gradually narrow down the set of alternatives until one is identified as the choice.

In our main result, Theorem 1, we establish the connection between these three choice models. First, we show that every choice pattern that is rationalizable by game trees also admits a sequential representation. We prove this by showing that we can equivalently formulate the model of rationalization by game trees using the following sequential heuristic. First, the DM observes a particular order over the binary menus of alternatives, i.e., over the two-option menus. Then, for any menu of available alternatives, she begins by considering the first binary menu within it. She then considers which of the two alternatives in the binary menu is the dominated one and removes it from the menu of available alternatives. She follows this sequential procedure in an ordered manner, removing one alternative at each step, until reaching a final alternative, which is the choice. Clearly, this choice procedure is a special case of sequential rationalizability based on an ordered sequence of single binary comparisons. We refer to the choice patterns that can be equivalently stated in terms of this heuristic as sequentially rationalizable by simple rationales. Given its procedural simplicity, this heuristic appears to offer particular interest from a behavioral perspective.

Then, we also show that every status-quo biased choice pattern admits a rationalization by game trees. To show this, we construct a tree in which the status-quo is a successor of those alternatives that are dominated by it, and these in turn are successors of the alternatives that dominate the status-quo. Interestingly, this order of the set of alternatives can also be understood as though the DM had considered a particular order over the binary menus of alternatives, and were to proceed by eliminating alternatives in the manner described above. The order over the binary problems would be lexicographic. It is therefore the case that status-quo biased choice is also sequentially rationalizable by simple rationales.

The question arises as to whether all sequentially rationalizable choice patterns can be equivalently formulated in terms of simple rationales. Theorem 2 shows that this is in fact not the case. There are choice patterns that are sequentially rationalizable but do not admit the simple, binary heuristic. In other words, there are cases in which we can explain choice through a sequence of asymmetric relations that cannot be decomposed into a sequence of simple rationales. We then show that only choice patterns using acyclic relations can be equivalently formulated in terms of simple rationales.

2. Choice by Sequential Procedures

Let X be a finite set of $n \geq 2$ objects. A choice function c assigns to every non-empty set $A \subseteq X$ a unique element $c(A) \in A$. Denote by P a binary relation on X. For any A, $M(A,P)$ refers to the set of maximal elements in A with respect to P, that is, $M(A,P) = \{x \in A : (y,x) \in P \text{ for no } y \in A\}$. Let $M(\emptyset, P)$ be equal to $\emptyset$.

In the classical approach a choice function c is said to be rationalizable if there is a binary relation P such that, for any choice problem A, $c(A) = M(A,P)$. We now formally introduce the three boundedly rational models of choice that we analyze in this paper.

In Masatlioglu and Ok's (2005, 2010), a choice problem is a pair (A, x) where A is the set of alternatives, and $x \in A$ or $x = \diamond$. When $x \in A$, the pair (A, x) represents a choice problem with a status-quo, while if $x = \diamond$, the choice problem is standard in the sense that it is without a status-quo. Thus, choice is defined over the collection of all choice problems (A, x). We adapt Masatlioglu and Ok's representation of status-quo biased choice to our setting with single-valued choice functions defined on the collection of all non-empty subsets $A \subseteq X$. Consider then the following definition, which is simply a reformulation of Theorem 1 in Masatlioglu and Ok (2005).

Status-Quo Biased Choice Function: A choice function c is status-quo biased or SQB if there exists an element $\overline{x} \in X$, a positive integer q, an injective function $u : X \to \mathbb{R}^q$ and a strictly increasing map $h : u(X) \to \mathbb{R}$ such that:

(1) For all $A \subseteq X$ with $\overline{x} \notin A$, $c(A) = \arg\max_{x \in A} h(u(x))$.

(2) For all $A \subseteq X$ with $\overline{x} \in A$:

$$c(A) = \begin{cases} \overline{x} & \text{if } A \cap \{x \in X : u(x) > u(\overline{x})\} = \emptyset, \\ \displaystyle\arg\max_{y \in A \cap \{x \in X : u(x) > u(\overline{x})\}} h(u(y)) & \text{if } A \cap \{x \in X : u(x) > u(\overline{x})\} \neq \emptyset. \end{cases}$$

Thus, in a status-quo biased choice function the presence of one status-quo $\overline{x}$ may perturb rational choices. When the status-quo is not present, the DM simply maximizes a utility function. When the status-quo is present, the DM maximizes the same utility function but only among the alternatives that dominate the status-quo in all the attributes. If there are no such alternatives, the DM chooses to remain at the status-quo.

Xu and Zhou (2007) study those choice functions that can be rationalized by extensive games with perfect information. More specifically, we say that $(G; R = (R_1, \ldots, R_K))$ denotes a game tree whenever: (i) G is an extensive game with perfect information that has alternatives X as terminal nodes, such that each alternative in X appears once and only once as a terminal node of G, and (ii) every node i in the tree G represents the possible choices of an agent i endowed with a linear order R_i over X. Denote by $G|A$ the reduced tree of G that retains all the branches of G leading to terminal nodes in A, and $SPNE(\Gamma)$ stands for the subgame perfect Nash equilibrium outcome of Γ.

Rationalizability by Game Trees: A choice function c is rationalizable by game trees or RGT whenever there exists a game tree $(G; R)$ such that $c(A) = SPNE(G|A; R)$ for all $A \subseteq X$.

Xu and Zhou's model aims to capture an internal procedure of choice-making guided by the conflict of several internal criteria.

Finally, consider a finite sequence of asymmetric relations $P_1, P_2, \ldots, P_K$ over X. Let $M_i^j(A)$ with $i \leq j$ denote the set $M_i^j(A) = M(M(\ldots M(M(A, P_i), P_{i+1}), \ldots, P_{j-1}), P_j)$. That is $M_i^j(A)$ is the set of alternatives surviving from A given the sequential application of relations $P_i, P_{i+1}, \ldots, P_{j-1}, P_j$. We will often refer to the relations involved in a sequential maximization as *rationales.*

Sequential Rationalizability: A choice function c is sequentially rationalizable or SR whenever there exists a non-empty ordered list $\{P_1, \ldots, P_K\}$ of asymmetric rationales on X such that $c(A) = M_1^K(A)$ for all $A \subseteq X$. In that case we say that $\{P_1, \ldots, P_K\}$ sequentially rationalizes c.

In other words, this model considers a DM who gradually narrows down the set of alternatives by applying a set of (possibly incomplete) criteria in a fixed order.

In our main result, we show that choice functions exhibiting a status-quo bias are also rationalizable by game trees, and the latter are sequentially rationalizable. Let $\mathcal{C}^{\omega}$ denote the class of choice functions that are ω.

Theorem 1. $\mathcal{C}^{SQB} \subset \mathcal{C}^{RGT} \subset \mathcal{C}^{SR}$.

Proof of Theorem 1: We start by showing $\mathcal{C}^{RGT} \subset \mathcal{C}^{SR}$. Let c be a choice function rationalizable by game trees. Consider a game tree $(G; R = (R_1, \ldots, R_K))$ that rationalizes c. Suppose, without loss of generality, that non-terminal nodes are indexed by a linear order $<$ such that $i < j$ if i is a successor of j in the game tree G. We say that i is resolute on $\{x, y\}$, if R_i determines the outcome over $\{x, y\}$. That is, whenever $SPNE(G|\{x, y\}; R) = SPNE(G|\{x, y\}; (R'_{-i}, R_i))$ for any other vector of criteria over the non-terminal nodes different than i, R'_{-i}. Define the minimal resolute player in $A \in \mathcal{P}(X)$ by $m(A) = \min\{i \in \{1, \ldots, K\} : i$ is resolute over a pair $\{x, y\}$ contained in $A\}$.

Consider any linear order $\lhd$ on the space of binary problems $\mathcal{B}(X)$ such that, for all $\{a, b\}, \{d, e\} \in \mathcal{B}(X)$, with $a = c(a, b)$ and $d = c(d, e)$, $\{a, b\} \lhd \{d, e\}$ whenever $m(\{a, b\}) < m(\{d, e\})$. Denote the ordered collection of binary problems according to $\lhd$ by $\{a_i, b_i\}_{i=1}^{n(n-1)/2}$, and without loss of generality let it be assumed that $a_i = c(a_i, b_i)$, $i = 1, \ldots, n(n-1)/2$. Define $P_i = \{(a_i, b_i)\}$. Clearly, $\{P_i\}_{i=1}^{n(n-1)/2}$ is a collection of rationales. We now show that such collection sequentially rationalizes c. Given that each rationale is composed by one single binary comparison this reduces to showing that $c(A) = c(A \setminus \{x^*\})$ where x^* is the dominated alternative in the first binary problem in A according to $\lhd$.

By rationalizability of game trees, $c(A) = SPNE(G|A; R)$, and also $c(A \setminus \{x^*\}) = SPNE(G|(A \setminus \{x^*\}); R)$, where x^* is the dominated alternative in the first binary problem in A according to $\lhd$. We show that $SPNE(G|A; R) = SPNE(G|(A \setminus \{x^*\}); R)$. By definition of subgame perfect Nash equilibrium, $SPNE(G|A;R) = SPNE(G|(A \setminus \{z\});R)$, where z is every dominated alternative in A according to the minimal resolute player in A. Clearly, by construction, x^* is dominated according to $m(A)$, and hence the claim follows.

To show that this inclusion is strict, consider the following example. Let $X = \{1,2,3,4\}$ and the choice function c_1 defined by $c_1(3,4) = c_1(2,4) = c_1(2,3,4) = c_1(1,2,4) = 4$, $c_1(2,3) = c_1(1,3) = c_1(1,2,3) = 3$, $c_1(1,2) = 2$ and $c_1(1,4) = c_1(1,3,4) = c_1(X) = 1$. It is easy to see that c_1 is a choice by sequential procedures. Consider for instance the following linear order $\lhd$ over the binary problems $\{3,2\} \lhd \{4,3\} \lhd \{2,1\} \lhd \{3,1\} \lhd \{1,4\} \lhd \{4,2\}$ and the associated rationales with one binary comparison each. It is trivial to see that c_1 is SR for this ordered collection of rationales. We now prove that $c_1 \notin \mathcal{C}^{RGT}$. Notice that the cycles involving alternatives $\{1,4,3\}$ and $\{4,2,1\}$ imply that the only possible game tree that explains choices has two branches; one comprising alternatives 1 and 2, and the other comprising alternatives 3 and 4. However, under this game tree, we should have $1 = c_1(X) = c_1(c_1(1,2), c_1(3,4)) = c_1(2,4) = 4$, a contradiction.

We now prove that $\mathcal{C}^{SQB} \subset \mathcal{C}^{RGT}$. Given $c \in \mathcal{C}^{SQB}$, there exists a status-quo $\overline{x} \in X$. Define the following two subsets of alternatives in X: $D = \{y \in X \setminus \{\overline{x}\} :$ it is not true that $u(y) > u(\overline{x})\}$ and $U = (X \setminus \{\overline{x}\}) \setminus D$. For notational convenience, denote the status-quo by x_1, the elements of D by $x_2, x_3, \ldots, x_r$ and the elements of U by $x_{r+1}, \ldots, x_n$. We now construct an extensive game $(G;R)$. This game has $n-1$ non-terminal nodes, each of them with two branches. The first non-terminal node points to alternatives x_1 and x_2. Every other non-terminal node i points to non-terminal node $i-1$ and to alternative x_i. Linear orders R_i are defined as follows. If $i \leq r$, then R_i places x_1 on top of the ranking and orders the rest of alternatives as the function $h(u(\cdot))$ does. If $i > r$, then R_i places alternative x_1 on the bottom of the ranking and orders the rest of alternatives as the function $h(u(\cdot))$ does. We now show that this game rationalizes c. This is proved by induction over the cardinality of set $A \subseteq X$. If $|A| = 1$, the claim is trivial. Suppose the claim is true for sets with cardinality equal to or lower than $t \geq 1$, and let $|A| = t+1$. Consider the first two elements in A according to the labeling of the alternatives. We distinguish the following cases:

- If one of the alternatives is x_1 and the other is x_l with $1 < l \leq r$ then $c(A) = c(A \setminus \{x_l\})$ follows from the fact that $c \in \mathcal{C}^{SQB}$ and x_l does not dominate x_1. By the inductive hypothesis, $c(A \setminus \{x_l\}) = SPNE(G|(A \setminus \{x_l\});R)$. Given that for every $i \leq r$, R_i places x_1 on top of the ranking, it is obvious that $SPNE(G|(A \cup \{x_1, \ldots, x_l\});R) = x_1 = SPNE(G|(A \cup \{x_1, \ldots, x_{l-1}\}\});R)$. As a consequence $SPNE(G|(A \setminus \{x_l\});R) = SPNE(G|A;R)$. This proves that $c(A) = SPNE(G|A;R)$

- If one of the alternatives is x_1 and the other is x_l with $r < l \leq n$, it follows that every alternative in A different than x_1 dominates x_1 according to u. Then $c(A) = \underset{y \in A \setminus \{x_1\}}{\arg\max}\, h(u(y)) = c(A \setminus \{x_1\})$. By the inductive hypothesis, $c(A \setminus \{x_1\}) = SPNE(G|(A \setminus \{x_1\});R)$. Given that all linear orders R_i in the reduced game place x_1 on the bottom of the ranking, it is obvious that $SPNE(G|A;R) = SPNE(G|(A \setminus \{x_1\});R)$. This proves that $c(A) = SPNE(G|A;R)$.

- Finally, if $x_1 \notin A$, take any of the non-chosen elements $a \in A$. By the fact that c is rational over subsets of A, $c(A) = c(A \setminus \{a\})$. By the inductive hypothesis $c(A \setminus \{a\}) = SPNE(G|(A \setminus \{a\});R)$. By the fact that all linear orders R_i in the reduced game are given by $h(u(\cdot))$, we trivially have $SPNE(G|A;R) = SPNE(G|(A \setminus \{a\});R)$. This proves that $c(A) = SPNE(G|A;R)$.

Hence, we have that $\mathcal{C}^{SQB} \subseteq \mathcal{C}^{RGT}$. Given that $\mathcal{C}^{RGT} \subset \mathcal{C}^{SR}$, it obviously follows that $\mathcal{C}^{SQB} \subset \mathcal{C}^{SR}$. Notice that the order over the alternatives constructed above naturally defines a lexicographic order over the binary problems that could be used to prove directly the latter inclusion.

To show that the inclusion is strict, consider the choice function c_2 defined as $c_2(A) = c_1(A)$ for all $A \neq \{1,3,4\}$ and $A \neq X$ and $c_2(1,3,4) = c_2(X) = 4$. We now show that there exists a tree $(G;R)$ that rationalizes c_2 and hence $c_2 \in \mathcal{C}^{RGT}$. This game has 3 non-terminal nodes, each of them with two branches. The first non-terminal node points to alternatives 1 and 2. Every other non-terminal node $i \in \{2,3\}$ points to non-terminal node $i-1$ and to alternative i. Linear orders R_i satisfy the following conditions: $2R_1 1$, $3R_2 1, 3R_2 2$, and $4R_3 3, 4R_3 2, 1R_3 4$. It is immediate to check that $(G;R)$ rationalizes c_2. However, from the cycle involving alternatives $1,3,4$ we can immediately deduce that the status-quo can only be alternative 3. At the same time, from the cycle involving alternatives $1,2,4$ we can deduce that the status-quo should be alternative 2, which is absurd. Hence, $c_2 \notin \mathcal{C}^{SQB}$.$\square$

3. DISCUSSION

Theorem 1 establishes the relation between RGT and SR by showing that any RGT model can be equivalently rationalized through a sequence of rationales, each composed of a unique binary comparison, which we refer to as simple rationales. Thanks to its simplicity, the sequential rationalization of choice by simple rationales appears a particularly attractive procedure from a behavioral perspective. For this reason, we wonder whether all SR choice functions could be equivalently stated using only simple rationales. We address this question here.

In Theorem 2, (i) we establish that the equivalence does not hold in general, i.e., there are SR choice functions without an equivalent simple rationalization and, (ii) we show that SR choice functions using acyclic rationales, and only these, can be equivalently formulated in terms of simple rationales.[1] For completeness, we also show that in the prominent case of two asymmetric rationales, called Rational Shortlist Methods (RSM) (see Manzini and

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[1] See Bossert, Sprumont and Suzumura (2005) and Ehlers and Sprumont (2008) for a thorough discussion of models of rationalization by a single asymmetric or a single acyclic rationale.

Mariotti, 2007), we can indeed transform the two asymmetric rationales into a collection of simple rationales.[2]

Formally, we say that the binary relation P is *simple* whenever P is a singleton set containing a pair of different elements of X, i.e., $P = \{(x, y)\}$ with $x, y \in X$, $x \neq y$. We say that P is α whenever the structure of α lies between the structure of a simple rationale and an acyclic one. Examples of α rationales are (strict) partial orders (transitive and asymmetric) or semiorders (a special case of a transitive and asymmetric binary relation; see Manzini and Mariotti, 2009). Denote by SR(S), SR(Ac) or SR(α) those choice functions that can be sequentially rationalized using simple rationales, acyclic rationales or α rationales, respectively.

Theorem 2. $\mathcal{C}^{RSM} \subset \mathcal{C}^{SR(Ac)} = \mathcal{C}^{SR(\alpha)} = \mathcal{C}^{SR(S)} \subset \mathcal{C}^{SR}$.

Proof of Theorem 2: We start by proving that $\mathcal{C}^{RSM} \subseteq \mathcal{C}^{SR(Ac)}$. Suppose c is an RSM. Then, there exists a pair of asymmetric rationales $\{P_1, P_2\}$ that sequentially rationalizes c. We now construct a collection of acyclic rationales $\{P'_1, \ldots, P'_K\}$ that also sequentially rationalizes c. Let $P'_1 = P_1$ and given $P_2 = \{(a^1_2, b^1_2), \ldots, (a^r_2, b^r_2)\}$, define the following simple rationales $P'_{j+1} = \{(a^j_2, b^j_2)\}$, $1 \leq j \leq r$. First, we prove that all rationales are acyclic. This is immediate for the simple rationales $P'_2, \ldots, P'_{r+1}$. Suppose, by way of contradiction, that $P'_1 = P_1$ is cyclic. Then, there exists a collection $x_1, \ldots, x_r \in X$, with $r > 1$, such that $(x_i, x_{i+1}) \in P_1$, $i = 1, \ldots, r-1$, and $(x_r, x_1) \in P_1$. Then $M(\{x_1, \ldots, x_r\}, P_1) = \emptyset$, contradicting the fact that $\{P_1, P_2\}$ sequentially rationalizes c. Therefore, all rationales are acyclic.

Second, we show that the collection $\{P'_1, \ldots, P'_{r+1}\}$ sequentially rationalizes c. Take any A. Suppose that $M^{r+1}_1(A)$ contains two or more distinct elements. Take any two such elements $x, y \in M^{r+1}_1(A), x \neq y$. Then, for $j = 1, \ldots, r+1$, it is neither the case that $(x, y) \in P'_j$ nor that $(y, x) \in P'_j$. But then, for $i = 1, 2$ it is neither true that $(x, y) \in P_i$ nor that $(y, x) \in P_i$. This contradicts the fact that $\{P_1, P_2\}$ rationalizes the choice in $\{x, y\}$ and therefore $M^{r+1}_1(A)$ contains at most one element. We now prove that $c(A)$ belongs to $M^{r+1}_1(A)$. Given that c is sequentially rationalized by $\{P_1, P_2\}$, it follows that $c(A) \in M(A, P_1) = M(A, P'_1)$ and, for any $y \in M(A, P'_1)$, it cannot be the case that $(y, c(A)) \in P_2$. Therefore, there is no P'_j, $j = 2, \ldots, r+1$, such that $(y, c(A)) \in P'_j$, and then $c(A) \in M^{r+1}_1(A)$. Hence, $c(A) = M^{r+1}_1(A)$ and $\{P'_1, \ldots, P'_{r+1}\}$ is a collection of acyclic binary relations that rationalizes c. Therefore, it follows that $\mathcal{C}^{RSM} \subseteq \mathcal{C}^{SR(Ac)}$.

We now show that the inclusion is strict, i.e., $\mathcal{C}^{SR(Ac)} \not\subseteq \mathcal{C}^{RSM}$. To do so, we define a choice function c_3 that is in $\mathcal{C}^{SR(Ac)}$, but is not an RSM. Let $X = \{1, \ldots, 4\}$ and c_3 be such that: $c_3(1, 3) = 1$, $c_3(1, 2) = 2$, $c_3(1, 2, 3, 4) = c_3(1, 2, 3) = c_3(2, 3, 4) = c_3(2, 3) = c_3(3, 4) = 3$, and $c_3(1, 2, 4) = c_3(1, 3, 4) = c_3(1, 4) = c_3(2, 4) = 4$. Consider the ordered collection of simple

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[2] Salant and Rubinstein (2008) also study the case of two rationales within the framework of a 'limited attention' model.

rationales $\{P_1, \ldots, P_6\} = \{\{(2,1)\}, \{(1,3)\}, \{(4,1)\}, \{(3,2)\}, \{(4,2)\}, \{(3,4)\}\}$. It is easy to check that this collection of simple rationales sequentially rationalizes c_3. Note, however, that $4 = c_3(1,2,4) = c_3(1,3,4)$ but $c_3(1,2,3,4) = 3$. This means that c_3 violates the classic Expansion property, and then by Theorem 1 in Manzini and Mariotti (2007), c_3 is not an RSM.[3] Therefore, $\mathcal{C}^{RSM} \subset \mathcal{C}^{SR(Ac)}$.

We now show that $\mathcal{C}^{SR(Ac)} = \mathcal{C}^{SR(S)}$. That $\mathcal{C}^{SR(Ac)} \supseteq \mathcal{C}^{SR(S)}$ is immediate. In the other direction, we show that if c is SR(Ac), it is also SR(S). Let $\{P_1, \ldots, P_K\}$ a collection of acyclic rationales that sequentially rationalizes c. First, construct the collection of rationales $\{P'_1, \ldots, P'_K\}$ from $\{P_1, \ldots, P_K\}$, as follows: for all $j = 1, \ldots, K$, $(x,y) \in P'_j$ if and only if $(x,y) \in P_j$ and there is no $i < j$ such that $(x,y) \in P_i$ or $(y,x) \in P_i$. Clearly, $\{P'_1, \ldots, P'_K\}$ is an ordered collection of acyclic rationales that sequentially rationalizes c. Assume, without loss of generality, that the constructed collection $\{P'_1, \ldots, P'_K\}$ is composed of non-empty rationales (otherwise, simply remove the empty rationales and re-number them).

Now, consider a rationale P'_j in the constructed collection of rationales $\{P'_1, \ldots, P'_K\}$, that contains more than one pair of alternatives. Since P'_j is acyclic, there is a pair of alternatives (a,b) in P'_j, such that $(b,d) \notin P'_j$ for every $d \in X$. We can split the rationale P'_j into two rationales, $\{(a,b)\}$ and $P'_j \setminus \{(a,b)\}$. We show that for every $A \subseteq X$, $M(A, P'_j) = M(M(A, \{(a,b)\}), P'_j \setminus \{(a,b)\})$. If either a or b is not in A, then clearly $M(A, P'_j) = M(A, P'_j \setminus \{(a,b)\})$, and since $M(A, \{(a,b)\}) = A$, it follows that $M(A, P'_j \setminus \{(a,b)\}) = M(M(A, \{(a,b)\}), P'_j \setminus \{(a,b)\})$. Thus, $M(A, P'_j) = M(M(A, \{(a,b)\}), P'_j \setminus \{(a,b)\})$. If, on the contrary, $a, b \in A$, then $M(A, \{(a,b)\}) = A \setminus \{b\}$. Given that (a,b) in P'_j and that $(b,d) \notin P'_j$ for every $d \in X$, $M(A, P'_j) = M(A \setminus \{b\}, P'_j)$. Hence, $M(A, P'_j) = M(M(A, \{(a,b)\}), P'_j \setminus \{(a,b)\})$, as desired. It then follows that the ordered collection of rationales $\{P'_1, \ldots, P'_{j-1}, \{(a,b)\}, P'_j \setminus \{(a,b)\}, P'_{j+1}, \ldots, P'_K\}$ also sequentially rationalizes c. By iterating this rationale-splitting process, we end up with a collection of rationales $\{P^*_1, \ldots, P^*_{n(n-1)/2}\}$, each of which contains one pair of alternatives and that sequentially rationalizes c. Thus, $c \in SR(S)$ and $SR(Ac) = SR(S)$.

Now, given that $\mathcal{C}^{SR(S)} = \mathcal{C}^{SR(Ac)}$, since an α rationale lies, in terms of structure, between a simple and an acyclic rationale, it follows immediately that $\mathcal{C}^{SR(Ac)} = \mathcal{C}^{SR(\alpha)} = \mathcal{C}^{SR(S)}$.

Lastly, we show that $\mathcal{C}^{SR(S)} \subset \mathcal{C}^{SR}$. The fact that $\mathcal{C}^{SR(S)} \subseteq \mathcal{C}^{SR}$ is obvious, so we now prove that the inclusion is strict. Consider the following example. Let $X = \{1, \ldots, 6\}$ and c_4 be such that: $c_4(1,3,4,5) = c_4(1,3,4) = c_4(1,3,5) = c_4(1,4,5) = c_4(1,3) = c_4(1,5) = 1$, $c_4(2,3,6) = c_4(2,5,6) = c_4(2,5) = c_4(2,6) = 2$, $c_4(2,3,4,5,6) = c_4(2,3,4,5) = c_4(2,3,5,6) = c_4(3,4,5,6) = c_4(2,3,4) = c_4(2,3,5) = c_4(3,4,5) = c_4(3,5,6) = c_4(2,3) = c_4(3,4) = c_4(3,5) = 3$, $c_4(2,4,5,6) = c_4(1,4,6) = c_4(2,4,5) = c_4(2,4,6) = c_4(1,4) = c_4(2,4) = c_4(4,6) = 4$, $c_4(1,3,4,5,6) = c_4(1,3,5,6) = c_4(1,4,5,6) = c_4(1,5,6) = c_4(4,5,6) =$

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[3] **Expansion**: For all $A, B \subseteq X$, $x = c(A) = c(B) \Rightarrow x = c(A \cup B)$.

$c_4(4,5) = c_4(5,6) = 5$, $c_4(1,3,4,6) = c_4(2,3,4,6) = c_4(1,3,6) = c_4(3,4,6) = c_4(1,6) = c_4(3,6) = 6$, and $c_4(A) = c_4(A \setminus \{2\})$ whenever $\{1,2\} \subseteq A$.

We first show that $c_4 \in \mathcal{C}^{SR}$. Consider for instance $P_1 = \{(1,2)\}$, $P_2 = \{(1,3),(3,4),(4,2),(2,5),(5,6),(6,1)\}$, $P_3 = \{(5,4),(1,5),(2,6),(6,3),(3,5),(4,6)\}$ and $P_4 = \{(4,1),(3,2)\}$. One can verify that all choices are sequentially rationalized by this ordered collection of asymmetric rationales.

Suppose that c_4 is sequentially rationalized by the ordered collection of simple rationales $\{P_1,\dots,P_K\}$ with $P_i = \{(a_i,b_i)\}$, $i = 1,\dots,K$. Let T be the smallest positive integer such that $P_T \neq \{(1,2)\}$ and $P_T \neq \{(2,1)\}$. T is well-defined since the collection of simple rationales must rationalize the choice in set $\{1,3\}$, for instance.

We now show that for every A with $\{a_T,b_T\} \subseteq A$ and $\{1,2\} \not\subseteq A$, it must be that $c_4(A) = c_4(A \setminus \{b_T\})$. If $T = 1$, it is immediate that $c_4(A) = M_1^K(A) = M_1^K(A \setminus \{b_1\}) = c_4(A \setminus \{b_1\})$. If $T > 1$, $\cup_{i<T}\{a_i,b_i\} = \{1,2\}$, and then it follows that $c_4(A) = M_1^K(A) = M_T^K(A) = M_T^K(A \setminus \{b_T\}) = M_1^K(A \setminus \{b_T\}) = c_4(A \setminus \{b_T\})$. We now show that $(a_T,b_T) \notin \{(1,2),(1,5),(3,5),(4,1),(6,1),(4,6),(5,6),(1,3),(6,3),(2,6),(3,4),(2,5),(3,2),(4,2),(5,4)\}$. To see this, simply notice that $c_4(1,3,4,5,6) \notin \{c_4(1,3,4,6), c_4(3,4,5,6), c_4(1,3,4,5)\}$ implies that $(a_T,b_T) \notin \{(1,5),(3,5),(4,1),(6,1),(4,6),(5,6)\}$. Analogously, $c_4(2,3,4,6) \notin \{c_4(2,3,4), c_4(2,3,6)\}$ implies that $(a_T,b_T) \notin \{(2,6),(3,4)\}$, and $c_4(2,4,5) \notin \{c_4(4,5), c_4(2,5)\}$ implies that $(a_T,b_T) \notin \{(4,2),(5,4)\}$. Finally, from $c_4(1,3,4,6) \neq c_4(1,4,6)$, $c_4(2,3,4,5,6) \neq c_4(2,3,4,6)$ and $c_4(2,3,6) \neq c_4(3,6)$ we have that $(a_T,b_T) \notin \{(1,3),(6,3),(2,5),(3,2)\}$.

Given the binary choices in c_4, (a_T,b_T) is then a pair such that $b_T = c(a_T,b_T)$. However, $M_1^K(a_T,b_T) = M_T^K(a_T,b_T) = a_T$ leading to a contradiction with the fact that the simple rationales sequentially rationalize c_4. It then follows that $c_4 \in \mathcal{C}^{SR} \setminus \mathcal{C}^{SR(S)}$, which concludes the proof of the theorem.$\square$

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