Employment Fluctuations with Downward Wage Rigidity: the Role of Moral Hazard[1]

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Abstract: This paper considers a dynamic matching model with imperfectly observable worker effort. In equilibrium, the wage distribution is truncated from below in order to satisfy a no-shirking condition. This downward wage rigidity induces the same type of inefficient churning and "contractual fragility" as in Ramey and Watson (1997). Nonetheless, the surprising lesson of our analysis is that workers' shirking motive reduces the cyclical fluctuations in job destruction, because firms are forced to terminate some marginal jobs in booms which they cannot commit to maintain in recessions. This time-inconsistency problem casts doubt upon the importance of inefficient churning as an explanation of observed employment fluctuations. On the other hand, the no-shirking condition implies that firms' share of surplus is procyclical, which can amplify fluctuations in job creation. Thus, our model is consistent with recent evidence that job creation is more important than job destruction in driving labor market fluctuations. Furthermore, unlike most models with endogenous job destruction, we obtain a robust Beveridge curve.

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1 Introduction

Matching models are by now possibly the most important framework for studying unemployment, job creation and job destruction, and labor market behavior in general.[1] Nonetheless, many recent papers, including Cole and Rogerson (1999), Fujita (2005), and Ravn (2006), have questioned their empirical success. One issue that has proved especially controversial is the claim that matching models fail to generate the degree of cyclical volatility in unemployment and vacancies observed in the data (Hall 2005A; Shimer 2004, 2005A; Costain and Reiter 2005; Hagedorn and Manovskii 2005).

While several alternatives have been proposed,[2] two possible mechanisms that might increase labor market volatility have been particularly influential. One line of research, advocated by Hall and Shimer,[3] suggests that rigid wages may generate volatility, because wage rigidity implies greater variation in profits, thus amplifying fluctuations in job creation and unemployment. However, attempts to model wage stickiness in a microfounded way are still in their infancy (Hall 2005A; Menzio 2005; Shimer and Wright 2005; Kennan 2006). It thus remains an open question whether rigid wages have large effects on volatility when an equilibrium model of wage stickiness is used. A second (but earlier) line of research, initiated by Ramey and Watson (1997), shows that incentive problems may amplify fluctuations in job destruction and unemployment. In these models production requires unobservable effort. As a result, the continuation value of a match needs to satisfy at least one and sometimes two incentive compatibility constraints to prevent shirking, and small perturbations of productivity may cause a wave of inefficient separations.[4] The merit of this second line of research is that it is based on a coherent theory of wage rigidity. However, the existing papers have focused on steady states, or analyzed the effects of a single productivity shock starting from arbitrary initial conditions.

In this paper, we construct a dynamic matching model with endogenous job creation and destruction which allows us to study an incentive problem like that of Ramey and Watson (1997) in the context of cyclical fluctuations in aggregate labor productivity.

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[1]Recent surveys of the matching literature include Mortensen and Pissarides (1999), Petrongolo and Pissarides (2001), Rogerson, Wright, and Shimer (2005) and Yashiv (2006).

[2]See for example Costain and Reiter (2005) and Mortensen and Nagypal (2005).

[3]See also Gertler and Trigari (2005), Walsh (2004), and Lubik and Schorfheide (2005).

[4]Other examples include Den Haan, Ramey and Watson (1999), Mortensen and Pissarides (1999) Rocheteau (2001), and Jansen (2001). The first paper studies a repeated prisoner's dilemma with shirking by managers and workers, while the other papers consider shirking by workers only.

We model creation and destruction like Mortensen and Pissarides (1994), by including both aggregate and match-specific productivity shocks. The incentive problem we consider is that of Shapiro and Stiglitz (1984): firms cannot perfectly monitor workers' effort, so the wage bargain must satisfy an incentive compatilibility constraint that prevents shirking.[5] The no-shirking condition truncates the wage distribution from below, and it is interesting to ask how the labor market fluctuates under this micro-founded form of real downward wage rigidity. On the job creation margin, because we assume labor productivity is cyclical but the disutility of effort is not, the no-shirking constraint implies that workers' share of surplus may be higher in recessions. Since this further encourages firms to hire in booms rather than recessions, our model's mechanism may help to amplify the variation in job creation. On the job destruction margin, the no-shirking constraint forces firms sometimes to terminate jobs with a strictly positive surplus. We can therefore also use our model to investigate whether recessions exhibit large waves of inefficient churning due to "contractual fragility".[6]

The debate on unemployment volatility is not the only reason it is interesting to combine the matching structure of Mortensen and Pissarides (1994) with the "efficiency wages" of Shapiro and Stiglitz (1984). Our model can also address several other prominent issues in recent literature. First, using new data, Shimer (2005B) and Hall (2005B) argue that variations in job creation matter more for explaining unemployment dynamics, and variations in job destruction matter less, than was previously thought. Our model of downward wage rigidity can shed light on the variations along both these margins. Second, one of the most robust stylized facts about the labor market is the negative correlation between unemployment and vacancies (the "Beveridge curve"). Yet previous papers with time-varying job destruction (Cole and Rogerson 1999; Mortensen and Pissarides 1994; Costain and Reiter 2005; Den Haan *et.al.* 2000) have often found that the Beveridge curve is delicate in the model. Third, on the theoretical side, our paper helps to correct a misconception about the dynamic properties of efficiency wage models. In models without matching frictions (Kimball 1994; Kiley 1997), it has been argued that efficiency wages serve to smooth the flow of profits to the firm, by driving down the wage in periods when unemployment is high. In a matching context, too, wages fall in recessions. But more importantly, a negative aggregate shock makes it more likely that the incentive compatibility constraint will

[5]Mortensen and Pissarides (1999), Rocheteau (2001), and Jansen (2001) make the same assumption, but they focus on a stationary environment without aggregate shocks.

[6]Ramey and Watson's (1997) term "contractual fragility" captures the idea that parties cannot write enforceable contracts that stipulate the desired effort level, so that firms may be forced to sever relationships that would survive under optimal employment contracts.

bind, decreasing firms' share of surplus, and thus amplifying changes in firms' profits. Finally, on the policy side, our economy has scope for welfare-improving employment protection regulation to reduce the extent of inefficient churning (*e.g.* Jansen, 2001).

Solving our model also involves some unusual technical issues, because firms' layoff decisions turn out not to be a standard optimal stopping problem. Under perfect information, our match surplus function would be continuous at the productivity level where separation becomes mutually optimal. But with unobservable effort, violation of the no-shirking condition implies that output drops suddenly to zero, causing a discontinuity in the surplus function. A related difficulty is that even taking as given aggregate labor market outcomes, the separation problem of a firm-worker pair may have multiple solutions, because match surplus and therefore also incentive compatibility both depend on the firm's reservation strategy. That is, if for any reason workers expect to be fired more frequently, firms will need to pay a higher wage to induce effort, and may therefore find it optimal to fire at the higher rate anticipated by workers. Due to this positive feedback between the reservation strategy and the minimum incentive compatible wage, the standard arguments based on contraction principles do not apply. Nonetheless, using a fixed point argument like that of Rustichini (1998), we show that there is a unique maximal surplus function and minimal vector of reservation productivities that satisfy the worker's no-shirking condition. In other words we characterize equilibria in which workers and firms only separate if there is no feasible way to induce worker effort with non-negative profits.

To preview our results, recall that our assumptions make it more costly for firms to provide incentives in recessions than in booms because the worker's minimum incentive compatible surplus is a larger share of the total surplus of a job. Nonetheless, the surprising lesson of our paper is that the shirking problem tends to smooth the cyclical fluctuations in job destruction. The intuition behind this finding is a *time-inconsistency problem*. In booms some marginal jobs survive that are destroyed when the economy enters into a recession. Since the firms cannot commit to maintain these jobs in the future, they need to pay a much higher flow surplus on top of workers' reservation wage than in the rest of the jobs. Hence, the shirking motive leads to more churning in all states, but we find that the layoff probability especially increases in good states. Thus, contractual fragility arguments fail to hold in an economy with cyclical fluctuations. We obtain smaller fluctuations in job destruction than in a comparable model without moral hazard, and for some parameter values the endogenous reservation productivity can even be constant over the cycle.

On the other hand, the fact that firms' share of surplus tends to rise in booms may increase the volatility of hiring expenditures and job creation. Therefore, the overall effect of the shirking motive on unemployment volatility is ambiguous: it tends to smooth fluctuations in job destruction but amplify fluctuations in job creation. While this means that our model makes limited progress on the unemployment volatility issue, it is nonetheless strikingly consistent with recent claims that unemployment variability is driven mostly by job creation, not by job destruction. Moreover, the fact that our model amplifies fluctuations in job creation while diminishing those of job destruction, also means that it tends to exhibit a robust Beveridge curve.

The rest of the paper is organized as follows. Section 2 describes the model. In Section 3 we characterize the unique value function of a firm-worker pair that maximizes the joint surplus subject to the no-shirking condition, and use this characterization to develop an algorithm for calculating a jointly-optimal no-shirking equilibrium. Section 4 discusses the intuition for the smoothing of the fluctuations in job destruction for the special case of 2 aggregate states and in section 5 we report some numerical results that corroborate our theoretical insights. Section 6 concludes.

2 Model

This section presents a continuous-time, infinite horizon matching model with imperfectly observable worker effort.

2.1 Preferences and production technology

Our economy is populated by a continuum of workers with measure normalized to one. There is also a continuum of firms; the number of firms is infinitesimal compared with the number of workers. All agents are risk-neutral and discount the future at the common rate r.

Workers are identical and derive utility from consumption and leisure. The instantaneous utility function of a worker is given by:[7]

$$U(c, n) = c + (1 - n)b, \tag{1}$$

where c denotes consumption, $n \in \{0, 1\}$ is the fraction of time devoted to work and b is the imputed value of leisure. Without loss of generality we assume that workers consume their entire income at any moment. During employment c is therefore equal to the worker's wage w. In addition, workers can obtain a private gain from shirking that is assumed to be equal to the leisure a worker would get from not going to work,

[7] Similar payoffs are assumed by Mortensen and Pissarides (1999), Marimon and Zilibotti (1999) and Rocheteau (2000).

b (this normalization is also without loss of generality). Accordingly, we can write the flow utility of a worker who exerts effort as $U(w,0) = w$, while the utility of a worker who shirks is $U(w,1) = w+b$. Unemployed workers, on the contrary, receive no income and just enjoy leisure $U(0,1) = b$. Finally, the discounted lifetime utility of a worker with income and working time paths $\{z(t); t \in \mathbf{R}^+\}$ and $\{n(t); t \in \mathbf{R}^+\}$ equals

$$\int_{\mathbf{R}^+} \exp(-rt) U[z(t), n(t)] dt. \tag{2}$$

All firms are identical and, in equilibrium, they have a continuum of jobs that are either filled with a worker or vacant. Besides effort, the productivity of a firm-worker pair depends on two components: a match-specific shock x and an aggregate shock X that affects all firms in the economy. Both the worker and firm observe x and X perfectly. Their flow output, denoted by $y(x, X; n)$, is given by

$$y(x, X; n) = \begin{cases} y(x, X) & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Notice that the productivity of a job is independent of the size of a firm. Thus firms are simply a collection of jobs. The output $y(x, X)$ produced by a worker who exerts effort is assumed to be a strictly increasing function of x and X, continuously differentiable in x, with $\lim_{x\to\infty} y(x, X) = \infty$.

Ongoing jobs may be hit by new idiosyncratic productivity shocks x, which arrive at Poisson rate λ. These shocks are *i.i.d.* draws from a distribution F with support $[\underline{x}, \bar{x}]$. In the main body of the text, we will assume that all newly created jobs have the highest possible idiosyncratic productivity $\bar{x}$. In Appendix 1 we consider an alternative specification, in which new jobs have random productivity.

New aggregate productivity shocks X arrive at rate μ, and are drawn from the discrete state space $\{1, 2, ...N\}$, where 1 is the worst state and N is the best state. Conditional on the arrival of a new shock, the probability that the state changes from X to Z is denoted by G_{ZX}, and we write the matrix of Markov transition rates as

$$G \equiv \begin{pmatrix} G_{11} & & G_{1N} \\ & \cdots & \\ G_{N1} & & G_{NN} \end{pmatrix}$$

Here, column j represents the probabilities of the N possible states that could follow state j, so each column sums to one.

2.2 Moral hazard

To introduce a shirking motive, we assume that firms cannot perfectly monitor individual effort. At any moment in time, the firm observes total output, but given that firms have a continuum of workers this does not reveal information about the effort of individual employees.

Faced with this moral hazard problem, firms offer incentives by paying workers a surplus and by committing to fire workers who are caught shirking. We assume that the firm's participation in the match causes it to observe worker's effort at the Poisson rate ϕ. Firing observed shirkers (off the equilibrium path) is an equilibrium strategy for the firm if failing to do so would cause all workers to shirk. Shirking by all workers (off the equilibrium path) is an equilibrium strategy for the workers since individual workers cannot demonstrate to the firm that they are not shirking.[8] In other words, an equilibrium within the firm involving effort by all workers, under a threat of firing, is sustained by trigger strategies involving a jump to a new equilibrium at that firm involving shirking by all workers, and therefore separation of all that firm's matches.

As in Shapiro and Stiglitz (1984), equilibria of this form must satisfy an incentive compatibility constraint. This constraint, referred to as the no-shirking condition (NSC), will act as a lower-bound on the outcomes during the wage negotiations. In the remainder of this section we embed our version of the shirking model into a matching model of unemployment.

2.3 Matching

Unemployed workers and firms are matched together in pairs through an imperfect matching technology (*e.g.* Pissarides 2000). The gross rate of formation of new matches m_t is given by

$$m_t = M(u_t, v_t) \tag{4}$$

where u_t is the number of unemployed workers, and v_t is the number of vacancies open, at time t. We assume M exhibits constant returns to scale. Therefore, the worker's probability of finding a match, per unit of time, can be written in terms of tightness $\theta_t \equiv v_t/u_t$ as

$$p(\theta_t) = \frac{M(u_t, v_t)}{u_t} = M\left(1, \frac{v_t}{u_t}\right) \tag{5}$$

[8] Assuming that the firm is capable of monitoring more often, at a cost, equilibria with monitoring rates higher than the exogenous rate ϕ might be sustainable. Such equilibria would depend on workers' ability to observe or infer the firm's monitoring rate. We will not enter into these complications here.

Similarly, the probability that an open vacancy finds a match is

$$q(\theta_t) \;=\; \frac{M(u_t, v_t)}{v_t} \;=\; M\left(\frac{u_t}{v_t}, 1\right) \tag{6}$$

so that $p(\theta) = \theta q(\theta)$.

2.4 The value of matching

Before stating the Bellman equations for workers' and firms' value functions, we assume two restrictions on the equilibrium which are known to be valid for related models (Mortensen and Pissarides 1994, Cole and Rogerson 1999). First, we assume that aggregate jump variables may depend on the aggregate productivity state X, and that match-specific jump variables may depend on x and X, but that neither may depend on other state variables, like the unemployment rate or the distribution of idiosyncratic productivities across existing jobs. We will see that the Bellman equations can be written in terms of x and X only, so it is not unreasonable to conjecture that such a minimum-state equilibrium exists. Second, we impose the reservation property. That is, we assume there exists a vector of reservation productivities $R(X)$ such that matches with idiosyncratic productivity x continue in state X if and only if $x \geq R(X)$. In our numerical work, we prove by construction that equilibria of this form exist, although this does not rule out other types of equilibria. For notational convenience, we will refer to the vector of reservation productivities as R, and the continuation region as $\mathcal{C}(R)$. That is, a match continues if productivity lies in the set $\mathcal{C}(R) \equiv \{(x, X) : x \geq R(X)\}$.

We can now spell out the Bellman equations. Call the wage $w(x, X)$, and let the value functions of employed and unemployed workers be $W(x, X)$ and $U(X)$, respectively. For any state $(x, X) \in \mathcal{C}(R)$, function W must satisfy:

$$\begin{aligned} rW(x,X) \;=\; & w(x,X) + \lambda\left[\int_{R(X)}^{\overline{x}} W(z,X)dF(z) + F(R(X))U(X) - W(x,X)\right] \\ & + \mu\left\{\sum_{Z:x\geq R(Z)} G_{ZX}\left[W(x,Z) - W(x,X)\right] \;+\; \sum_{Z:x<R(Z)} G_{ZX}\left[U(Z) - W(x,X)\right]\right\} \end{aligned} \tag{7}$$

This equation states that the flow of returns to a matched worker includes the wage, plus two flows of expected capital losses and gains: the gains from drawing a new idiosyncratic shock z, at rate λ; and the gains from switching to a new aggregate state Z, drawn with conditional probability G_{ZX}, at rate μ. Conditional on an idiosyncratic shock, the separation probability is $F\left(R(X)\right)$, and conditional on an aggregate shock, separation occurs if the current idiosyncratic x is less than the new reservation productivity, $R(Z)$.

The unemployed search for jobs while obtaining a constant flow payoff b from leisure. Let $\theta(X)$ be labor market tightness, and suppose the rate of job finding is $p(\theta(X))$. Then for any X, the value of unemployment satisfies:

$$rU(X) = b + p(\theta(X))\, N^W(X) + \mu \sum_Z G_{ZX}\left[U(Z) - U(X)\right] \tag{8}$$

where $N^W(X)$ is the worker's expected increase in value from a new job offer. Since we assume new jobs are drawn from the top of the distribution, the gain from a job offer is

$$N^W(X) = \left[W(\overline{x}, X) - U(X)\right] \tag{9}$$

and new jobs are always accepted in equilibrium.

Now consider the value functions associated with vacancies, $V(X)$, and filled jobs, $J(x, X)$. For any state (x, X) in the continuation region $\mathcal{C}(R)$, the value of a filled vacancy satisfies:

$$\begin{aligned} rJ(x,X) \;&=\; y(x,X) - w(x,X) + \lambda\left[\int_{R(X)}^{\overline{x}} J(z,X)\,dF(z) + F(R(X))V(X) - J(x,X)\right] \\ &+ \mu\left\{\sum_{Z:x\geq R(Z)} G_{ZX}\left[J(x,Z) - J(x,X)\right] + \sum_{Z:x<R(Z)} G_{ZX}\left[V(Z) - J(x,X)\right]\right\} \end{aligned} \tag{10}$$

Thus the flow of profits to the matched firm consists of output minus wages, plus two flows of expected losses and gains analogous to those of the worker.

Next, suppose that maintaining a vacancy costs c per period, and that vacancies are filled at rate $q(\theta(X))$. Then for each X, the value of a vacancy must satisfy:

$$rV(X) = -c + q(\theta(X))\, N^F(X) + \mu \sum_Z G_{ZX}\left[V(Z) - V(X)\right] \tag{11}$$

where $N^F(X)$ is a firm's expected increase in value resulting from finding a possible match. Since new jobs come from the top of the productivity distribution, we have

$$N^F(X) = \left[J(\overline{x}, X) - V(X)\right] \tag{12}$$

Lastly, we assume that firms are free to open any number of vacancies. Thus, in equilibrium, the value of a vacancy is zero in any aggregate state X:

$$V(X) = 0 \tag{13}$$

so hereafter V will disappear from our equations.

2.5 Incentive compatibility

We are now in a position to derive the NSC. A worker will never shirk if the gain from shirking during a short interval dt is less than the expected cost of a disciplinary layoff in case the worker is detected. The logic also works in the opposite direction. If it pays to shirk during a short period dt, then workers will always choose this option.

Formally, let $W^s(x, X)$ denote the value function for a worker who shirks during the interval dt. Assuming that the worker exerts effort during the rest of the time the firm-worker pair remains together, we obtain

$$\begin{aligned} rW^s(x,X)\,dt &= w(x,X)\,dt + bdt + \phi dt\,[U(X) - W(x,X)] \\ &+\lambda dt\left[\int_{R(X)}^{\overline{x}} W(z,X)\,dF(z) + F(R(X))U(X) - W(x,X)\right] \\ + \mu dt &\left\{ \sum_{Z:x\geq R(Z)} G_{ZX}\,[W(x,Z) - W(x,X)] + \sum_{Z:x<R(Z)} G_{ZX}\,[U(Z) - W(x,X)] \right\} + o(dt) \end{aligned} \tag{14}$$

where $o(dt)$ signifies a quantity which becomes negligible compared to dt as $dt \to 0$.

Comparing this equation to (7), dividing by dt and taking the limit as $dt \to 0$, we find that the only difference between shirking and not shirking is

$$rW^s(x,X) - rW(x,X) = b + \phi\,(U(X) - W(x,X))\,.$$

Hence, workers (weakly) prefer not to shirk as long as their surplus exceeds b/ϕ:

$$W(x,X) - U(X) \geq \frac{b}{\phi}, \tag{15}$$

where b/ϕ is the expected gain in leisure before the worker is caught shirking.[9] The above inequality acts as an incentive-compatibility constraint that must be satisfied at all states $(x, X) \in \mathcal{C}(R)$ since we rule out temporary layoffs.

2.6 Wages and turnover

The worker and firm bargain over the wage flow $w(x, X)$. The wage can be renegotiated after any shock, and could also be renegotiated more frequently, but there is no incentive to do so in our equilibrium. Other transfers that could alleviate the moral hazard problem of workers, such as shirking penalties or bond payments, are ruled out.

[9] In what follows we vary ϕ to generate different values for b/ϕ. This is why assuming the gain from shirking equals the value of unemployed leisure is just a notational simplification that implies no loss of generality.

As usual, we assume that the flow wage is determined through Nash bargaining. For any state (x, X), we define the total surplus relative to the threat point of separation, as follows:

$$S(x, X) = W(x, X) - U(X) + J(x, X) \tag{16}$$

We assume the worker receives a fraction β of this total surplus, unless the incentive compatibility constraint binds, in which case the wage must rise until the constraint is satisfied.[10] Thus, for states (x, X) in the continuation region $\mathcal{C}(R)$, the worker's surplus is given by:

$$W(x, X) - U(X) = \max\{\beta S(x, X), b/\phi\} \tag{17}$$

while the surplus of the firm satisfies:

$$J(x, X) = \min\{(1-\beta)S(x, X), S(x, X) - b/\phi\} \tag{18}$$

Of course, the firm also has the possibility of separating from the match. So for any $(x, X) \in \mathcal{C}(R)$, the firm's surplus must satisfy

$$J(x, X) \ \geq \ 0 \tag{19}$$

which, together with (15) and (16) implies that

$$S(x, X) \ \geq \ b/\Phi \tag{20}$$

for (x, X) in the continuation region $\mathcal{C}(R)$.

Given that the surplus is split according to the rules (17) and (18), (20) is both a necessary and sufficient condition for match continuation. Since workers and firms are better off separated outside the continuation region, for $(x, X) \notin \mathcal{C}(R)$ we can define

$$W(x, X) - U(X) \ = \ J(x, X) \ = \ S(x, X) \ = \ 0 \tag{21}$$

2.7 Privately optimal outcomes

In our economy separations correspond to layoffs. A firm severs a relationship when it is no longer profitable to pay the worker an incentive compatible wage. Workers, on the other hand, base their effort decisions on their beliefs about the duration of their jobs. From existing studies we know that this non-cooperative choice of effort and reservation strategies may lead to multiple Pareto rankable outcomes (Den Haan *et al.*

[10] It is relatively straightforward to derive these conditions from a Nash bargaining game that determines the wage over a short interval dt. Details are available upon request.

1999; Mortensen and Pissarides 1999). This multiplicity is due to a positive feedback between the reservation productivities and the minimum incentive-compatible wage chosen inside a particular match at a given labor market tightness. Intuitively, suppose a worker anticipates an increase in the reservation productivity for the current state. Since this implies shorter expected job duration, the worker needs a higher wage than before in order to earn the surplus b/ϕ, and the higher wage floor may force firms to fire at the higher reservation productivity.

Shortly, in Prop. 1, we will show that there exists a unique R with lower reservation productivities, and therefore higher expected duration and surplus, than any other. In other words, a deviation by the worker and firm alone— without any change in aggregate conditions— can select a unique reservation policy which is mutually optimal, subject to incentive compatibility, for both worker and firm. Therefore, we think it makes sense to rule out other types of equilibria and to focus on reservation policies that are constrained optimal for the firm-worker pair.[11] In the next section we will characterize these privately optimal outcomes.

3 Analysis

3.1 The match surplus equation

As a first step towards defining equilibrium, we now perform some simplifications for a more concise description of our economy. In the continuation region $\mathcal{C}(R)$, equation (7) can be rewritten as:

$$(r+\lambda+\mu)\,W\left(x,X\right) = w\left(x,X\right) + \lambda\left[\int_{R(X)}^{\overline{x}} W\left(z,X\right)dF(z) + F(R(X))U(X)\right]$$
$$+\mu\left[\sum_{Z:x\geq R(Z)} G_{ZX}W\left(x,Z\right) + \sum_{Z:x<R(Z)} G_{ZX}U\left(Z\right)\right]$$

Therefore, an employed worker's surplus $W\left(x,X\right)-U\left(X\right)$ satisfies

$$\begin{aligned}(r+\lambda+\mu)\left(W\left(x,X\right)-U\left(X\right)\right) \;=\; & w\left(x,X\right)+\lambda\int_{R(X)}^{\overline{x}}\left(W\left(z,X\right)-U\left(X\right)\right)dF(z) \\ & -b-p(\theta(X))N^{W}(X)+\mu\sum_{Z:x\geq R(Z)}G_{ZX}\left(W\left(x,Z\right)-U(Z)\right)\end{aligned} \tag{22}$$

[11] For a thorough analysis of this problem, see Jansen (2001).

where we have used (8) to eliminate $rU(X)$ on the right hand side. The surplus of a filled job is similar, but can be simplified further by setting $V(X) = 0$ for all X:

$$(r+\lambda+\mu)\, J\,(x,X) \;=\; y(x,X) - w(x,X) + \lambda \int_{R(X)}^{\overline{x}} J(z,X)dF(z) + \mu \sum_{Z:x\geq R(Z)} G_{ZX} J(x,Z) \tag{23}$$

Summing equations (22) and (23), we obtain:

$$(r+\lambda+\mu)\, S\,(x,X) \;=\; y(x,X) - b - p\,(\theta(X))\, N^{W}(X) + \lambda \int_{R(X)}^{\bar{x}} S\,(z,X)\, dF(z) + \mu \sum_{Z:x\geq R(Z)} G_{ZX} S(x,Z) \tag{24}$$

This expression is fairly intuitive: we see that the surplus includes the flow payoff y minus the flow payoff b associated with unemployment and minus the gains that accrue to unemployed workers from finding new jobs, plus capital gains due to individual and aggregate shocks. The next subsections explain how we solve (24) to find S, which is the key to solving the model. Readers who wish to skip these technicalities may prefer to jump to the equilibrium definition in Sec. 3.4 and then to Sec. 4 where we solve the simpler special case of just two aggregate states.

3.2 Jointly optimal continuation

Ever since we wrote down the value functions in Sec. 2.4, we have been assuming that separation is governed by a reservation strategy. Given N aggregate states, there must be N reservation productivities $\{R_1, ...R_N\}$. However, these need not all be distinct: some aggregate states could share the same reservation productivity. For notational convenience, we will number the reservation productivities backwards, as

$$R_{N+1} \leq R_N \leq \quad \ldots \quad \leq R_1 \leq R_0$$

where we have also defined $R_{N+1} \equiv \underline{x}$ and $R_0 \equiv \overline{x}$. We can then divide up the support $[\underline{x}, \overline{x}]$ of the idiosyncratic shock into $N+1$ intervals of the form $I_i \equiv [R_i, R_{i-1})$. (If some of the reservation productivities are equal, then some of these intervals are empty.) Since we are assuming that all new jobs have the best productivity, it is helpful to treat the maximum productivity as a separate interval $I_0 \equiv \{\overline{x}\}$.

We also assumed, in Sec. 2.7, that the worker-firm pair chooses a mutually optimal reservation strategy. Basically, this means that the pair must never separate as long as there is enough surplus to prevent shirking. But we must define this condition carefully, because the surplus itself depends on the reservation strategy which the pair expect to follow in the future.

So to describe optimal continuation, we must calculate the surplus associated with continuing in the match, which we will call T. More precisely, suppose a worker-firm pair anticipate continuing as long as their current state (x, X) remains unchanged. And suppose that after (x, X) changes (due to an idiosyncratic or aggregate shock), they expect to follow reservation strategy R and expect the value of their match to be $S(x, X)$, where S is a nonnegative function, weakly increasing in x, defined for $x \geq \underline{x}$. Moreover, suppose aggregate conditions are given by nonnegative N-dimensional vectors θ and N^W.[12] The value T of remaining matched as long as current conditions are unchanged can be calculated from the right-hand side of (24), as follows:

$$T(x, X; S, R, \theta, N^W) \equiv$$

$$(r+\lambda+\mu)^{-1}\left\{y(x,X) - b - p\left(\theta(X)\right)N^W(X) + \lambda\int_{R(X)}^{\bar{x}} S\left(z,X\right)dF(z) + \mu \sum_{Z: x \geq R(Z)} G_{ZX}S(x,Z)\right\} \tag{25}$$

For the proofs of the following propositions, one especially important property of the continuation value T is that if S nonnegative, then an increase in R decreases T.

Joint efficiency requires that the pair continue as long as incentive compatibility is satisfied: that is, in any state (x, X) such that $T(x, X; S, R, \theta, N^W)$ is at least equal to b/ϕ. But note that T itself depends on the reservation strategy R. Therefore, given any candidate reservation strategy R, we can calculate a new reservation strategy $\tilde{R}$ as follows:

$$\tilde{R}(X) = \min\{x \in [\underline{x}, \infty) : T(x, X; S, R, \theta, N^W) \geq b/\phi\} \tag{26}$$

The true reservation strategy associated with a given surplus function S must be a fixed point of the mapping (26). One unambiguously lowest fixed point exists, as the following lemma shows.

Lemma 0. *Given any aggregate conditions $\theta \geq \overrightarrow{0}$ and $N^W \geq \overrightarrow{0}$, and given any nonnegative surplus function $S(x, X)$ that is weakly increasing for $x \in [\underline{x}, \infty)$, there exists a unique vector $\underline{R}$ such that:*

1. *$\underline{R}$ solves (26) given S, θ, and N^W*
2. *If there exists another fixed point R' of (26) then $\underline{R}(X) \leq R'(X)$ for all X.*

[12] For now we are looking at the pair's partial equilibrium decision, given arbitrary aggregate conditions θ and N^W. Therefore we need not yet impose consistency between S and R and θ and N^W.

Proof. See Appendix 2.

We will use the notation $R^*(S, \theta, N^W)$ to refer to the lowest fixed point $\underline{R}$ identified in Lemma 0, showing its dependence on S, θ, and N^W. Note that an increase in S increases T, causing $R^*(S, \theta, N^W)$ to (weakly) decrease.

Like R, we can also think of the surplus function S as the solution to a fixed point problem. Inside the continuation region, the surplus function is given by T; outside, by definition, it is zero. But T depends on S. Therefore given any candidate surplus function S, we can define a new surplus function $\tilde{S}$ as follows:[13]

$$\tilde{S}(x, X) = \begin{cases} 0 & \text{for} \quad x < R(X) \\ T(x, X; S, R, \theta, N^W) & \text{for} \quad R(X) \le x \le \bar{x} \\ \tilde{S}(\bar{x}, X) & \text{for} \quad x > \bar{x} \end{cases} \tag{27}$$

The surplus function must be a fixed point of the mapping (27). Our next proposition shows that there is a unique fixed point of (27) which maximizes surplus; associated with it is a unique, lowest possible vector of reservation productivities. In other words, for any aggregate conditions, there is a unique reservation strategy that is jointly optimal for the worker-firm pair.

Proposition 1. *For any aggregate conditions $\theta \ge \overrightarrow{0}$ and $N^W \ge \overrightarrow{0}$, there exists a unique pair $\bar{S}$ and $\underline{R}$ such that:*

1. *$\underline{R}$ solves (26) given surplus function $\bar{S}$*
2. *$\bar{S}$ solves (27) given reservation vector $\underline{R}$*
3. *If there exists another pair (S', R') that solve (26) and (27), then $\underline{R}(X) \le R'(X)$ and $\bar{S}(x, X) \ge S'(x, X)$ for all x and X.*

Proof. See Appendix 2.

The proof of Prop. 1 is closely based on Rustichini's (1998) method for incentive-constrained problems. It constructs a monotone sequence of functions S^i, for $i \in \{0, 1, 2, \ldots\}$, by iterating on (27). The initial function S^0 is weakly increasing in x, and (27) preserves this property. In fact, (27) maps weakly increasing functions into functions that are strictly increasing in the continuation interval. Function S^0 is also weakly increasing in X, and two conditions suffice to ensure that (27) preserves this property. First, higher X has a sufficiently large effect on productivity:

$$y(x, X+1) - y(x, X) > p(\theta(X+1))N^W(X+1) - p(\theta(X))N^W(X) \tag{28}$$

[13]It is convenient to define S for $x > \bar{x}$, even though these values of x never occur, because this ensures that the "min" in (26) is always well-defined.

for all $x \in [\underline{x}, \bar{x}]$ and $X \in \{1, 2, ...N-1\}$. Second, aggregate shocks X exhibit first-order stochastic dominance, so that a higher X now tends to predict higher X in the near future. The transition matrix for aggregate states over a short period dt can be written as $(1-\mu dt)I + \mu G dt$. First-order stochastic dominance of G itself suffices for first-order stochastic dominance of $(1-\mu dt)I + \mu G dt$, but is not necessary.

Since the monotonicity properties of S are preserved at each step i, they also hold in the limit, and this also implies that the limiting reservation productivities are increasing in X. Therefore we have proved the following corollary.

Corollary 2. *Suppose $\theta \geq \overrightarrow{0}$ and $N^W \geq \overrightarrow{0}$ satisfy (28), and that aggregate transitions across states X exhibit first-order stochastic dominance. Then the jointly optimal fixed point pair $(\bar{S}, \underline{R})$ of (26) and (27) has the following properties:*

1. *Function $\bar{S}$ is strictly increasing in x for $x \in [R(X), \bar{x}]$*
2. *Function $\bar{S}$ is weakly increasing in X*
3. *The vector of reservation productivities $\underline{R}$ is weakly decreasing in X.*

From now on, we will assume aggregate shocks exhibit first-order stochastic dominance, and we will restrict attention to equilibria satisfying (28), so that the reservation productivities are monotonic. Therefore the surplus function will be increasing in both arguments, which immediately implies that the reservation productivities are decreasing. Hence, the N reservation productivities which we called $R_N \leq R_{N-1} \leq ... \leq R_1$ correspond, in order, to the N aggregate states: $R(N) \leq R(N_{N-1}) \leq ... \leq R(1)$. Thus we can use the notation R_X interchangeably with $R(X)$, and we know that on the (possibly empty) interval $I_i \equiv [R_i, R_{i-1})$, all states $X \geq i$ will continue.

3.3 Characterizing the surplus function

We are now ready to describe in more detail what the solution to the surplus equation (24) looks like.

3.3.1 Calculating the slope of the surplus function

Within the segments I_i, we can differentiate equation (24) to calculate the slope of the surplus function (for each X) on that segment; we obtain

$$(r+\lambda+\mu)\frac{\partial S(x,X)}{\partial x} = \frac{\partial y(x,X)}{\partial x} + \mu \sum_{Z:R_Z \leq x} G_{ZX}\frac{\partial S(x,Z)}{\partial x} \tag{29}$$

Notice that this equation contains just one value of x. Therefore, the equations on any segment I_i can be solved independently from those on all other segments, and the possible existence of empty segments is irrelevant for the solution. Since the reservation productivities are monotonic in X, on any non-empty segment I_i (29) constitutes a system of $N+1-i$ differential equations in the $N+1-i$ unknown functions $S(x,X)$, for $X \geq i$. The equations for segment I_i can be simplified as follows:

$$\begin{pmatrix} \frac{\partial S(x,i)}{\partial x} \\ \cdots \\ \frac{\partial S(x,N)}{\partial x} \end{pmatrix} = ((r+\lambda+\mu)I - \mu M_i)^{-1} \begin{pmatrix} \frac{\partial y(x,i)}{\partial x} \\ \cdots \\ \frac{\partial y(x,N)}{\partial x} \end{pmatrix} \tag{30}$$

where I is an identity matrix of order $N+1-i$ and M_i is the matrix

$$M_i = \begin{pmatrix} G_{ii} & & G_{Ni} \\ & \cdots & \\ G_{iN} & & G_{NN} \end{pmatrix}$$

(M_i is the transpose of the last $N+1-i$ rows and columns of the matrix G.)

Thus, changes in S can be calculated explicitly on each segment I_i as long as we choose a productivity function $y(x,X)$ that can be integrated explicitly with respect to the distribution of idiosyncratic shocks F. Similarly, we can integrate the surplus functions $W-U$ and J segment by segment, with one additional caveat: workers must receive surplus B/Φ when the incentive compatibility constraint binds. Given that S is strictly increasing in the continuation interval, we can uniquely define the cutoff point $\hat{x}(X)$ below which incentive compatibility is binding, by

$$\hat{x}(X) \ = \ \min\{x \in [\underline{x}, \infty) : \beta S(x,X) \geq b/\phi\} \tag{31}$$

Thus the formula for the worker's surplus, and likewise that for the firm's surplus, will differ depending on whether x is less or greater than $\hat{x}(X)$.

3.3.2 Calculating the discontinuities in the surplus function

Now that we know how to integrate the surplus inside the segments $[R_i, R_{i-1})$, we must next ask what happens to the surplus at the endpoints of these segments. As (20) and (21) show, the surplus function may be discontinuous at the reservation productivities. To be precise, let us define the jump in $S(x,X)$ at $x = R_i$ as

$$j(R_i, X) \equiv \lim_{dx \to 0} S(R_i + dx, X) - S(R_i - dx, X)$$

It turns out that two different types of jumps may occur.

First, if there is continuation in state X on both sides of R_i, then jobs with $x \geq R_i$ have a strictly lower probability of destruction than those with $x < R_i$. Therefore jobs with $x \geq R_i$ are strictly more valuable than those with $x < R_i$: that is, $S(x,X)$ jumps up discontinuously at $x = R_i$. In this case, equation (24) must hold on both sides of R_i, so the jumps at R_i satisfy an equation system similar to (30):

$$(r+\lambda+\mu)\, j(R_i, X) \;=\; \mu \sum_{Z:R_Z \le R_i} G_{ZX}\, j(R_i, Z) \tag{32}$$

This equation shows that the jumps at R_i are nonzero except in two possible cases. If there is no incentive problem, so that $S(R_i, i) = 0$, then (32) is solved by $j(R_i, Z) = 0$ for all Z. The jump would also be zero if G_{ZX} were zero for all Z satisfying $R_Z \le R_i$.

So far, we have characterized the jumps in $S(x, X)$ at points x strictly inside the continuation interval $[R_X, \bar{x}]$ (that is, at other reservation productivities R_i, for $i > X$). However, since $S(x, X)$ is zero outside of $\mathcal{C}(R)$ and satisfies (20) inside it, there must also be a jump of at least b/ϕ at the reservation productivity R_X in state X.[14] In other words, the surplus jumps up from zero to at least b/ϕ when we reach an x high enough that incentive compatibility is satisfied.

In fact, the jump in $S(x, X)$ at $x = R_X$ may sometimes be strictly greater than b/ϕ. Since T is continuous in x as long as x is not one of the reservation productivities, (26) and (27) imply that $S(R_X, X) = b/\phi$ for any X such that $R_X < R_{X-1}$ strictly: a larger (smaller) jump would mean R_X was too high (low). But when $R_X = R_{X-1}$, it is possible that $T(R_X, X, S, R, \theta, N^W) > b/\phi$ even though $T(R_X - \epsilon, X, S, R, \theta, N^W) < b/\phi$ strictly for arbitrarily small ϵ.[15] In other words, inequality (20) and equation (21) imply a set of complementary slackness conditions governing the reservation productivities R_X and the corresponding surpluses $S(R_X, X)$. For any $i \in \{2, 3, ...N\}$, monotonicity of the surplus implies:

$$R_i \le R_{i-1}$$

and incentive compatibility implies:

$$S(R_i, i) \ge \frac{b}{\phi}$$

The fact that the surplus is differentiable away from the reservation productivities implies that if $dR_i \equiv R_i - R_{i-1}$ is strictly negative, then $dS_i \equiv S(R_i, i) - b/\phi$ must be

[14] Note that in our setup, all surplus and value functions are upper semicontinuous (that is, continuous from the right). In particular, we have implicitly assumed that workers exert effort when (15) holds with equality, and that firms stay in the match when (19) holds with equality, so that matches continue when (20) holds with equality. Thus $S(R_X, X) \ge b/\phi$, while $S(x, X) = 0$ at all $x < R_X$.

[15] Recall that $S(x, X)$ is discontinuous at R_{X-1}. If the discontinuity there is large enough so that continuation in state X is incentive incompatible at all $x < R_{X-1}$, then R_X will be the same as R_{X-1}. In this case, $S(x, X)$ will jump up by more than b/ϕ at $x = R_X = R_{X-1}$, because $S(R_X, X) = S(R_{X-1}, X) \ge S(R_{X-1}, X-1) \ge b/\phi$.

zero. Therefore at least one of the inequalities must hold with equality:

$$(R_i - R_{i-1})\left(S(R_i, i) - \frac{b}{\phi}\right) \equiv dR_i dS_i = 0$$

Notice therefore that we can now summarize the entire surplus function by a vector of N numbers: first R_1, and then for each $i \in \{2, 3, ...N\}$, either dR_i or dS_i. The two possible cases for each of these last $N-1$ numbers can be easily distinguished, since dR_i is necessarily nonpositive, while dS_i is nonnegative.

3.4 Equilibrium

As we have seen, the surplus functions can be defined in terms of the productivity pair (x, X) without reference to the current distribution of employment and unemployment. Therefore, it suffices to define (and calculate) an equilibrium in terms of the minimum state variable (x, X) before considering other state variables. We therefore postpone for later the discussion of the dynamics of unemployment.

Obviously this model has trivial equilibria in which workers always shirk, and therefore firms never hire them. But we are interested in **no-shirking equilibria** in which the worker's surplus is sufficiently large to provide incentives not to shirk. Summarizing the relationships discussed so far, such an equilibrium can be defined in terms of just four objects, S, R, θ, and N^W.

Definition. *A no-shirking equilibrium is a surplus function $S(x, X)$, a vector of reservation productivities R, a labor market tightness vector θ, and a vector of new job values N^W that satisfy the following conditions:*

1. *For each X, the surplus function satisfies the system of differential equations (24) for all $x \in [R(X), \bar{x}]$, and is zero for $x \in [\underline{x}, R(X))$.*
2. *For each X, the surplus function satisfies the boundary condition (26) at the reservation productivity $R(X)$.*
3. *Labor market tightness $\theta(X)$ and the new job value $N^W(X)$ are given by*

$$c = q(\theta(X)) \min\{S(\bar{x}, X) - b/\phi, (1-\beta)S(\bar{x}, X)\} \quad (33)$$

$$N^W(X) = \max\{b/\phi, \beta S(\bar{x}, X)\} \quad (34)$$

3.5 An algorithm to calculate equilibrium

By now it should be clear that our main challenge is the partial equilibrium problem of solving (24) to find the surplus function S and reservation strategy R. Then to find a general equilibrium we only need to make sure that S and R imply a tightness vector consistent with zero profits from vacancy creation. Lastly, given S, R, and θ, the simulation of employment and productivity dynamics is straightforward.

One way to solve for S would be by backwards induction, conditional on a given θ. The results of Rustichini (1998) guarantee that this converges to the surplus associated with the worker and firm's optimal reservation strategy. But we would have to repeat this process for many different tightness vectors θ, which could be very slow. Therefore, we propose a faster algorithm, based on the fact that the entire surplus function can be summarized by one N-dimensional vector which we will call Q. We define

$$Q_1 \equiv R_1 \tag{35}$$

$$Q_i \equiv dR_i \equiv R_i - R_{i-1} \text{ if } R_i < R_{i-1} \tag{36}$$

$$Q_i \equiv dS_i \equiv S(R_i, i) - b/\phi \text{ if } R_i = R_{i-1} \tag{37}$$

This definition takes advantage of the complementary slackness relations that govern the surplus at the reservation productivities. If (for $i > 1$) Q_i is negative, then this indicates that R_i is strictly less than R_{i-1}, and therefore that $S(R_i, i) = b/\phi$. In this case, $Q_i \equiv R_i - R_{i-1}$. If (for $i > 1$) Q_i is positive, then this indicates that $R_i = R_{i-1}$, and in this case Q_i equals the excess jump $S(R_i, i) - b/\phi$ of the surplus function in state i. $Q_i = 0$ indicates the knife-edge case in which $R_i = R_{i-1}$ and $S(R_i, i) = b/\phi$.

All equilibrium quantities can be constructed from a candidate value of Q, including S, R, and θ. It then suffices to check whether Q implies an optimal reservation strategy, given aggregate conditions. Thus, instead of repeatedly solving a dynamic programming problem for each value of θ, we solve one N-dimensional root-finding problem to calculate S, R, and θ simultaneously. The steps are as follows.

1. Loop over aggregate states X from 1 to N, using the information in Q to calculate R_X and $S(R_X, X)$.
2. For each X from 1 to N, loop over intervals $I_Z = [R_Z, R_{Z-1})$ from $Z = X$ to $Z = 1$.
    - (a) If R_{Z-1} differs from R_Z, solve the differential equations (30) to calculate the increase in S on interval I_Z.
    - (b) If R_{Z-1} differs from R_Z, and $Z \geq 2$, use the equations (32) to calculate the jump in $S(x, X)$ at $x = R_{Z-1}$.

Given these two steps, we have constructed the (strictly increasing, upper semi-continuous) surplus function S implied by Q. The next steps are:

3. Use equation (33) to calculate the firm's probability of job finding q.
4. Use (6) to calculate labor market tightness θ.
5. Use (5) to calculate the worker's job finding probability p.
6. Use (34) to calculate the worker's value N^W of a new job.

We now know all the objects that appear in the surplus equation (24). On the left-hand side of (24), Q tells us directly the value of $S(R_X, X)$:

$$S(R_X, X) = \begin{cases} b/\phi \text{ if } X = 1 \text{ or } Q(X) < 0 \\ b/\phi + Q(X) \text{ if } X > 1 \text{ and } Q(X) \geq 0 \end{cases} \tag{38}$$

To see whether separation is optimal, we can now check whether (24) holds with the desired accuracy at the reservation productivity $x = R_X$ for each X:[16]

$$(r + \lambda + \mu) S(R_X, X) = y(R_X, X) - b + \lambda \int_{R_X}^{\overline{x}} S(z, X)\, dF(z) + \mu \sum_{Z: R_X \geq R_Z} G_{ZX} S(R_X, Z) - p(\theta(X)) N^W(X) \tag{39}$$

If we find a vector Q that satisfies (39), then we have found the equilibrium surplus function S. With it, we have also found R and θ, which we can use to simulate the dynamics of the distribution of employment and productivity.

3.6 Employment dynamics

This is a heterogeneous agent model in which the state variable includes the full distribution of idiosyncratic productivities. Nonetheless, the model can be solved quickly to arbitrary accuracy in two steps. First, as in Mortensen and Pissarides (1994), the equations defining values, surpluses, and tightness can be solved without reference to the unemployment rate or to the productivity distribution, as explained in Sec. 3.5. Second, given the reservation productivities and the tightness vector, we can simulate the dynamics of employment and productivity.

Given tightness and the unemployment rate u_t, the new matches formed in a short time interval dt are $p(\theta(X))u_t dt$. By assumption, these new matches all have the highest productivity $\bar{x}$. To describe the dynamics of the productivity distribution in continuing jobs, we keep track of the mass of employment in each interval $I_i \equiv [R_i, R_{i-1})$

[16] Checking this equation involves integrating $S(x, X)$. The integral can be evaluated piecewise using the derivative information from step 2a.

separately. Jobs with productivity in $[R_1, \overline{x}]$ are *stable*: no productivity shock X can destroy them. But when the current state is $X \geq 2$, there may be some other continuing jobs that are *fragile*, because they can be destroyed if X decreases; these jobs have individual productivity less than R_1. Finally, any job that receives an individual shock $x < R_N$ will always be immediately destroyed.

Now let $e_t(\mathcal{X})$ be the measure of employed workers whose productivities lie in set $\mathcal{X}$. Thus the mass of jobs with the highest productivity can be written as $e_t(I_0) \equiv e_t(\{\overline{x}\})$, and the mass of jobs in any other interval I_i, for $i \in \{1, 2, ...N\}$, is $e_t([R_i, R_{i-1}))$.[17] Total employment is $e_t \equiv e_t([R_N, \overline{x}]) = \sum_{i=0}^{N} e_t(I_i)$. Then the change in the mass of individuals in each of these employment states over a short time interval dt can be calculated as follows, dropping terms of order $o(dt)$:

$$de_t(\{\overline{x}\}) = p(\theta(X_t))u_t dt - \lambda e_t(\{\overline{x}\})\, dt \tag{40}$$

$$de_t([R_i, R_{i-1})) = \mathbf{1}(X_{t+dt} \geq i)\left[\lambda(F(R_i) - F(R_{i-1}))e_t - \lambda e_t([R_i, R_{i-1}))\right] dt - \mathbf{1}(X_{t+dt} < i)\, e_t([R_i, R_{i-1})) \tag{41}$$

$$du_t = \lambda F(R(X_{t+dt}))e_t([R(X_{t+dt}), \overline{x}])\, dt - p(\theta(X_t))u_t dt + \mathbf{1}(X_{t+dt} < X_t)\, e_t([R(X_t), R(X_{t+dt}))) \tag{42}$$

It is straightforward to verify that these flows sum to zero. Note that the terms $\mathbf{1}(X_{t+dt} < i)\, e_t([R_i, R_{i-1}))$ and $\mathbf{1}(X_{t+dt} < X_t)\, e_t([R(X_t), R(X_{t+dt})))$, which appear as outflows from fragile employment and an inflow to unemployment, are not of order dt. These terms represent the spike of destruction of fragile jobs that occurs any time the aggregate state X decreases.

4 Intuition: two aggregate states

In this section we illustrate the main features of the model for the case of two aggregate states, called 1 (recessions) and 2 (booms). We assume F is uniform and productivity is linear, with additive aggregate shocks, so that output is $y(x, X) = x + \zeta_X$. These last two assumptions guarantee that the surplus functions are linear. For further simplification, we assume a symmetric Markov process where $G_{12} = G_{21} = 1$: that is, any aggregate shock takes us from state X to the opposite state, called $-X$. Thus the two aggregate states each occur 50% of the time, on average.[18] Finally, as in the

[17] Equations (40)-(42) are correct even if there are empty intervals $R_i = R_{i-1}$.

[18] This G matrix does not exhibit first-order stochastic dominance. However, this (μ, G) pair is equivalent to the alternative transition matrix $\tilde{G} \equiv [0.5\ 0.5;\ 0.5\ 0.5]$ with a faster shock arrival rate $\tilde{\mu} \equiv 2\mu$. Since this alternative representation exhibits first-order stochastic dominance, the transitions across aggregate states do too, so Corollary 2 applies.

preceding sections we assume new jobs have the highest idiosyncratic productivity $\bar{x}$. Accordingly, we can write the surplus equation (24) as

$$(r+\lambda+\mu)\,S(x,X) \;=\; x+\zeta_X-b-p\left(\theta(X)\right)\max\left\{\frac{b}{\phi},\beta S(\overline{x},X)\right\}+\lambda\int_{R_X}^{\bar{x}}\frac{S(z,X)}{\bar{x}-\underline{x}}dz+\mu\mathbf{1}\left(x\geq R_{-X}\right)S(x,-X) \tag{43}$$

where $\mathbf{1}\left(x\geq R_{-X}\right)$ is an indicator function which takes value 1 if the job survives after an aggregate shock, and zero otherwise.

Depending on the size of the minimum incentive compatible surplus, job destruction in this economy is either countercyclical (low values of b/ϕ) or acyclical (high values of b/ϕ). Both cases are analyzed below.

4.1 No moral hazard

It is instructive to start with the case of perfect information, which makes our model equivalent to that of Mortensen and Pissarides (1994). That is, we assume no moral hazard ($b/\phi = 0$), so that matches continue as long as surplus is nonnegative. Therefore $S(R_1, 1) = S(R_2, 2) = 0$ at the reservation productivities, and equation (32) is consistent with continuous surplus functions (jumps $j(R_i, Z) = 0$ for all i and Z).

Assuming (28) is satisfied, we also have $S(R_1, 2) > S(R_1, 1)$, which implies R_2 must be strictly less than R_1. Therefore, recessions are periods of cleansing in which the least productive jobs disappear. The jobs with idiosyncratic match component $x < R_1$ are *fragile jobs* that only survive during booms. Using (30), we can calculate that the surplus functions are piecewise linear with slope

$$\frac{\partial S(x,X)}{\partial x}=\frac{1}{r+\lambda}\qquad\text{for } X=1,2,\text{ if } \ x>R\left(1\right) \tag{44}$$

and

$$\frac{\partial S(x,2)}{\partial x}=\frac{1}{r+\lambda+\mu}\qquad\text{if } \ x<R\left(1\right) \tag{45}$$

Equation (45) shows that the output of fragile jobs is discounted at a higher rate than that of other jobs, because they are destroyed whenever a recession begins.

Given the reservation productivities R_1 and R_2, we can use these slope formulas to write the surplus during recessions as

$$S(x,1)=\frac{x-R_1}{r+\lambda} \tag{46}$$

while the surplus function during booms satisfies

$$S(x,2)=\frac{x-R_2}{r+\lambda+\mu}\qquad\text{for}\quad x<R_1 \tag{47}$$

and

$$S(x,2) = \frac{x - R_1}{r + \lambda} + \frac{R_1 - R_2}{r + \lambda + \mu} \qquad \text{for} \quad x \geq R_1 \tag{48}$$

These functions are illustrated in Fig. 1. The upper curve is the surplus function in booms; the lower curve is that in recessions.

4.2 Countercyclical job destruction

With moral hazard, the slope formulas (44) and (45) are still correct as long as $R_2 < R_1$. What differs in this case is that matches can only continue if surplus is at least b/ϕ, so the surplus function $S(x, X)$ jumps discontinuously from 0 to b/ϕ at $x = R_X$.

Interestingly, these are not the only discontinuities in the surplus function caused by moral hazard. Intuitively, the fact that marginal jobs have strictly positive surplus makes job fragility more costly. A matched pair in a boom with x just below R_1 know they will separate as soon as a recession arrives. Without moral hazard, this is irrelevant, since marginal jobs have zero value. But with moral hazard, they expect to lose surplus b/ϕ when a recession starts, which makes jobs just below R_1 substantially less valuable than those just above it. Using (32), the jump in $S(x, 2)$ at $x = R_1$ is

$$j\left(R_1, 2\right) = \frac{\mu b/\phi}{r + \lambda + \mu} \tag{49}$$

Thus the surplus function during recessions is:

$$S(x,1) = \frac{x - R_1}{r + \lambda} + \frac{b}{\phi} \tag{50}$$

while the surplus function during booms satisfies

$$S(x,2) = \frac{x - R_2}{r + \lambda + \mu} + \frac{b}{\phi} \qquad \text{for} \quad x < R_1 \tag{51}$$

and

$$S(x,2) = \frac{x - R_1}{r + \lambda} + \frac{\mu b/\phi + R_1 - R_2}{r + \lambda + \mu} + \frac{b}{\phi} \qquad \text{for} \quad x \geq R_1 \tag{52}$$

These functions are illustrated in figure 2. We see that the surplus is always at least b/ϕ in the continuation region, and also has a discontinuity at $x = R_1$ in booms.

The preceding formulas take R_1 and R_2 as given, but it is also helpful to see how they can be calculated. We obtain two equations from the zero profit condition (33), setting $X = 1$ and $X = 2$. Two more equations are obtained by evaluating the surplus equation (43) at the reservation productivities and substituting $S(R_X, X) = b/\phi$ for $X = 1, 2$. Since $S(x, X)$ can be substituted out of (33) and (43) using formulas (50)-(52), these four equations contain just four unknowns: $\theta(1)$, $\theta(2)$, R_1, and R_2.

Analyzing these equations, (43) shows that job destruction decisions are always driven by the no-shirking constraint, because in each state X the surplus at the reservation productivity is $S(R_X, X) = b/\phi$. Job creation, on the other hand, may or may not involve a binding NSC, as the "min" in (33) shows. At the job creation margin, wages may share surplus according to the Nash bargain (if $\beta S(\bar{x}, X) \geq b/\phi$), or the minimum incentive compatible wage may be paid (if $\beta S(\bar{x}, X) < b/\phi$). In the latter case, the wage distribution is degenerate since the NSC binds on all jobs. One important possibility is that the NSC could bind for new jobs in recessions but not in booms, making firms' surplus share pro-cyclical. This might increase the cyclical volatility of job creation by strengthening firms' incentive to hire in booms rather than recessions.[19]

4.3 Acyclical job destruction

Eq. (49) showed that if $R_2 < R_1$ strictly, then the surplus in booms, $S(x, 2)$, must jump up by $\frac{\mu b/\phi}{r+\lambda+\mu}$ at $x = R_1$. But if shocks arrive too quickly (large μ) or moral hazard is too strong (large b/ϕ), then there may not be enough surplus to permit such a large jump. In that case firms cannot maintain fragile jobs because the wage needed to induce effort in those jobs would result in negative profits. That is, when moral hazard is very strong, the reservation productivities R_1 and R_2 are likely to collapse to a single value R, making the job destruction rate constant over the cycle. This possibility is illustrated in Fig. 3, which also shows the value $T(x, 2, S, R, \theta, N^W)$ of continuing a match with $x < R_1$ in a boom.

Acyclical destruction can be seen as the result of a time inconsistency problem, because if a firm could commit to maintaining jobs with $x < R_1$ during recessions, those jobs might become sufficiently valuable to prevent shirking in booms, which might make both firm and worker better off *ex ante*. But we rule out commitment and instead require that incentive compatibility be satisfied at all times for both agents, causing the reservation productivities to collapse when the surplus is small relative to the strength of the moral hazard problem. Inspecting (52), a sufficient condition for acyclical job destruction is $S(R_1, 2) < \left(\frac{r+\lambda+2\mu}{r+\lambda+\mu}\right) b/\phi$.

With acyclical job destruction, the slope of the surplus function in recessions and booms is given by (44). Since the reservation productivity is the same in both aggregate states, we can write the surplus functions as

$$S(x, 1) = \frac{x - R}{r + \lambda} + \frac{b}{\phi} \tag{53}$$

[19] If new matches have random productivity, as discussed in Appendix 1, then firms' surplus share is always procyclical if there is moral hazard. That is, since $S(x, 1) < S(x, 2)$, the NSC will bind on a larger fraction of new jobs in recessions than in booms.

$$S(x,2) = S(x,1) + dS_2 \tag{54}$$

where $dS_2 \equiv S(R,2) - b/\phi$ is the amount by which the surplus exceeds that needed to prevent shirking at the reservation productivity in booms.

To actually calculate R, we would again use the two zero profit conditions (33) for $X = 1$ and $X = 2$. In addition, we would evaluate the surplus equation (43) at the reservation productivity R in state $X = 1$, setting $S(R,1) = b/\phi$. Finally, we would evaluate (43) at R in state $X = 2$, setting $S(R,2) = b/\phi + dS_2$. Thus we obtain four equations in the four unknowns $\theta(1)$, $\theta(2)$, R, and dS_2. By subtracting (43) evaluated at $(R,1)$ from (43) evaluated at $(R,2)$, we can also obtain an intuitive expression for the excess jump dS_2:

$$(r + \lambda F(R) + 2\mu)\, dS_2 \;=\; (\zeta_2 - \zeta_1) - [p(\theta(2))N^W(2) - p(\theta(1))N^W(1)] \tag{55}$$

This equation interprets the excess jump dS_2 as the increase in aggregate productivity associated with booms, minus the increase in the value of unemployment associated with booms, both appropriately discounted.

5 Numerical results

In this section we perform some illustrative simulations, mainly to explore how labor market volatility varies with the degree of moral hazard. Sec. 4 showed that moral hazard may tend to amplify the volatility of job creation but to smooth job destruction. It is quantitatively interesting to compare these effects and see how they interact.

To illustrate how labor market fluctuations change as we tighten the incentive compatibility constraint, we start from a benchmark model without moral hazard which replicates the mean and the cyclical volatility of the postwar U.S. unemployment rate. To match both the mean and the volatility of u our benchmark model assumes a very high value of b, an assumption that has been criticized by Shimer (2005A). In a second set of exercises we instead impose a lower b like that Shimer advocates. This helps us see how far downward wage rigidity due to moral hazard can take us in explaining unemployment volatility. For both calibrations of b, we first consider the case of deterministic initial match values, but we also present results for random initial match values.

The baseline parameters are stated in Table 1. For comparison with most business cycle literature, we state the parameters and the results at quarterly frequency.[20] In

[20]However, since the model is defined in continuous time, the simulations are performed with very short periods (two weeks) after an appropriate rescaling of the parameters. Each set of results reported in Tables 2-5 is generated by simulating 1000 histories with a length of 240 quarters. The first 40 quarters of every series are discarded, so that the remaining 200 quarters correspond roughly to the length of the U.S. post-war period.

Table 1: Baseline parameters

Parameter	Values
ζ_X	$[-0.053, 0, 0.053]$
x	$U \sim [0.9625, 1.0375]$
r	0.01
b	0.897
$m(u, v)$	$u^{0.5}v^{0.5}$
β	0.5
c	0.125
μ	0.067
λ	0.081

the baseline simulation, new jobs start with the highest productivity. We assume the economy cycles through three aggregate states with the ζ_X values given in the table, which are chosen to replicate the proportional standard deviation of U.S. labor productivity. Aggregate transitions obey the following Markov matrix:

$$G \equiv \begin{pmatrix} 0 & 0.25 & 0 \\ 1 & 0.5 & 1 \\ 0 & 0.25 & 0 \end{pmatrix}$$

The quarterly probability of an aggregate shock is set to $\mu = 0.067$. The matching technology is assumed to be Cobb-Douglas, with elasticity 0.5, consistent with the estimates surveyed in Petrongolo and Pissarides (2001). We also assume for convenience that β is equal to 0.5, which guarantees that our benchmark equilibrium is constrained efficient. The rest of the parameters (in particular, the high b) are chosen to obtain a mean unemployment rate of 6 percent and the best possible fit for the proportional standard deviation of unemployment and vacancies in U.S. post-war data.

Table 2 presents the baseline results. The series documented are the unemployment rate u, the vacancy rate v, the cross-sectional mean of labor productivity y, the cross-sectional mean of wages w, job creation jc, and job destruction jd. In the first column we present the post-war U.S. business cycle facts reported in Shimer (2005A). Comparing these data to the efficient decentralized equilibrium ($b/\phi = 0$) confirms that our baseline model performs well in many respects. The next three columns report the results for three higher values of b/ϕ. As our analysis of jointly optimal separation in Sec. 4.3 suggested, we find that the proportional standard deviation of job destruction (Row 6) decreases monotonically from 0.1557 in the efficient case to 0.0346 when $b/\phi = 0.15$. This reduction in the job destruction variability is accompanied by an increase in all three reservation productivities, but especially in those

Table 2: Deterministic Initial Match Value

	Data	**Model**			
Minimum surplus b/ϕ**:**	—	0	0.05	0.1	0.15
Coefficients of variation					
u	0.188	0.1712	0.1437	0.1284	0.2005
v	0.183	0.0835	0.1233	0.1072	0.2480
y	0.0306	0.0256	0.0261	0.0256	0.0266
w	0.013	0.0238	0.0176	0.0182	0.0197
jc	0.117	0.0846	0.0546	0.0345	0.0550
jd	0.197	0.1557	0.1082	0.0719	0.0346
Correlations					
$\mathrm{corr}(u, y)$	-0.367	-0.8876	-0.9127	-0.9118	-0.9282
$\mathrm{corr}(v, y)$	0.362	0.6354	0.8735	0.9553	0.9706
$\mathrm{corr}(u, v)$	-0.896	-0.2990	-0.6533	-0.8049	-0.8455
$\mathrm{corr}(jc, jd)$	-0.65	0.4833	0.4386	0.0261	-0.0910

associated with good aggregate states. In our benchmark, $R_3 = 1.0081$, $R_2 = 1.01433$ and $R_1 = 1.01986$, while for $b/\phi = 0.15$, we find $R_3 = R_2 = 1.02829$ and $R_1 = 1.03107$. This is consistent with our observation that increasing moral hazard eventually causes the reservation productivities to collapse, smoothing job destruction.

By contrast, Row 5 of Table 2 shows that the coefficient of variation of job creation has a U-shaped relation with moral hazard. Starting from $b/\phi = 0$, rising moral hazard causes the cyclical volatility of job creation to decrease. Note that for low b/ϕ, job creation and destruction are positively correlated, because waves of job destruction make it profitable to hire, causing an "echo effect" in job creation. Thus the initial decline in job creation variability is probably caused by the sharp decrease in the variability of job destruction. However, beyond $b/\phi = 0.10$, increases in moral hazard tend to amplify variation in job creation. For these parameters, the NSC binds on all jobs in state 1.[21] As a result, firms' surplus share is higher in booms than in recessions, giving them an incentive to increase the cyclical volatility of vacancies and job creation. For $b/\phi = 0.15$, this effect is strong enough that unemployment fluctuations increase substantially even though job destruction is almost acyclical.

These simulations also shed light on the Beveridge curve relation. The introduction of downward wage rigidity greatly improves the model's performance in this respect. In the efficient benchmark, the negative correlation between u and v is only around one third of its observed value, whereas for values of b/ϕ between 0.10 and 0.15 it is

[21] In fact, for $b/\phi > 0.14$ the NSC always binds, so there is a degenerate wage distribution in all three states.

Table 3: Random Initial Match Value

	Data	Model				
Minimum surplus b/ϕ**:**	—	0	0.05	0.1	0.2	0.3
Coefficients of variation						
u	0.188	0.5970	0.5176	0.4458	0.3554	0.3350
v	0.183	0.2800	0.1973	0.1624	0.1983	0.3466
y	0.0306	0.0248	0.0255	0.0257	0.0245	0.0263
w	0.013	0.0488	0.0485	0.0483	0.0470	0.0498
jc	0.117	0.3283	0.2401	0.1838	0.1359	0.1769
jd	0.197	0.8107	0.6678	0.5747	0.4671	0.4382
Correlations						
corr(u, y)	-0.367	-0.8154	-0.8314	-0.8396	-0.8351	-0.8188
corr(v, y)	0.362	-0.4805	-0.2978	0.0846	0.7128	0.8895
corr(u, v)	-0.896	0.7471	0.6068	0.2953	-0.3310	-0.5282
corr(jc, jd)	-0.65	0.2970	0.2404	0.1316	-0.0629	-0.0735

almost as high as in the data. The model's failure to produce a strong Beveridge curve when $b/\phi = 0$ is typical of models with endogenous job destruction, due to the "echo effect" mentioned earlier: the high rates of unemployment occurring in recessions make them especially cheap times to hire. But with sufficient moral hazard, our model instead implies that the reservation productivities collapse, so that even though the job destruction rate is endogenous it turns out to be constant over the cycle. As occurs in simpler models with exogenous destruction (e.g. Pissarides 2000), this implies a strong Beveridge curve as hiring responds procyclically to productivity, driving unemployment countercyclically with a lag. Likewise, introducing moral hazard improves the correlation between job creation and job destruction.

Our next example shows the effect of incentive problems when the productivity of new jobs is drawn from the same distribution F as that of continuing jobs, which is the case described in Appendix 1. There are at least two reasons to think that this might help generate greater unemployment volatility than the case where all jobs are born at the top of the idiosyncratic productivity distribution. First, only some new matches will be accepted, so there is an extra margin for variation in the rate of job creation. Second, firms' share of surplus will always be procyclical in this case (even if the NSC is never binding on the best jobs), because the expected fraction of matches for which the no-shirking condition binds will be larger in recessions. This will also tend to increase the procyclicality of job creation.

The results are reported in Table 3. All parameters are the same as before except that we recalibrate b to 0.9285 so that we again obtain a 6 percent unemployment rate

Table 4: Deterministic Initial Match Value with $b = 0.4$

	Data	Model				
Minimum surplus b/ϕ:	—	0	0.1	0.15	0.5	0.75
Coefficients of variation						
u	0.188	0.0191	0.0144	0.0140	0.0123	0.0219
v	0.183	0.0137	0.0154	0.0162	0.0176	0.0312
y	0.0306	0.0210	0.0201	0.0204	0.0204	0.0205
w	0.0130	0.0179	0.0186	0.0193	0.0201	0.0194
jc	0.117	0.0072	0.0035	0.0032	0.0013	0.0057
jd	0.197	0.0130	0.0058	0.0049	0.0032	0.0018
Correlations						
corr(u, y)	-0.367	-0.9427	0.9611	0.9637	-0.9641	-0.9572
corr(v, y)	0.362	0.8969	0.9671	0.9742	0.9805	0.9760
corr(u, v)	-0.896	-0.7159	-0.8772	-0.8945	-0.9090	-0.9118
corr(jc, jd)	-0.65	0.5030	0.2155	0.2192	0.2307	0.2192

in the absence of moral hazard. This specification greatly increases the volatility of most variables, tripling the proportional standard deviation of unemployment when $b/\phi = 0$ compared with Table 2. Nonetheless, the qualitative effects of moral hazard are the same. The introduction of downward wage rigidity smoothes job destruction, while the proportional standard deviations of vacancies and job creation are both U-shaped. Furthermore, the introduction of wage rigidity greatly improves the Beveridge curve and the correlation between job creation and destruction.

Our results so far illustrate many of the basic qualitative and quantitative implications of our model. But our high benchmark cost of working b is controversial (Shimer 2005A; Costain and Reiter 2005; Hagedorn and Manovskii 2005). Therefore it is interesting to investigate whether imposing moral hazard, by itself, can generate reasonable levels of unemployment volatility under Shimer's parameterization $b = 0.4$. Table 4 reports the results when $b = 0.4$ and all new matches have the best idiosyncratic productivity. Table 5 considers $b = 0.4$ with random productivity of new matches.

To start again from a 6% unemployment rate when $b = 0.4$, the simulations in Table 4 recalibrate $c = 0.665$ and $x \in [0.65, 1.35]$. The values of b/ϕ considered in Table 4 (as in Table 5 below) are chosen so that they encompass the full range equilibrium configurations from an efficient economy without moral hazard, to a pure efficiency wage model in which the NSC binds for all jobs even in booms (Case IV). As before, the variability of job destruction is monotonically decreasing with moral hazard (there are three separate reservation productivities when $b/\phi = 0.1$; for $b/\phi = 0.15$ we have $R_3 = R_2$; and for $b/\phi \geq 0.5$ all the reservation productivities are equal). The volatilities

Table 5: Random Initial Match Value with $b = 0.4$

	Data	**Model**				
Minimum surplus b/ϕ:	—	0	0.1	0.5	1.0	2.5
Coefficients of variation						
u	0.188	0.0508	0.0389	0.0174	0.0163	0.0227
v	0.183	0.0256	0.0166	0.0231	0.0281	0.0466
y	0.0306	0.0226	0.0232	0.0234	0.0217	0.0216
w	0.0130	0.0156	0.0178	0.0216	0.0200	0.0181
jc	0.117	0.0290	0.0193	0.0054	0.0074	0.0143
jd	0.197	0.0670	0.0465	0.0064	0.0025	0.0055
Correlations						
corr(u, y)	-0.367	-0.8522	-0.8684	-0.9375	-0.9094	-0.8640
corr(v, y)	0.362	-0.2958	0.1289	0.9668	0.9703	0.9718
corr(u, v)	-0.896	0.7287	0.3285	-0.8349	-0.8004	-0.7431
corr(jc, jd)	-0.65	0.3907	0.3396	-0.0670	0.1944	0.2871

of vacancies, job creation, and unemployment are all U-shaped functions of b/ϕ, but even in the extreme cases the proportional standard deviation of unemployment is much lower than in the data.

Table 5 considers random initial match productivities with $b = 0.4$. This time, calibrating to 6% unemployment when $b/\phi = 0$ requires $c = 0.5$ and $x \in [0.7, 1.3]$. Imposing random initial productivity multiplies unemployment volatility by a factor of 2.5, but this is far from sufficient to match the data. The qualitative behavior of all series is similar to our other simulations, except that in this case the variability of job destruction is also U-shaped; the increase in the volatility of destruction observed in the last column of Table 5 may be a response to the increased volatility of creation.

Thus our numerical results indicate that our model of downward wage rigidity based on moral hazard amplifies unemployment fluctuation much less than the *ad hoc* wage stickiness of Shimer (2005A), and can even go in the opposite direction. This is partly because our mechanism does not make wages perfectly sticky, and also partly because our framework allows for endogenous job destruction. One of our clearest numerical findings is that job destruction becomes much less volatile as incentive problems increase, contrary to the hypothesis of Ramey and Watson (1997). Another clear numerical result is that random initial match productivity greatly increases labor market volatility, a factor not considered by Shimer (2005A) and related studies that assume exogenous destruction.

6 Conclusions

This paper has characterized the dynamics of a matching model in which workers' effort is unobservable. We were motivated to begin this project by Shimer (2005A) and related papers claiming that matching and bargaining models feature insufficient propagation of shocks because they imply an excessively flexible wage. It seemed to us that a "no-shirking constraint" could be an appropriate microfoundation for a form of downward wage rigidity applicable not only to continuing jobs but also to new jobs, as would be required to increase the volatility of hiring. Besides job creation, we felt we should also endogenize job destruction, as in Mortensen and Pissarides (1994), since one important implication of downward wage stickiness could be its potential to cause inefficient separation.

Having solved this model, we have also learned about a number of issues beyond those we initially planned to address. First, incorporating a no-shirking constraint for workers in a matching model implies that firms' share of surplus may decrease in recessions. This tends to smooth the wage and increase the volatility of firms' hiring, in stark contrast with the effects of a no-shirking constraint in a frictionless labor market (Kimball 1994), where increased unemployment makes it cheaper for firms to hire in recessions. This suggests that macroeconomists may have underestimated the potential of "efficiency wage" models to account for business cycle facts.

Nonetheless, as a solution to Shimer's unemployment volatility puzzle, unobservable worker effort gives mixed results at best. While the no-shirking constraint tends to make job creation more volatile, we have also found that it tends to make job destruction less volatile, because it pushes the firing threshold up more in booms than in recessions, due to the time inconsistency problem. Thus the overall implications for unemployment volatility are ambiguous. However, by focusing on endogenous destruction we have also identified another potential source of unemployment volatility, namely, random productivity of new jobs. Random initial match productivity increases job creation volatility because the fraction of new matches that actually result in employment is procyclical. It also strengthens the effect of moral hazard on the volatility of hiring by ensuring that firms' share of surplus is always procyclical.

Our results also have strong implications for the "contractual fragility" mechanism advocated by Ramey and Watson (1997). It is true that a negative shock may inefficiently destroy jobs in our model by tightening incentive constraints. But this lowers the value of relatively bad jobs *ex ante*, causing the time inconsistency problem mentioned above: workers are costly to motivate at the time of hiring if the firm cannot commit to maintain them in the future. Thus workers are likely not to be hired into potentially fragile jobs in the first place. Therefore, when the full dynamics of the model are studied, strong variations in job destruction caused by contractual fragility fail to arise. This is just one more way of saying that the no-shirking constraint tends to cause reservation productivities to collapse.

As all these points emphasize, our model has rich implications for both the job creation and destruction margins. The fact that unobservable effort tends to amplify fluctuations in job creation while smoothing job destruction means it may be consistent with Shimer (2005B) and Hall's (2005B) evidence that the creation margin matters more than the destruction margin for explaining employment fluctuations. By shrinking the changes in the reservation productivity, moral hazard makes our model act more like one with an exogenous separation rate.[22] This may help explain why the negative correlation between unemployment and vacancies is such a robust stylized fact, in spite of the fact that it often vanishes in models where job destruction is endogenized. We therefore believe that further quantitative work on moral hazard in labor effort could be fruitful for explaining patterns of variation in employment and unemployment, and job creation and destruction.

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[22]Thus although our model is consistent with the that creation matters more than destruction for labor market fluctuations, this fact should not be interpreted as evidence in favor of efficient separations, as Hall (2005B) does.

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Appendix 1: Random productivity of new jobs

In the main text, we assumed that all new jobs started with the maximum possible idiosyncratic productivity $\bar{x}$. Here we consider the alternative case in which new jobs have random idiosyncratic productivity. For simplicity, we assume the idiosyncratic component of productivity has distribution F with support $[\underline{x}, \bar{x}]$, which is the same as the distribution of productivity shocks to continuing jobs.

The analysis of the case with random new jobs is almost the same as the case considered in the main text, with the following changes. Equation (9) describing the worker's expected increase in value from a new job offer, $N^W(X)$, is replaced by

$$N^W(X) = \int_{R(X)}^{\overline{x}} (W(z, X) - U(X))\, dF(z)$$

Note that in this case, some new jobs are rejected, and the value $N^W(X)$ reflects this. Likewise, considering the firm's expected increase in value from finding a new match possibility, equation (12) is replaced by

$$N^F(X) = \int_{R(X)}^{\overline{x}} (J(z, X) - V(X))\, dF(z)$$

Also, in Section 3.2, if new jobs are random then there is no need to define an interval $I_0 \equiv \{\bar{x}\}$ consisting of the best jobs only. Instead, productivity level $\bar{x}$ should be included in the first interval, defining $I_1 \equiv [R_1, \overline{x}]$. Thus in this case, I_1 is closed at both ends, unlike the other intervals, which are open at the upper endpoint.

In the definition of a no-shirking equilibrium, point 3. is replaced by the following:

3′. *Labor market tightness* $\theta(X)$ *and the new job value* $N^W(X)$ *are given by*

$$c \;=\; q(\theta(X))\int_{R(X)}^{\bar{x}} \min[S(z,X)-b/\phi,(1-\beta)S(z,X)]dF(z) \qquad (56)$$

$$N^W(X) \;=\; \int_{R(X)}^{\bar{x}} \max[b/\phi,\beta S(z,X)]dF(z) \qquad (57)$$

Thus in step 3 of the algorithm of Sec. 3.5, q is calculated using (56). In step 6, N^W is calculated using (57). Evaluating the integral in (57) requires us to know the cutoffs $\hat{x}(X)$, which can be calculated using (31).

The three equations (40)-(42) describing the dynamics of employment and unemployment are replaced by these two equations.

$$\begin{aligned} de_t\left([R_i,R_{i-1})\right) \;=\; & \mathbf{1}\left(X_{t+dt}\geq i\right)\left[\left(F(R_i)-F(R_{i-1})\right)\left(\lambda e_t+p(\theta(X_t))u_t\right)-\lambda e_t\left([R_i,R_{i-1})\right)\right]dt \\ & -\mathbf{1}\left(X_{t+dt}<i\right)e_t\left([R_i,R_{i-1})\right) \qquad (58) \end{aligned}$$

$$\begin{aligned} du_t \;=\; & \lambda F(R(X_{t+dt}))e_t\left([R(X_{t+dt}),\overline{x}]\right)dt \;-\; (1-F(R(X_{t+dt})))p(\theta(X_t))u_t dt \\ & +\mathbf{1}\left(X_{t+dt}<X_t\right)e_t\left([R(X_t),R(X_{t+dt}))\right) \qquad (59) \end{aligned}$$

In (58)-(59), the notation $[R_1,R_0)$ refers to the first interval, which is actually the closed interval $[R_1,\bar{x}]$. But since productivity exactly equal to $\bar{x}$ occurs with probability zero, this distinction is immaterial.

Appendix 2: Proofs.

Proof of Lemma 0. We prove Lemma 0 by constructing a monotone, bounded sequence R_i of reservation productivity vectors.

Define the N-dimensional vector $R_0 \equiv (\underline{x},\underline{x},...\underline{x})$. Given S, θ, and N^W, define a new vector R_1 by iterating once on (26) evaluated at $R=R_0$. By construction, since the minimum in (26) is selected from $x\geq\underline{x}$, we have $R_1(X)\geq R_0(X)$ for each X.

Define R_2 by iterating once on (26) evaluated at $R=R_1$. Note that since S is weakly increasing in x and y is strictly increasing and unbounded in x, T is strictly increasing and unbounded in x. Also, since S is nonnegative, T is weakly decreasing in R. Since $R_1\geq R_0$, these monotonicity properties of T imply that $R_2(X)$ exists, and satisfies $R_2(X)\geq R_1(X)$, for all X. By induction, if we define R_{i+1} by iterating once on (26) evaluated at $R=R_i$, we obtain $R_{i+1}(X)\geq R_i(X)$ for all X and all $i\geq 0$.

We can find an upper bound for R by constructing a lower bound for T. Since S is nonnegative and y is strictly increasing, each element $R(X)$ must be less than $\hat{R}(X)$, defined as follows:

$$\hat{R}(X) \;=\; \min\left\{x\in[\underline{x},\infty):\frac{y(x,X)-b-p(\theta(X))N^W(X)}{r+\lambda+\mu}\geq b/\phi\right\} \qquad (60)$$

So the increasing sequence of vectors R_i is bounded above by the vector $\hat{R}$, and therefore the sequence R_i converges to a limit $\underline{R}$.

Finally, suppose there is another fixed point R' of (26). By construction, we have $R' \geq R_0$. Applying (26) once to both sides of this inequality, we obtain $R' \geq R_1$. Applying (26) again and again to both sides, we obtain $R' \geq R_i$ for all i, and therefore $R' \geq \underline{R}$. ***Q.E.D.***

Proof of Prop. 1. Rustichini (1998) advocates solving incentive-constrained models by constructing a bounded, monotone sequence of value functions. This proof adjusts Rustichini's method to deal with our surplus function and reservation productivities simultaneously. It is formally very similar to the proof of Lemma 0.

Note that $S^0(x, X) = y(\bar{x}, N)/r$ is a nonnegative function, and is an upper bound to the true surplus function. Let $R^*(S^0, \theta, N^W)$ be the minimum fixed point of (26) identified in Lemma 0. Set $R_0 \equiv R^*(S^0, \theta, N^W)$.

Define $S^1(x, X)$ by iterating once on (27), evaluated at $S = S^0$ and $R = R_0$. By construction, $S^1(x, X) \leq S^0(x, X)$ for all x and X. Also, for $R_0(X) \leq x \leq \bar{x}$, $T(x, X, S^0, R_0, \theta, N^W) \geq b/\phi$. Therefore S^1 is a nonnegative function, weakly increasing in $x \in [\underline{x}, \infty)$. Thus by Lemma 0, there exists a fixed point $R_1 \equiv R^*(S^1, \theta, N^W)$ of the mapping (26) evaluated at $S = S^1$. Since $S^1 \leq S^0$, $R_1 \geq R_0$.

Now define $S^2(x, X)$ by iterating once on (27), evaluated at $S = S^1$ and $R = R_1$. Since T is increasing in S, and $S^1 \leq S^0$, and since T is decreasing in R, and $R_1 \geq R_0$, we conclude that $S^2(x, X) \leq S^1(x, X)$ for all x and X. Also, for $R_1(X) \leq x \leq \bar{x}$, $T(x, X, S^1, R_1, \theta, N^W) \geq b/\phi$. Therefore S^2 is a nonnegative function, weakly increasing in $x \in [\underline{x}, \infty)$. Thus by Lemma 0, there exists a fixed point $R_2 \equiv R^*(S^2, \theta, N^W)$ of the mapping (26) evaluated at $S = S^2$. Since $S^2 \leq S^1$, $R_2 \geq R_1$. By induction, we can define a decreasing sequence of surplus functions $S^{i+1} \leq S^i$ which all satisfy the assumptions of Lemma 0, and are therefore associated with an increasing sequence of reservation vectors $R_{i+1} \geq R_i$.

The functions S^i are all bounded below by zero. Therefore the sequence S^i converges to a limit $\bar{S}$, which is also a nonnegative function, weakly increasing in $x \in [\underline{x}, \infty)$, which has associated with it a reservation vector $\underline{R} \equiv R^*(\bar{S}, \theta, N^W)$. Since $\bar{S} \leq S^i$ for all i, and R^* is decreasing in S, $\underline{R} \geq R_i$ for all i.

Now suppose there exists another fixed point pair (S', R'). Since S^0 is an upper bound for all other fixed points of (27), and since R^* is decreasing in S, we have $S^0 \geq S'$ and $R_0 \leq R'$. Note that the mapping defined by (27) is increasing in S and decreasing in R. Iterating once on (27), we obtain $S^1 \geq S'$ and $R_1 \leq R'$. Now by induction, S^i and R_i bound S' and R' for all i, and thus in the limit we have $\bar{S}(x, X) \geq S'(x, X)$ for all x and X and $\underline{R}(X) \leq R'(X)$ for all X. ***Q.E.D.***

Fig. 1. Surplus functions without moral hazard

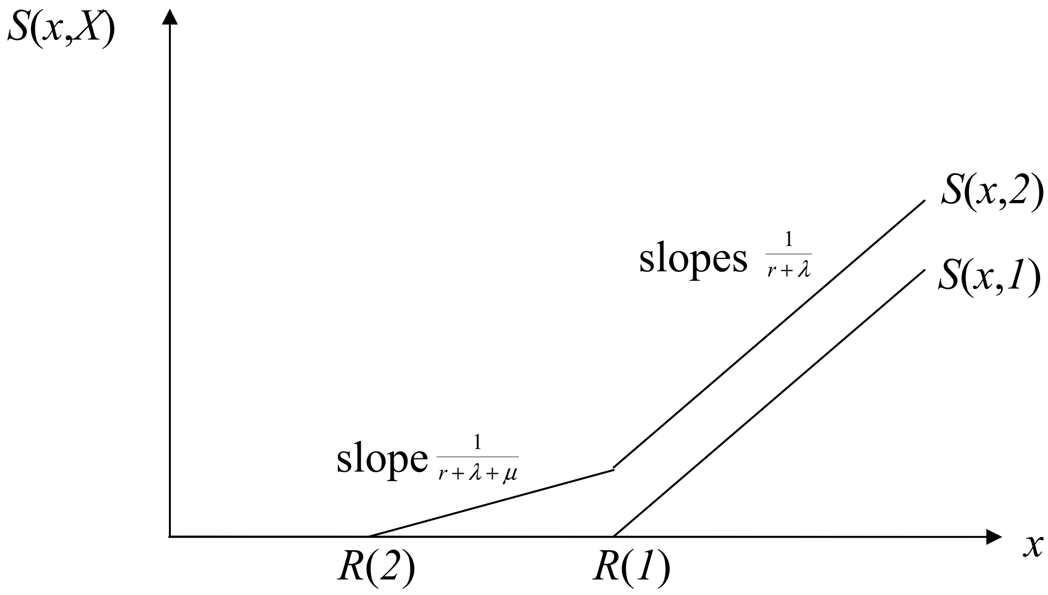

Fig. 2. Surplus functions under moral hazard: Countercyclical job destruction

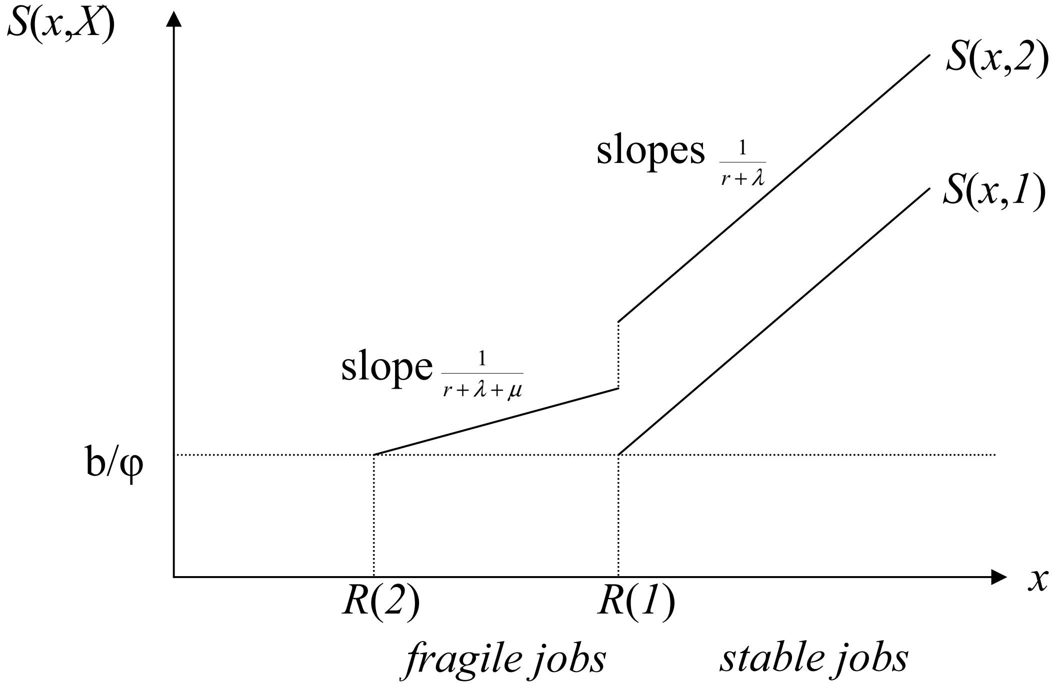

Fig. 3. Surplus functions under moral hazard: Acyclical job destruction