Information Gathering in Organizations: Equilibrium, Welfare and Optimal Network Structure*†

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August, 2008

Abstract

We model an organization as a game in which all agents share a common decision problem and some level of coordination is necessary between individual actions. Agents have individual private information on the task they have to perform and they share this private information through pairwise channels of communication. We analyze how this communication pattern, modelled by means of a network structure, impacts individual behavior and aggregate welfare. In the unique equilibrium of this bayesian game each agent's optimal action depends on a properly defined knowledge index that measures how the aggregation of information helps others to infer higher-order beliefs on his information after communication. Adding communication channels is not always beneficial for the organization since it can lead to miscoordination. We single out the geometry of communication links among agents that the manager would like to implement to improve the organization's performance.

*Toni passed away in November of 2007, after the first version of the paper was written. His friendship, energy and talents are sorely missed.

†We are grateful to the editors, Douglas Gale and Xavier Vives, to three anonymous referees and to Wouter Dessein, Jan Eeckhout, Faruk Gul, Andrea Prat and Yves Zenou and audiences in the Conference on Complementarities and Information (IESE, 2007) and in Stockholm University for helpful comments. We also thank Antonio Cabrales for helpful conversations and discussions. Financial support from the Fundación BBVA, from the Spanish Ministry of Science and Education and FEDER through grants SEJ2005-01481/ECON and SEJ2006-09993/ECON, from the Juan de la Cierva Program, and from the Barcelona Economics Program XREA is gratefully acknowledged.

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1 Introduction

Coordination, together with appraisal and planning, is a major concern in modern organizations. An example is provided by the modern corporation, which depends on the decentralized activity of its multiple divisions. Managing a complex, decentralized organization requires one to devise mechanisms, such as clear and stable lines of authority and communication, to sustain coordination.

Communication is an essential element for coordination. As Milgrom and Roberts (1992) note:

> "The coordination problem is to determine what things should be done, how they should be accomplished, and who should do what. At the organizational level, the problem is also to determine who makes decisions and with what information, and how to arrange communication systems to ensure that the needed information is available."

Indeed, this interdependence was already recognized by the builders of the modern firm, like Alfred Sloan, who restructured General Motors in the 1920's, as highlighted in Chandler (1962).[1]

Given that communication is one of the most important elements of the organization, we aim to analyze how information sharing mechanisms influence the decentralized activity and overall performance within the organization. We focus particularly on understanding the implications of different network structures of communication for the organization's success in achieving it's goals and coordinating the activity of its different units. In this paper, we discover the optimal structure of communication among these units in order to ensure efficient decentralized behavior. Some real examples drawn from a variety of contexts can help to illustrate the role of communication in fostering coordination and the organizational inefficiencies that can originate in ineffective or missing communication channels.

A recent example of how a lack of communication can induce coordination failure is found in the air transport industry. In 2006 Airbus announced that the A380 superjumbo would be delayed for two years, incurring losses of at least 500,000 million euros per year, in the years to come. The reason given by the company was that the engineers in Germany and the engineers in France were using different versions of a computer-aided design software. When transmitting files from one center to the other, information was lost and, in particular, measurement units were changed. When the first bundles of wiring arrived at the assembly plant, workers realized that the units were incompatible with the rest of the fuselage structure.

Airbus could have avoided these losses if all engineering units had used the same software version. This would have required a costly and time-consuming implementation (installing the new version and training engineers in Germany) but this would have been nothing compared to the

[1] "Sloan first turned the interdivisional committee as a means of improving communication and coordination [...]" (Chandler, 1962, pg. 154). "The committees, therefore, provided a systematic and regular means by which the line, staff, and general officers could meet monthly or even more often to exchange information and consider common problems." (Chandler, 1962, pg. 155).

losses of billions of euros generated by the impossibility of communicating effectively across units any advance in design.

Organizations other than firms also face a natural need for coordination. For example, armies are obviously obliged to coordinate their movements in war if they wish to achieve victory, and a lack of communication among them almost surely leads to defeat. This is reflected in the Battle of Tanneberg, an early episode of the First World War. This battle brought together the German and Russian forces in East Prussia in 1914. The Russian front was composed of the First and Second Armies. Poor communication between the two was shown to be a crucial element in their defeat in the battle. The reasons for the communication failure were varied, ranging from their inability to use modern communication tools[2] to enmity among the generals of the Russian army.[3] While the Russian army was probably superior,[4] a coordination failure between the two armies, largely due to an absence of communication, caused 100,000 human losses in the Russian side.

We model organizations as a group of agents facing a common task and with an incentive to coordinate with respect to others' actions. More precisely, individuals choose actions while facing both an external and an internal motive. The external motive corresponds to the common task assigned to the group, about which there is some degree of uncertainty. The internal motive reflects the need to coordinate one's action with the actions of the rest of group.

We model the communication structure by means of a network. The network nodes are the members of the organization (they can be, for example, individual workers or different departments), and the network links keep track of who communicates with whom. The communication network represents the pattern of ongoing communications inside the group. Altogether, an organization is characterized by its information-processing needs and its information-processing capabilities. Information-processing needs correspond to the nature of the information uncertainty about the task to be performed and to the exact balance between the external and the internal motives faced by each member of the group. Information-processing capabilities correspond to the communications network. We are going to focus on how to structure communication capabilities to enhance organizational effectiveness.

The Impact of Network Geometry

For explanatory purposes, one representation of Airbus' internal communication structure among the French unit, the German unit and the Headquarters in the above example is provided by the

[2] "The Russian II Army had twenty-five telephones, a few Morse coding machines, and one Hughes apparatus, a primitive teleprinter capable of discharging 1.200 words per hour, which broke down and forced the commander to move around on horseback to find out what was going on.The Germans themselves had only forty wireless-stations for their whole armed forces; the Russians had even fewer, and in any case men did not know how to use them." Stone (1998), pg. 51.

[3] "Communications, particularly between Zhilinkski and Rennenkampf, were confused to the point where Zhilinkski, nominally commander of the front, sometimes barely knew what was happening." Stone, pg. 58.

[4] A few weeks later the Russians recovered and invaded East Prussia.

following graph.

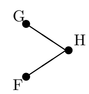

Figure 1.

The missing link from F to G represents the inability to effectively communicate local information from the French unit to the German one, and vice versa. Obviously, the organization as a whole would be better off if F and G could sustain permanent and clear communication about their respective activities. In that case, all agents would share all available information within the organization, facilitating coordination and increasing knowledge about the task to perform. As we will show later on, this result can be generalized to any other network structure. Closing triads is always a welfare enhancing intervention in the geometry of communication channels. Hence, in this example, the creation of a communication line between F and G is, indeed, an adequate policy irrespective of the rest of Airbus' organizational structure.

One might be tempted to postulate that not only is closing triads beneficial for the organization but that adding any other link is beneficial, too. This would be an error: increasing information-processing capabilities within the organization can be detrimental to the organization as a whole. A simple example will provide the main intuition to sustain this statement. Consider the two following networks::

Figure 2.

In the first network, leaving agent E isolated, agents F, G and H can achieve full local coordination due to the fact that they share symmetric positions in the network and hold the same information. In the second network, agent H has a new informational source, agent E. Agent H has more precise information about the task to be performed and part of this information is helpful to better coordinate with agent E. Therefore, he reacts by adopting a different action than before. This decision destroys complete coordination with the other two members in the triad. This miscoordination can be harmful to the organization when the coordination motive is predominant.

Model and Results

In our model, agents act in a decentralized manner and, hence, the internal coordination problem together with the uncertain external motive define an incomplete information game. In the first part of the paper, we provide an analysis of equilibrium and welfare for general information-sharing schemes. In the second part of the paper, we build on the results already developed to embed a network structure of communication channels. The communication network disseminates within the group any private information that individuals hold about the task. We show that the simultaneous-choice game has a unique Bayes-Nash equilibrium for any communication network. Information overlap is crucial to build cross inferences about each others' information and, ultimately, to compute the (arbitrarily) high order beliefs that enter the equilibrium determination.

In this equilibrium, individuals use a strategy that is a linear function of a knowledge index that we define. The knowledge index is computed using the high order beliefs and thus depends on the communication network. Beyond reflecting the geometry of communication possibilities, the knowledge index also depends on the exact balance between internal and external motives for the organization members—their information-processing needs—that enters the fixed-point equilibrium calculation. This knowledge index has some connection with standard centrality indexes in sociology, except that it is computed with an information correlation matrix (induced by informal communication) rather than with a sociometric matrix.

We also provide a closed-form expression for equilibrium payoffs and work out their comparative statics with respect to the exogenous payoff parameters, comprising the communication network, the balance of the internal versus the external motives, and the accuracy of the private signals about the task to be performed. When considering aggregate welfare, our analysis extends the conclusion derived from the three-agent example presented above to any other network structure: in any internal communication structure in which two disconnected agents share a common interlocutor, the organization would be strictly better off if these two agents were connected. This proposition is a natural but not completely obvious result: other agents not involved in this triad entertain differing beliefs about the information the actors in the triad possess, and a higher level of communication activity within the triad might make it even more difficult to keep track of these pieces of information. This could be the cause of more severe mis-coordination between members of the triad and members outside it. However, within the triad, higher complete communication activity increases local common knowledge and helps the members of the triad to acquire a more precise picture of the task to be performed. It can be proved that the latter effects overwhelm the potentially negative externalities exerted on the other members of the organization.

This comparative statics of the equilibrium payoffs with respect to the communication network is inherited from that of the knowledge index. Overall, the knowledge index shows to be a helpful analytical object that resumes in a simple and computable form all strategic complexity derived from the environment.

Given the aforementioned result that not all link additions are welfare-enhancing, the recommendation of closing triads provides a partial ordering within the set of all possible networks. One implication of this partial order is that the optimal communication structure is the complete graph, where everyone can communicate with everyone else. However, communication *urbi et orbi* need not be costless or even possible. Furthermore, empirical evidence tracking down the structure of informal communication structures shows that the complete network is a very rare exception rather than the rule (see for example Krackhardt and Hanson, 1993).

We inquire about the optimal communication network for a fixed supply of communication links. Because network closure defines a pre-order that does not allow to rank all networks unambiguously, we cannot make use of our previous comparative statics result to single out the optimal geometric arrangement of a fixed supply of network links. When the internal coordination problem is not too overwhelming (compared to the external concern), and the uncertainty about the characteristics of the group task to be performed is large enough, the optimal fit between information-processing needs and capabilities is achieved by a centralized and clustered network. More precisely, the optimal geometric arrangement is the one that maximizes a network span index that we define.[5] Instead, when the internal coordination concern is more demanding, the optimal network is regular (and thus distributed). Our theoretical analysis is closely related to a number of research branches such as the study of games with strategic complementarities (Vives, 2005) and the welfare and social value of information in this class of games (Angeletos and Pavan, 2007; Morris and Shin, 2002), as well as the literature on belief convergence in networks (DeGroot, 1974; DeMarzo *et al.*, 2003). The links with these papers, as well as with other related works, are more precisely stated in the next section.

2 Related Literature

Our analysis relates to Angeletos and Pavan (2007). In that paper the authors provide a neat taxonomy of quadratic games with strategic complementarities in terms of informational effects in equilibrium, welfare and social value of information, under the assumption of a continuum population and independent (Gaussian) information structure. In our setup payoffs also remain quadratic and the information structure is still Gaussian for tractability purposes. Even so, we depart from their analysis in both assumptions: from the former, in the fact that we assume a finite population, and from the latter because we allow for correlated information. With a finite population we can introduce a well-defined network structure linking agents. The correlated information structure expresses the common information received when communication is allowed. Our analysis however is obviously more restrictive than theirs because we do not attempt here to analyze the effects of communication in all kind of quadratic game with strategic complementarities. We focus our attention in the so-called beauty contest games, a variation of the payoff structure in Morris and Shin (2002). Later on, once the main theoretical results are introduced we are going to come back again to the comparison with Angeletos and Pavan, to see which kind of conclusions

[5] For instance, the star is the network with maximum span among the minimally connected networks -the trees.

are maintained and where our analysis differ.

Since the payoff structure of the Bayesian game we analyze is supermodular we inherit some of the nice properties these kind of games have both about existence of equilibria and comparative statics results. In particular, since the payoffs also satisfy the property of increasing differences in type, the game is monotone supermodular in the sense of Van Zandt and Vives (2006). This ensures that there exists an equilibrium for this game and that, whenever the equilibrium is unique, as it is in our case, optimal actions are monotone in type. In accordance with this conclusion, our characterization shows that equilibrium strategies are linear and increasing in owns' communication report, i.e. type. Vives (2005) provides an exhaustive survey of the literature of complementarities in games that highlights, within many other applications, that global games and team problems use to lie within the realm of monotone supermodular Bayesian games. Our game can be seen both as a global game and as an (almost) team theory problem. Hence, existence and the monotone comparative statics result of monotonicity on types are not very surprising under this perspective. To establish uniqueness and linearity of the equilibrium play we rely on a central result in team theory due to Radner (1962).

The analysis in the last subsection of the paper, that deals with communication in networks for a number of periods, bears some resemblance with DeMarzo *et al.* (2003). We borrow from the literature of beliefs convergence that traces back to the seminal work of DeGroot (1974) (see also Golub and Jackson, 2007, for an extensive survey and additional results with an economic flavour, and Moreau, 2005, for a recent contribution to this literature in the area of computer science). This research explores the effects of different dynamic information gathering procedures in the final believes each agent has about a parameter that represent the state of the world. In particular, DeMarzo *et al.* analyze the effects of the repeated iteration of the communication procedure we have specified above, i.e. agents aggregate each period, in a boundedly rational manner, the information they receive from their neighbours in the network. If some agents are more connected than others, their initial information spreads more recurrently through all available links, and since agents do not take into account for this repetition of information, agents share a final common opinion about the state of the world that overweights the information of more connected individuals. The analysis of beliefs convergence in this literature tends to be mechanical and isolated from the possible strategic motives that could sustain it. Observe that these belief dynamics analyzed by this literature does not coincide with the analysis of higher-order beliefs in our incomplete information game. It simply characterizes the final level of correlation among communication reports of different agents, and hence its effects is in the matrix of correlations. We analyze in the final section of this paper the equilibrium and welfare implications of DeMarzo *et al.* communication procedure in our setup, comparing it to the one-period communication process, in which repeated accumulation of information can not arise.

Besides the aforementioned works, our paper is related to other literatures.

In this paper we analyze a bayesian game with strategic complementarities with a unique equilibrium. Our payoff structure can be alternatively represented as a team problem. The unicity result relies on the seminal work of Radner (1962) on the theory of team decisions. Team theory complements the principal-agent view of organizations and it has been used to answer a variety of questions on the theory of organizations. For example, building on Marshak and Radner (1972), there is an extensive literature that analyzes in a team-theoretical framework the optimal inner structure of an organization that begins with Crémer (1980) and is followed by Aoki (1986), Geanakoplos an Milgrom (1991), Dessein and Santos (2006) and Dessein *et al.* (2006). In particular, Dessein and Santos and Dessein *et al.* highlight some communicational aspects of organizations, as we try to do in this work. However, the communication structure in these set of papers differs from ours and we answer complementary questions to those they analyze.

Another question analyzed in the team-theoretical literature of organizations, and also closely related to our work, is the study of the optimal information structure of an organization. Crémer (1993) analyzes, in a two agents family of quadratic payoff functions, when shared knowledge, in which all agents receive the same signal about the state of the world, is superior to decentralized knowledge, in which each agent receives a different signal of the state of the world. Building on this work, Prat (2002) extends this analysis to more general setups, allowing for any finite number of agents and general team payoff structures. We depart from this literature in that we do not have to introduce extreme information assumptions such as that only no communication or full communication is possible. We can analyze a variety of intermediate situations in our setup.

Morris and Shin (2002) analyze in a game with a similar payoff structure with the one we analyze here the impact of public information on social welfare. They show that in this game with strategic complementarities public information can reduce welfare. Morris and Shin (2006) complements this analysis with the introduction of semi-public signals that act as a restricted communication process, and provide some comparisons between public and semi-public information with respect to welfare. Our analysis extends the analysis in this last work for a discrete setup an heterogeneous information structures.

From a more abstract perspective, our work also relates to the literature on global games, higher-order beliefs and common knowledge in games (Rubinstein, 1989; Geanakoplos, 1992, Morris and Shin, 2003). The knowledge index we introduce provides a tractable tool to aggregate higher order beliefs into a scalar value that measures how informative is his communication report to strategically internalize both the decision problem and the coordination motive.

Finally, there are alternative approaches to communication in organizations that are interested in different questions than those we pursue here. Crémer *et al.* (2007), building in some early discussions by Arrow on communication in organizations (see Arrow, 1974), study the language with which information is transmitted in organizations. They analyze which is the optimal organizational language and how it affects firm structure. Dewatripont and Tirole (2005) analyze a model of communication in which there are strategic interactions between the communication efforts of the members of an organization. Efforts determine if communication is informative or not. We abstract from this question and we assume in the networked communication process we introduce that all information agents communicate is hard, meaning that there is no loss of information or noise in the communication channel.

Besides the more general literature relating communication and information structures to the performance of organizations just surveyed, an increasing amount of research in economics and related areas is devoted to analyze more precisely the particular inner network structure of organizations.

The experimental psychology literature has long ago documented the crucial role of communication time and communication pattern for information aggregation purposes and is reported in the seminal work by Bavelas (1950) followed by Leavitt (1951). These works initiated a plethora of empirical research comparing centralized versus decentralized network organizational structures but it was not accompanied by much theoretical development. Shaw (1964) provides a review of this literature.

Organizational theorists have recently emphasized the important role of networks within organizations, and try to analyze with the use of tools from social network analysis (see Wasserman and Faust, 1994, for a formal approach to this topic) which are the relevant variables of the network structure of an organization for management purposes. Krackhardt and Hanson (1993), Ahuja and Carley (1999) and Cummings and Cross (2003) are some relevant examples in which the particular networked communication structure of real-world organizations is sketched. A common finding of this literature is that the network structure of communication substantially differs from the formal chart of the organization. Furthermore, communication networks tend to be clearly incomplete.

The literature on the optimal formal organization has highlighted the role of hierarchies, a particular form of network structures, to reduce costs associated to communication transmission and information processing. Sah and Stiglitz (1987) is an early example on this direction, in which the authors compare two different organizational structures, polyarchies and hierarchies, and its respective benefits to reduce possible errors when processing the information the organization receives and communicates. Besides, building on Radner (1993), the work by Bolton and Dewatripont (1994), Van Zandt (1999a), Garicano (2000), Guimerà *et al.* (2002) and Doots *et al.* (2003) highlights the importance of hierarchies, and more general network structures, to diminish the costs related to processing information that flows through the network of contacts. This literature is surveyed by Van Zandt (1999b) and Ioannides (2003). Our work is complementary to this one, and analyzes which are the informational externalities communication generate, how these interplay with payoff externalities, and how do they shape equilibrium outcomes.

From a more theoretical perspective, our work also relates to the literature of games of incomplete information played in a network, such as Morris (2000) and Chwe (2000). In particular, Chwe

(2000) is closer in spirit to our work. Chwe analyzes a game in which agents want to coordinate their binary decisions and guess the action of others from the local information that agents communicate to their neighbours in the network. Anyhow, our model, and the procedure of our analysis, differs in several points from Chwe. We analyze a game with a continuum of possible actions and agents do not only pursue to coordinate, but also to attain the real state of the world. These differences are reflected in the way agents rely on the network to form higher order expectations in both models.

3 The game

Actions and payoffs Consider an organization formed by n agents, that can represent for example workers or different divisions of the organization, each one choosing an action $a_i \in \mathbb{R}$. Payoffs depend on own and others actions, and on some exogenous parameter $\theta \in \mathbb{R}$. More precisely, we focus on payoffs that reflect two different concerns. On one hand, players want to match their action a_i to the value of θ, that represents the optimal task the organization should undertake. On the other hand, the work of the different members of the organization has to show some level of coordination. Hence, they all want to align their choice with that of the others.

We consider the following quadratic payoffs:[6]

$$u_i\left(a_1, ..., a_n; \theta\right) = -\left(1-r\right)\left(a_i - \theta\right)^2 - r\frac{1}{n-1}\sum_{j\neq i}\left(a_i - a_j\right)^2, \ i = 1, ..., n. \tag{1}$$

The first term is a quadratic loss between own's action and the optimal task −the external concern. The second term is the average discrepancy (or distance) between own's and others' actions for all possible pair-wise comparisons −the internal concern. The parameter $0 \leq r \leq 1$ measures the balance of the external target concern (that binds at $r = 0$) versus the internal coordination concern (that binds at $r = 1$).

If the exact value of θ is known to everybody, $a_i^* = \theta$ for all players i is the unique (first-best) Nash equilibrium. We analyze instead the case of incomplete information, where the exact value of θ is not known. This represents a situation in which the different actors in the organization do not know the exact task to be performed but only an approximate signal of it.

Information structure In the incomplete information case, the value of θ is determined by nature, $\theta \sim N\left(\theta_0, \phi_0\right)$.

[6] An alternative measure of mis-coordination is the quadratic loss between own action and the *average* of others' action. Formally, $v_i\left(a_1, ..., a_n; \theta\right) = -\left(1-r\right)\left(a_i - \theta\right)^2 - r\left(a_i - A_{-i}\right)^2, \ i = 1, ..., n,$where $A_{-i} = \left(\sum_{j\neq i} a_j\right) / \left(n-1\right)$ which is reminiscent of the discrete population counterpart of the beauty contest game for a continuum of agents in Morris and Shin (2002). One can readily check that the Bayes-Nash equilibria of the incomplete information game with these payoffs is the same than that with payoffs (1). Equilibrium payoffs, however, differ, but all results obtained here remain qualitatively unperturbed under the alternative specification.

Players don't know the exact realization of θ. Initially, they hold some information about θ that communicate to other agents through a well-defined information gathering scheme. This process delivers some signal $y_i = \theta + \varepsilon_i$ to each agent. This signal is, precisely, the result of aggregating all information agent i holds after communication. Since we interpret these signals as the result of some information aggregation procedure, that can involve communication or partially public signals, it is natural to expect some level of correlation among most of them. The networked communication procedure presented in the introduction provides a simple and compelling example of such information aggregation procedures. We abstract from the details of each information sharing scheme by considering its outcome as a primitive in our model though, later on, we are going to provide more structure to the mapping from initial information to signals.

We assume that the vector of signals follows a multinormal distribution:

$$\mathbf{y}|\theta \sim N\left(\theta \mathbf{1}_n, \mathbf{\Sigma}\right), \tag{2}$$

where $\mathbf{1}_n$ is the $n-$dimensional vector of 1s, and $\mathbf{\Sigma} = [\sigma_{ij}]$ is a general variance-covariance matrix. The elements σ_{ij} measure the correlation between signals received by i and j.

This general specification of signals allows for any correlation pattern across them. For instance, when signals are (conditionally) independent, we have:

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_{nn} \end{bmatrix},$$

with $\sigma_{11} = \cdots = \sigma_{nn}$ if signals are identically distributed. Instead, when output signals have a common public component, we have:

$$\mathbf{\Sigma} = \sigma \begin{bmatrix} 1 & \cdots & \rho \\ \vdots & \ddots & \vdots \\ \rho & \cdots & 1 \end{bmatrix},$$

where $0 \leq \rho \leq 1$ is the correlation across signals induced by the public component.

More generally, the variance-covariance matrix $\mathbf{\Sigma}$ can take any value. In Section 7, we analyze a particular family of communication processes by which players mix their conditionally independent private signals to obtain individual communication reports (the output signals) that can exhibit arbitrary correlation patterns.

Denote by $E_i[\cdot] = E[\cdot|\widehat{y}_i]$ the expectation operator by player i conditional on his signal realization being $\widehat{y}_i$. Standard algebra on normal distributions leads to:[7]

$$E_i[\theta] = (1 - f_i)\,\theta_0 + f_i \widehat{y}_i \qquad \text{and} \qquad E_i[y_j] = (1 - \omega_{ji})\,\theta_0 + \omega_{ji} \widehat{y}_i, \tag{3}$$

[7] See the appendix for more details.

where

$$f_i(\mathbf{\Sigma}) = \frac{\phi_\theta}{\phi_\theta + \sigma_{ii}} \qquad \text{and} \qquad \omega_{ji}(\mathbf{\Sigma}) = \frac{\phi_\theta + \sigma_{ij}}{\phi_\theta + \sigma_{ii}},$$

for all $i = 1, ..., n$ and $j \neq i$.

We gather together all the possible values (3) for pair-wise inferences about each other's signals into an $n-$square matrix:

$$\mathbf{\Omega}(\mathbf{\Sigma}) = \begin{bmatrix} 0 & & \omega_{ij}(\mathbf{\Sigma}) \\ & \ddots & \\ \omega_{ij}(\mathbf{\Sigma}) & & 0 \end{bmatrix}.$$

We set $\omega_{ii} = 0$, for all i. Each of the $n(n-1)$ out-of-diagonal cells in $\mathbf{\Omega}(\mathbf{\Sigma})$ gives the factor by which the row player i multiplies his own report to compute his forecast of every other column player j's report. Notice that the matrix $\mathbf{\Omega}(\mathbf{\Sigma})$ is symmetric only when the diagonal cells of $\mathbf{\Sigma}$ are identical, i.e, $\sigma_{11} = \cdots = \sigma_{nn}$. In words, i need not make the same inferences about j's reports than j about i.

From now on, we omit the dependency on the parameter $\mathbf{\Sigma}$ when no confusion is possible.

4 High-order beliefs and the knowledge index

Best-responses and high-order beliefs A strategy maps output signals to actions. At equilibrium, each player maximizes his expected payoffs, where the expectation is taken over the target parameter θ as well as others' actions, and is conditional on own information.

Let $\rho = r/(n-1)$. Given that payoffs are concave in own action, player i's best response is obtained from $\partial E_i[u_i(a;\theta)]/\partial a_i = 0$. We get:

$$BR_i(\tilde{a}_{-i}) = (1-r)E_i[\theta] + \rho \sum_{j \neq i} E_i[\tilde{a}_j], \text{ for all } i = 1, ..., n. \tag{4}$$

The Bayes-Nash equilibria are given by the signal functions or strategies $\tilde{a}_i(\widehat{y}_i)$ that solve (4). It turns out that the linearity of both best-responses and the Gaussian information structure imply that the only signal functions that solve (4) are linear in own information.

Before stating and proving formally this result, we provide a simple heuristic that suggests that equilibrium strategies should indeed be linear, and that paves the way towards our closed-form equilibrium characterization.

Nesting best-responses into each other in (4) gives:

$$BR_i(\tilde{a}_{-i}) = (1-r)E_i[\theta] + (1-r)\rho \sum_{j \neq i} E_i E_j[\theta] + \rho^2 \sum_{j \neq i} \sum_{k \neq j} E_i E_j[\tilde{a}_k],$$

so that optimal actions depend on first and second-order iterated expectations about θ.

Recursively iterating the process, one can easily check that $BR_i(\cdot)$ can be written as an infinite sum of arbitrarily high-order iterated expectations of the target value θ:

$$E_{i_1}E_{i_2}\cdots E_{i_p}[\theta], \tag{5}$$

which we denote as $E_{i_1,i_2,\ldots,i_p}[\theta]$, weighted by the geometrically decaying factor ρ^p.

For the sake of illustration, suppose that $\theta_0 = 0$. Then, using (3) recursively, we can write these iterated expectations as a function of the inferences players make about each others' output signals:

$$E_{i_p,\ldots,i_1}[\theta] = f_{i_1}\omega_{i_1i_2}\ldots\omega_{i_{p-1}i_p}\widehat{y}_{i_p}. \tag{6}$$

Two comments are in order.

First, arbitrarily high-order iterated expectations by player i can be expressed as a linear function of his own output report $\widehat{y}_i$, which suggests a linear solution to (4). Indeed, we characterize below a linear equilibrium, which turns out to be the unique equilibrium of the game (provided $r < 1$).

Second, high-order iterated expectations (6) depend both on the number of iterations and on the ordered identities of the players along the chain of iterated expectations (expectations about whom expectations, about whom expectations ...). This is not surprising. Indeed, the entries of $\mathbf{\Omega}$ allow for up to $n(n-1)$ different inferences technologies of each others's signals depending on the identity of the two involved players, the inferring one and the inferred one.

When $\theta_0 = 0$, using the expression for high-order iterated expectations (6), we define high-order beliefs about the value of θ by agent i as follows:

- order zero beliefs are $E_i^0[\theta] = E_i[\theta] = f_i y_i$;
- order one beliefs are $E_i^1[\theta] = \sum_{j\neq i} E_{j,i}[\theta] = \sum_{j\neq i} f_i\omega_{ij}y_j$
- order $p \geq 2$ beliefs are $E_i^p[\theta] = \sum_{i_p\neq\cdots\neq i_2\neq i} E_{i_p,i_{p-1},\ldots,i_2,i}[\theta] = \sum_{i_p\neq\cdots\neq i_2\neq i} f_i\omega_{ii_2}\ldots\omega_{i_{p-1}i_p}y_p$

Order zero beliefs differ for different pairs of agents i,j whenever $\sigma_{ii} \neq \sigma_{jj}$. Differences remain for any other level of p-order beliefs unless we deal with a very symmetric information structure $\mathbf{\Sigma}$, that we do not impose. Hence, in general, we cannot invoke nor construct some average belief operator to compute high-order beliefs.[8]

The knowledge index We now define a *knowledge index* that proves useful to provide a closed-form expression for the equilibrium.

The inference matrix $\mathbf{\Omega}$ keeps track of the pair-wise inference coefficients ω_{ij} across all pairs of players. The pth power $\mathbf{\Omega}^p = \mathbf{\Omega}^{(p\ \overset{times}{\cdots})}\mathbf{\Omega}$ keeps track of the inference coefficients $\omega_{i_1i_2}\ldots\omega_{i_{p-1}i_p}$ for all $p-$order chains of iterated inferences that enter the calculation of order p beliefs.

[8] As it is the case for example in Morris and Shin (2002).

The coordinates of $\mathbf{\Omega 1}_n$ sum, for each row player, the pair-wise inference coefficients other player's use to infer his signal, $\omega_{i1} + ... + \omega_{in}$. More generally, the coordinates of $\mathbf{\Omega}^p \mathbf{1}_n$ sum, for all row player, all the compound inference coefficients that enter the calculation of $p-$order iterated expectations other players are doing to infer the information available to the respective row player (6).

Consider the following infinite sum:

$$\mathbf{1}_n + \rho\mathbf{\Omega 1}_n + \rho^2\mathbf{\Omega}^2\mathbf{1}_n + \rho^3\mathbf{\Omega}^3\mathbf{1}_n + \cdots = \sum_{p=0}^{+\infty} \rho^p \mathbf{\Omega}^p \mathbf{1}_n = \left[\mathbf{I}_n - \rho\mathbf{\Omega}\right]^{-1} \mathbf{1}_n, \tag{7}$$

where $\mathbf{I}_n$ is the $n-$identity matrix. The coefficients of the vector $\left[\mathbf{I}_n - \rho\mathbf{\Omega}\right]^{-1} \mathbf{1}_n$ sum all compound inference coefficients that enter the calculation of arbitrary high-order iterated expectations for each player, weighted by the geometrically decaying factor ρ^p.[9]

Notice that (7) is well-defined for low enough values of ρ. It turns out that an exact strict upper bound for convergence to obtain is the inverse of the largest eigenvalue of $\mathbf{\Omega}$ (Debreu and Herstein, 1953). This largest eigenvalue is sometimes called the spectral index of the matrix.

Suppose that $\sigma_{ij} \leq \sigma_{ii}$, for all $i \neq j$. Then, $0 \leq \omega_{ij} \leq 1$, for all $i \neq j$. Together with the fact that $\omega_{ii} = 0$, for all i, we conclude that an upper bound for the spectral index of $\mathbf{\Omega}$ is $n-1$. Therefore, the infinite series (7) converges when $\rho < 1/(n-1)$ or, equivalently, $r < 1$.

Definition 1 *Let $r < 1$ and $\mathbf{\Sigma}$ such that $\sigma_{ij} \leq \sigma_{ii}$, for all $i \neq j$. The vector of individual knowledge indexes is:*

$$\mathbf{k}(r, \mathbf{\Sigma}) = (1-r)\left[\mathbf{I}_n - \rho\mathbf{\Omega}\left(\mathbf{\Sigma}\right)\right]^{-1} \mathbf{1}_n. \tag{8}$$

Suppose first that output signals are perfectly informative about the true value of θ, that is, $\mathbf{\Sigma} = \mathbf{0}_{n\times n}$. Then, $\omega_{ij} = 1$ for all $i \neq j$ and $k_i(r) = 1$, for all i. With complete information, the knowledge index is one for all players.

Under incomplete information, instead, the knowledge index takes a value smaller than one for every player, $0 \leq k_i(r) \leq 1$, for all i. The fact that $\mathbf{k}(r)$ is non-negative derives from the fact that the matrix $\left[\mathbf{I}_n - \rho\mathbf{\Omega}\right]^{-1}$ is non-negative when the spectral index condition holds (Debreu and Herstein, 1953), which is true in this case by the discussion above. When there is no coordination concern, the knowledge index is one for all players, $k_i(0) = 1$, for all i.

We have that

$$\frac{f_i}{1-r} k_i = \underbrace{f_i}_{\text{coeff. in } E_i^0[\theta]} + \rho \underbrace{\sum_{j\neq i} f_i \omega_{ij}}_{\text{coeffs. in } E_i^1[\theta]} + \rho^2 \underbrace{\sum_{i_3 \neq i_2 \neq i} f_i \omega_{ii_2}\omega_{i_2 i_3}}_{\text{coeffs. in } E_i^2[\theta]} + \cdots \tag{9}$$

[9] The first term $\mathbf{1}_n$ corresponds to the individual forecast of θ based on own information. The higher order terms correspond to individual forecast of θ that involve inferences about some other players' reports.

From (9), it is straightforward to obtain that the knowledge index is increasing with respect to $\mathbf{\Omega}$ (for the component-wise partial ordering), whenever the infinite sum remains well-defined. This means that the knowledge index, as we could expect, is increasing in the abilities of agents to make better inferences of others information.

This knowledge index is formally reminiscent of standard centrality measures in sociology, but is computed with an information correlation matrix rather than with socio-metric data.

The knowledge index also depends on the value of r. Indeed, it is the coordination concern that triggers the high-order iterated expectations that boil down to (8). How the knowledge changes with r is not trivial, as $[\mathbf{I}_n - \rho\mathbf{\Omega}]^{-1}$ is an increasing (infinite) polynomial in r whereas the multiplicative factor $(1-r)$ decreases with r. The next result shows that the latter effect dominates. However, because of these two conflicting effects, the elasticity of the knowledge index with respect to r is smaller than one.

Proposition 1 $\mathbf{k}(r)$ *is non-increasing with* r*.*

5 Equilibrium

We are now ready to state the next result that establishes uniqueness of the Bayes-Nash equilibrium, fully characterized in terms of the knowledge index.

Theorem 1 *Let* $r < 1$*. The unique Bayes-Nash equilibrium has strategies linear in output signals given by:*

$$\tilde{a}_i^* (\widehat{y}_i) = (1 - k_i)\,\theta_0 + k_i E_i [\theta] = (1 - f_i k_i)\,\theta_0 + f_i k_i \widehat{y}_i, \tag{10}$$

for all $i = 1, ..., n$*, where* k_i *is the knowledge index defined in (8).*

When $r = 0$, players face a simple decision problem, $u_i(a;\theta) = -(a_i - \theta)^2$. The optimal action is their individual forecast $E_i[\theta] = (1 - f_i)\theta_0 + f_i\widehat{y}_i$ and the knowledge index is $k_i(0) = 1$.

When $0 < r < 1$, instead, payoffs are interdependent. Players now need to conciliate the decision problem with the coordination concern. At equilibrium, they rely on their own signal in proportion to their knowledge index. Otherwise, the mean prior acts as a focal action. In particular, the knowledge index of agent i aggregates in a single idiosyncratic, with a form that takes into account,value how well other agents are able to infer his communication report through all possible chains of higher-order beliefs (see the expression in (9)). The knowledge index provides a necessary fine-tuning between the common information available from the prior and all information gathered. The equilibrium behavior reflects the anisotropy of the chains of pair-wise inferences, itself related to the details of the variance-covariance matrix of individual signals.

Notice that the focal action (here, the mean prior θ_0 of the target value) serves very well the purpose of minimizing the coordination loss. However, it induces an individual decision loss equal to $-(1-r)\phi_\theta$, proportional to the variance of the prior distribution. This is exactly the same loss that players would incur if no output signal were available. Instead, if players use some of the information contained in their output signal, they can reduce this decision loss. On top of that, and even more when individual signals are correlated, players can also use individual signals to draw inferences about others' information. Output signals can thus also be useful on the coordination front. The knowledge index, which depends both on the salience of the coordination problem r, and on the variance-covariance matrix of output signals $\mathbf{\Sigma}$, reflects the optimal equilibrium use of private information to reduce both decision and coordination losses. The focal action is used in proportion with the lack of knowledge.

The uniqueness and linearity result follows from a central theorem by Radner (1962) on teams, and the fact that our quadratic game payoffs admit a potential that represents common (team) interests for all players, as pointed out by Ui (2004). The particular closed-form for the equilibrium strategy, that involves explicitly the knowledge index, exploits the intimate connection between quadratic games and centrality indexes in sociology established by Ballester *et al.* (2006).

In general, the knowledge index varies across players in a way that reflects the anisotropy of high-order iterated expectations. Given the close connection between knowledge index and equilibrium strategies, we can not expect symmetric behavior from the part of the players. Rather, high-order iterated expectations are an-isotropic, and change with the particular ordered chain of pair-wise inferences. Therefore, beyond the identity of the players in the chain of iterated expectations, the order also matters. This an-isotropy very likely sustains asymmetric choices across players[10] and it rules out symmetric equilibria, in general.

Combining Theorem 1 and Proposition 1, we can conclude that when the coordination problem becomes more acute (that is, r increases), players shift weight from their information to the focal action. Note, however, that the decrease in the information weight is smaller than the increase in the coordination concern. Indeed, the elasticity of $\mathbf{k}$ with respect to r is smaller than one. This is because, as explained above, information also serves (partly) the purpose of coordinating actions.

One more comment is in order. Observe that we are analyzing a game with strategic complementarities, since

$$\frac{\partial^2 u_i}{\partial a_i \partial a_j} = \frac{2r}{n-1} > 0.$$

Furthermore it is a game with increasing differences in types whenever

$$\frac{\partial^2 E_i[u_i]}{\partial a_i \partial y_i} = 2(1-r) f_i > 0 \qquad \text{for all } i \tag{11}$$

that always holds, for all i. Hence, we know that the following monotone comparative statics result is satisfied: if the equilibrium is unique, it is going to be increasing in type (Van Zandt and

[10] Note that $\mathbb{E}_{i,j}[\theta] = f_j\omega_{ji}\widehat{y}_i \neq f_i\omega_{ij}\widehat{y}_j = \mathbb{E}_{j,i}[\theta]$, unless $\widehat{y}_i = \widehat{y}_j$ (notice that $f_j\omega_{ji} = f_i\omega_{ij}$). More generally, $\mathbb{E}_{i_1,i_2,\ldots,i_p}[\theta] \neq \mathbb{E}_{\sigma(i_1),\sigma(i_2),\ldots,\sigma(i_p)}[\theta]$ for all non-trivial permutation σ of $\{i_1,\ldots,i_p\}$. Therefore, beyond the identity of the players in the chain of iterated expectations, the order also matters.

Vives, 2006). The proof for uniqueness of equilibrium only depends on the form of the payoff functions and does not assume any condition on the information structure $\mathbf{\Sigma}$. In particular, Theorem 1 establishes existence, uniqueness and linearity of the Bayes-Nash equilibrium when $r < 1$ and expresses it using the knowledge index k_i. A sufficient condition for convergence of the infinite sum in (7) is $\sigma_{ij} \leq \sigma_{ii}$, for all $i \neq j$. This last condition on $\mathbf{\Sigma}$ is always true if signal accuracies are all the same, $\sigma_{11} = \cdots = \sigma_{nn}$. Otherwise, it imposes bounds on the signal correlations. Indeed, writing covariances as $\sigma_{ij} = \gamma_{ij}\sqrt{\sigma_{ii}\sigma_{jj}}$, where $0 \leq \gamma_{ij} \leq 1$ is the correlation coefficient, the condition becomes $\gamma_{ij}^2 \leq \sigma_{jj}/\sigma_{ii}$, for all i, j. This condition on $\mathbf{\Sigma}$ can be relaxed, as stated below.

Remark 1 *Let r and $\mathbf{\Sigma}$ such that the largest eigenvalue of $\rho\mathbf{\Omega}$ is strictly smaller than one. Then, (8) is well-defined and the unique Bayes-Nash equilibrium has strategies linear in output signals given by (10).*

However, none of this conditions is necessary to ensure that the knowledge index k_i is positive. Since (11) is always satisfied and we can ensure that equilibrium strategies are monotone increasing on type, we can ensure that all k_i's are positive for all i even if the sum in (7) diverges. Of course, the knowledge index can always be computed directly with the matrix expression in (8), irrespective of whether (7) converges or not.

6 Welfare

We also assume from now on that $\theta_0 = 0$. This simplifies computations without altering the conclusions.

We compute the aggregate ex ante equilibrium payoffs given by:

$$U^*(r) = E_\theta E_{(y_1,\ldots,y_n)|\theta}[\sum_{i=1}^{n} u_i(\tilde{a}^*(y))|\theta]. \tag{12}$$

Define the following diagonal matrix of forecast coefficients (3), with zeros off-diagonal:

$$\mathbf{F} = \begin{bmatrix} f_1 & & 0 \\ & \ddots & \\ 0 & & f_n \end{bmatrix}.$$

Straight algebra leads to the following expression.

Proposition 2 *Let $r < 1$. The aggregate ex ante equilibrium payoffs (12) are:*

$$U^*(r) = \phi_\theta(1-r)\left[\mathbf{k}^t(r)\mathbf{F}\mathbf{k}(r) - n\right] \leq 0. \tag{13}$$

There are two sources of welfare losses, coordination losses and decision losses.

Decision losses correspond to the external target concern. They reflect the inaccuracies of the individual information to correctly forecast the true value of θ. When $r = 0$, the expression for ex ante payoffs (13) becomes:

$$U^{*}(0) = \phi_{\theta}\left[\sum_{i=1}^{n} f_i - n\right],$$

which is an increasing function of the forecast coefficients f_i, $i = 1, ..., n$. The forecast coefficients are monotone increasing with the individual signal accuracies $1/\sigma_{ii}$.

Coordination losses correspond to the internal coordination concern. Straightforward algebra shows that these losses are proportional to $\sum_i f_i \left(k_i - k_i^2\right)$. Hence, coordination losses are small if k_i is close to zero for all i, that is, all players take an action very close to the focal action. Coordination losses are also small when k_i is close to one for all i, that is, either coordination concerns vanish ($r = 0$) or there is no incomplete information ($\mathbf{\Sigma} = \mathbf{0}_{n\times n}$).

The social optimum We compare here the equilibrium use of information with the efficient use of information. The next result characterizes the optimal actions a social planner would implement.

Proposition 3 *Let $r < 1$. The socially optimal action for each player is:*

$$\tilde{a}_i^S(\widehat{y}_i) = \left(1 - k_i\left(\frac{2r}{1+r}\right)\right)\theta_0 + k_i\left(\frac{2r}{1+r}\right)E_i[\theta].$$

The social optimal action is equal to the equilibrium action of a game where the weight assigned to the coordination loss is increased from r to $2r/(1+r)$. Notice that Proposition 1 implies that $\mathbf{k}(2r/(1+r)) \leq \mathbf{k}(r)$. At equilibrium, agents overuse their private information compared to the social optimum. Indeed, at the socially optimal actions each player relies more on the prior information and less on the private information than at the equilibrium action.

A more interesting feature of the shape of social optimal actions is that they remain heterogeneous and strongly dependent on the information structure through the knowledge index operator. In particular, it can be illustrative to compare this feature with the characterization of socially optimal actions for beauty contest games without communication and a continuum of players, as in Angeletos and Pavan (2007). In that case, the socially optimal action is $\tilde{a}_i^S = E_i[\theta]$. This is because given the continuum of players in the social welfare function the coordination motive completely vanishes and the remaining decision problem is optimally solved by setting $\tilde{a}_i^S = E_i[\theta]$. On the contrary, our assumption of a finite number of players induces that the social welfare function still internalizes part of the coordination concern. Two diverging forces are in order then in our characterization. Firstly, since $2r/(1+r)$ is monotonely increasing in r, *ceteris paribus* social optimum actions approach θ_0. Prior information acts as a focal point that helps to resolve the coordination problem. Secondly, given that $k_i' \left(2r/(1+r) \right) \geq k_i \left(2r/(1+r) \right)$ if $\mathbf{\Omega}' \geq \mathbf{\Omega}$ (for the component-wise partial order, see Proposition 4 below), then, *ceteris paribus,* if communication activity increases in a way that ensures that all correlations among pairs of communication reports do not decrease and communication reports become more precise, then socially optimal actions are tilted towards $E_i [\theta]$. Communication procedures that increase correlations diminish the possibility of miscoordination between agents because agents rely on similar pieces of information to make their estimate of the target θ, and hence their estimates are going to be closer.

Of course, as we are going to show later on, there are many communication procedures in which increasing communication does not translate in a clear increase in the matrix Ω. In these cases, the knowledge index once again fine-tunes, like in equilibrium strategies, the effects payoff and informational externalities generate.

Angeletos and Pavan (2007) analysis of games with strategic complementarities and substitutabilities is performed with the use of two concepts related to the information structure: *accuracy,* that measures the precision of agents to correctly forecast the common task, and *commonality,* that measures the ability of agents to correctly forecast the information of others. These two values determine the whole informational structure. Unfortunately, in our case this two-dimensional description of information does not suffice to fully characterize informational effects. Angeletos and Pavan define accuracy as the inverse of the variance of $v_i = \theta - E_i [\theta]$. In our case this value is equal to $1/Var(v_i) = 1/Var \left(\theta - (1 - f_i) \theta_0 - f_i y_i \right) = 1/\phi_0$, that still remains a common characteristic of all agents. But with commonality, defined as the covariance $Cov(v_i, v_j)$, we obtain that

$$
\begin{aligned}
Cov(v_i, v_j) &= Cov(\theta - (1 - f_i) \theta_0 - f_i y_i, \theta - (1 - f_j) \theta_0 - f_j y_j) \\
&= (1 - f_i - f_j) \phi_0 + f_i f_j (\phi_0 + \sigma_{ij}) \\
&= \frac{\phi_0^2 \sigma_{ij}}{(\phi_0 + \sigma_{ii}) (\phi_0 + \sigma_{jj})}
\end{aligned}
$$

Hence, commonality is indeed not a common characteristic of the information structure. Instead, it is strongly pairwise dependant. Therefore the map from informational structure, defined in our setup by $\mathbf{\Sigma}$, to the space $(accuracy, commonalities)$ is not effective in reducing the dimensionality of the elements that enter in the analysis. The reason is simple. In a framework where information is partially and heterogeneously shared, as it happens when communication arises, information available to each one of the agents involved in the game is strongly dependent on the way information has been transmitted, and characteristics of the final communication reports remain fairly heterogeneous. Instead, as it can be seen in the determination of equilibrium strategies, payoffs and socially optimal actions, the idiosyncratic pair of values (f_i, k_i) seem to do the right job, and clearly imply a reduction in the dimensionality of elements that are into play. When there is information sharing, communication reports are a kind of interlaced semipublic signals, in which different agents share different pieces of information. The knowledge index operator is able to embed the strategic implications of this interlaced nature in a single value for each agent, and it is the tool that in our framework seems to help better to measure the ability of agents to forecast other agents information.

Information structure and welfare We now analyze some comparative statics results relating information structure and welfare. Since we are now giving very few structural assumptions on the kind of information gathering scheme that is being used, these comparative statics provide some general statements about the impact of changes on communication patterns in outcomes. Later on, when we analyze a more detailed communication procedure, we are going to be able to provide more precise statements on how different interventions on this procedure affect equilibrium and welfare.

Expression (13) implies that $U^*(r, \mathbf{\Sigma})$ is an increasing function of $\mathbf{k}^t(r)\,\mathbf{F}\mathbf{k}(r)$, from which we derive the following result.

Theorem 2 *Let* $\mathbf{\Sigma}, \mathbf{\Sigma}'$ *such that* $\mathbf{F}' \geq \mathbf{F}$ *and* $\mathbf{\Omega}' \geq \mathbf{\Omega}$*. Then,* $U^{*\prime}(r) \geq U^*(r)$*.*

We can understand a change from $\mathbf{\Sigma}$ to $\mathbf{\Sigma}'$ as a change in the communication process the organization is using to share information. The theorem provides a sufficient condition that ensures that the new process is better than the first one. If this new communication process improves the accuracy of the communication report of each agent (this is, indeed, the unique way by which the elements f_i can increase) and helps to improve the abilities of all agents to forecast others information (remember that a property of the knowledge index is that it increases whenever $\mathbf{\Omega}$ increases), it produces an unambiguous increase in total payoffs.

Imagine for example the case of an organization in which everybody communicates with everybody else with some common level of noise. If the manager were able to reduce this noise in the pairwise communication channels, the information gathered by each agent would obviously be more precise (implying an increase in the entries f_i), and the ability of agents to infer others' information would improve, because the reduction in noise ensures that the information any two agents gather is more similar than in the previous situation (mathematically, this translates in an increase of all out-diagonal terms in $\mathbf{\Omega}$). Theorem 2 ensures then that this noise reduction implies a desired welfare improvement.

Alternatively, other possible variations in the communication process can come for example from differences in the network structure that determines communication capabilities within the organization. We provide later on an analysis of the effects of this kind of structural interventions on welfare. We show that adding a link has an ambiguous effect. It certainly increases the accuracy of all agents' signals but it can negatively interfere with the inference process to forecast others' information. As presented in the introduction there are cases in which the second effect dominates the first one and the addition of the link shows to be detrimental for the organization. We also provide a result that ensures that some kinds of link addition align positively both effects, ensuring that some link additions are unambiguously welfare-improving.

The general comparative statics result of equilibrium welfare with respect to the variance-covariance matrix in Theorem 2 has a number of implications that we now explore.

We first establish comparative statics result with respect to information accuracy and correlations.

We write covariances as $\sigma_{ij} = \gamma_{ij}\sqrt{\sigma_{ii}\sigma_{jj}}$, where $0 \leq \gamma_{ij} \leq 1$ is the correlation coefficient. In other words, the variance-covariance matrix $\mathbf{\Sigma}$ is uniquely defined by a vector of accuracies:

$$\boldsymbol{\sigma} = (\sigma_{11}, ..., \sigma_{nn})$$

and a symmetric correlation matrix:

$$\mathbf{\Gamma} = \begin{bmatrix} 1 & & \gamma_{ij} \\ & \ddots & \\ \gamma_{ij} & & 1 \end{bmatrix}.$$

Proposition 4 *$U^*(r)$ is non-decreasing with $\mathbf{\Gamma}$ and non-increasing with $\boldsymbol{\sigma}$ (for the component-wise partial order)*

This is a natural an desirable effect. Aggregate welfare increases when individual information becomes more accurate and/or correlation increases.

Remark 2 *Monotonicity with respect to $\mathbf{\Gamma}$ always holds. The monotonicity with respect to $\boldsymbol{\sigma}$ holds if correlations are not too high, that is, $\gamma_{ij} \leq 2\sqrt{\sigma_{ii}/\sigma_{jj}}$, for all i, j. A sufficient condition is that individual accuracies are not too different, that is, $\max_i\{\sigma_{ii}\} \leq 4\min_i\{\sigma_{ii}\}$.*

We now establish a comparative statics result with respect to the correlation pattern.

Proposition 5 *Suppose that $\sigma_{11} = \cdots = \sigma_{nn}$. Let j, l such that $k_l(r) > k_j(r)$ and $i \neq j, l$ such that $\gamma_{ij} > \gamma_{il}$. Let $\mathbf{\Sigma}'$ obtained from $\mathbf{\Sigma}$ by swapping γ_{ij} and γ_{il}. Then $\mathbf{k}'(r) > \mathbf{k}(r)$ and $U^{*\prime}(r) > U^*(r)$.*

In words, the knowledge index increases for all players when players with a higher knowledge index make better inferences about every other player's information than players with a lower knowledge index. As a consequence, aggregate welfare is higher.

Payoffs and welfare Payoffs are a weighted sum of an external target concern and an internal coordination concern, with weights $1-r$ and r, respectively. The external target concern induces a loss that depends on the forecast accuracy of the value of θ. This loss increases when players receive noisier information. The coordination concern measures the level of complementarity in the game and induces a loss that depends on the discrepancy of actions across players. When the dispersed information available to players differs widely across them, the coordination loss is higher the more individual actions are sensitive to private information.

As a matter of fact, Proposition 1 together with (10) imply that players shift weight from their information towards the focal action when r increases. If the information available to them is not very accurate to start with, this allows players to decrease coordination losses while alleviating the burden of the decision loss.

Proposition 6 *$U^*(r)$ is monotone increasing with r whenever $\sigma_{ii} \geq \underline{\sigma}$, for all $i = 1, ..., n$ for some $\underline{\sigma}$.*

In words, when the information available to players is noisy, decreasing the forecast problem and increasing the coordination problem reduces equilibrium welfare loss. This has a clear implication for the manager of an organization. If he knows that the environment is so noisy that, even after communication, the information the different members possess is not sufficiently precise, then he should try to convince them that the coordination concern the organization faces is larger than the true one.

7 A network communication process

So far, the communication process is characterized by the distribution over its output signals. We now describe a particular instance of a communication process for which we can explicitly compute this distribution over output signals.

The networked class of decentralized communication devices we consider is the following. Agents connected through a given network, and only them, communicate in pairs. Agents average the stream of signals received from their network of contacts. Here, we analyze how this decentralized information-sharing scheme shapes individual and organization decisions and outcomes when the network geometry varies.

The networked communication process. The agents receive a conditionally independent private signal, $x_i|\theta \sim N(\theta, \phi_\varepsilon)$.

We model the communication possibilities within the organization by a network g. We set $g_{ij} = g_{ji} = 1$ if i and j communicate with each other, and $g_{ij} = 0$ otherwise. Of course, $g_{ii} = 1$, while $g_i = g_{i1} + \cdots + g_{in}$ is the total number of interlocutors to player i, including oneself.

This communication process disseminates idiosyncratic signals in the population. After communication the information available to player i is:

$$y_i(g) = \frac{1}{g_i} \sum_{j=1}^{n} g_{ij} x_j. \tag{14}$$

The communication report $y_i(g)$ averages private signals across all information sources available to i, which include communication partners in the network g and oneself.

More generally, denote by $\mathbf{G} = [g_{ij}]$ the network adjacency matrix of communication links, and by $\overline{\mathbf{G}} = [g_{ij}/g_i]$ its row normalization. A compact notation for the communication report after the communication round is $\mathbf{y} = \overline{\mathbf{G}}\mathbf{x}$.

The covariance between the output signals $y_i(g)$ and $y_j(g)$ is then readily computed:

$$\sigma_{ij}(g) = \phi_\varepsilon \sum_{k=1}^{n} \overline{g}_{ik}\overline{g}_{jk}, \text{ for all } i, j,$$

Let $\alpha = \phi_\varepsilon/\phi_\theta$. Then, it is readily checked that

$$f_i(g) = \frac{1}{1+\frac{\alpha}{g_i}} \qquad \text{and} \qquad \omega_{ij}(g) = f_j(g)\left(1+\alpha\frac{c_{ij}(g)}{g_i g_j}\right),$$

where $c_{ij}(g) = \#\{k : g_{ik}g_{kj} = 1\}$ is the number of common interlocutors to both i and j in the network of communication g.[11] In particular, $c_{ij}(g) = 0$ for all two agents that do not share any common information source with each other, and $c_{ij}(g) \geq 1$, otherwise.

Consider, for instance, a regular communication network such that $g_i = d$ for every i. Then, $\sum_{j=1}^{n} c_{ij} = d^2$, for all i,[12] and thus $\sum_j \omega_{ij} = \varpi(d)$ for all i, where

$$\varpi(d) = \frac{n-1+\alpha(d-1)}{d+\alpha}.$$

Therefore, each of the terms in the infinite sum (7) can be written as $\rho^p\mathbf{\Omega}^p = \rho^p\varpi^p(\mathbf{\Omega}/\varpi)^p = \rho^p\varpi^p(\mathbf{\Omega}/\varpi)$, where the last equality uses the fact that $\mathbf{\Omega}/\varpi$ is a double-stochastic matrix, that is, a non-negative matrix with all rows and columns adding up to one. The knowledge index in a regular network is then readily computed:

$$k_i(\rho, d) = \frac{1-r}{1-\rho\varpi(d)}, \text{ for all } i. \tag{15}$$

In general, the impact of adding new links on both the knowledge index and the equilibrium payoffs depends very critically on the geometry of both the original and the resulting network. To fix ideas, let $n = 3$, and consider the three following networks in Figure 3.

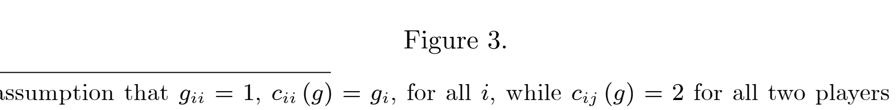

Figure 3.

[11] Given our assumption that $g_{ii} = 1$, $c_{ii}(g) = g_i$, for all i, while $c_{ij}(g) = 2$ for all two players who are one-link away from each other in the network of communication ($g_{ij} = 1$).

[12] The adjacency matrix of a regular network of degree d is $\mathbf{G} = d\mathbf{M}$, where $\mathbf{M}$ is a double stochastic matrix. Hence, $\mathbf{G}^2 = d^2\mathbf{M}^2$, where $\mathbf{M}^2$ is also a double stochastic matrix.

Adding link 23 leads from g to g', while adding link 13 leads from g' to g''.

Consider first the change from g' to g''. With the new link 13, both the number of information sources for players 2 and 3, and the sources overlap for all three players increase. As a result, all the inference coefficients increase, that is, $\mathbf{\Omega}'' \geq \mathbf{\Omega}'$, and so do the knowledge indexes of each player, that is, $\mathbf{k}''(\rho) \geq \mathbf{k}(\rho)$.

Consider now the change from g to g'. With the new link 23, players 2 and 3 widen their information sources, thus making better inferences about others' reports. Formally, $\omega'_{2i} \geq \omega_{2i}$ and $\omega'_{3i} \geq \omega_{3i}$, for all i. Instead, player 1 retrieves information from the same sources in g and in g', but loses grip on 2's true value of the report in the new network g' compared to the old one g. Formally,

$$\omega'_{12} = \frac{1 + \frac{\alpha}{3}}{1 + \frac{\alpha}{2}} < 1 = \omega_{12}.$$

As the previous example illustrates, the inference matrix need not be monotonic to link addition. New links that close triples (e.g., from g' to g'') create a common grounding in information sources, and thus allow for better cross inferences. Instead, new links that lead to open triples (e.g., from g to g') spread away information sources and dampen the accuracy of cross reports by adding noise on each others' awareness. The impact of link addition on the knowledge index and equilibrium payoffs is not clear (see Proposition 4) and depends on the relative balance of both forces.[13]

To keep track of the impact of link addition on the number of close and open triples, we define:

$$\tau(g) = \#\{(i,j,l) : g_{ij} g_{jl} g_{li} = 1, i \neq j \neq l \neq i\},$$

and

$$\iota(g) = \#\{(i,j,l) : g_{ij} g_{jl} = 1, g_{li} = 0, i \neq j \neq l \neq i\}.$$

By definition, $\tau(g)$ (resp. $\iota(g)$) gives the number of close (resp. open) triples in g.

Definition 2 *Let g, g' be two different networks. We say that g' is a closure of g, denoted $g' \trianglerighteq g$, when both $g' \supseteq g$ and $(\tau(g'), -\iota(g')) \geq (\tau(g), -\iota(g))$.*

In words, a network closure amounts to adding links such that the number of close triples (weakly) increases whereas that of open triples (weakly) decreases.

Suppose, for instance, that g is a star with one hub and $n-1$ spokes. Then, $\tau(g) = 0$ and $\iota(g) = (n-1)(n-2)$. Let g' deduced from g by adding ℓ spoke-to-spoke links. Then, $\tau(g') = 2\ell$ and $\iota(g') = \iota(g) - 2\ell$. Therefore, $g' \trianglerighteq g$, that is, g' is a closure of g.

We have the following result.

[13] For the case of $n = 3$ players, one can readily check that $\mathbf{k}(\rho, \mathbf{\Omega}') \geq \mathbf{k}(\rho, \mathbf{\Omega})$ although $\omega'_{12} < \omega_{12}$. Below we give an example where link addition actually *decreases* the knowledge index.

Proposition 7 *If $g' \trianglerighteq g$, then $\mathbf{k}'(r) \geq \mathbf{k}(r)$ and $U^{*\prime}(r) \geq U^*(r)$. In particular, denoting by g^N the complete network where all communication links are active, $U^*\left(r, \mathbf{\Sigma}^1\left(g^N\right)\right) \geq U^*\left(r, \mathbf{\Sigma}^1(g)\right)$, for all network g.*

The next example shows that adding links without satisficing the closure condition can indeed be detrimental not only for the knowledge index of some players but for the organization as a whole.

Consider the two networks in Figure 4, where g' is deduced from g by adding the link 14.

Figure 4.

Using (7) and (8), we can write, at a first order in r:

$$\mathbf{k}(r) - \mathbf{k}'(r) = \frac{r}{n-1}\left[\mathbf{\Omega} - \mathbf{\Omega}'\right] \cdot \mathbf{1}_n + O\left(r^2\right).$$

We have that $\sum_{j \neq 1} \omega_{1j} = (3 + 2\alpha) / (1 + \alpha)$ and $\sum_{j \neq 1} \omega'_{1j} = \left(24 + 15\alpha + 2\alpha^2\right) / \left(12 + 10\alpha + 2\alpha^2\right)$. Hence, for sufficiently large α the difference $k_1\left(r, \mathbf{\Sigma}^1(g')\right) - k_1\left(r, \mathbf{\Sigma}^1(g)\right)$ is close to an expression of the form

$$k_1(r) - k'_1(r) \simeq \frac{r}{n-1}(2-1) + O\left(r^2\right)$$

This suggests that it is possible that for large enough α (i.e. individual information is much more precise than prior information) and large enough r (i.e. agents share a strong coordination concern) the knowledge index of agent 1 can decrease. Of course, this heuristic approach is not enough for proving that. However, we can demonstrate through numerical simulations that this is indeed the case. And not only this but also that the knowledge indexes of agents 2 and 3 decrease for a suitable range of parameters. Consider for example an extreme case in which $\alpha = 250$ and $r = 0.99$. Then

$$\begin{aligned} \frac{1}{1-r}\mathbf{k}(r) &= (2.95, 2.95, 2.95, 1.03) \\ \frac{1}{1-r}\mathbf{k}'(r) &= (2.74, 2.85, 2.85, 1.92) \end{aligned}$$

Therefore, for r sufficiently close to 1 and a high enough α, we have $k'_i(r) < k_i(r)$ for $i = 1, 2, 3$ despite $g \subset g'$. Note that $\tau(g) = \tau(g')$, while $\iota(g') > \iota(g)$, implying that g' is not a closure of g. Network g' opens two triples with respect to g. Furthermore,

$$\mathbf{k}^t(r)\,\mathbf{F}\mathbf{k}(r) = 0.313 > 0.310 = \mathbf{k}'^t(r)\,\mathbf{F}'\mathbf{k}'(r)$$

This implies that $U^{*\prime}(r) \geq U^{*}(r)$. Hence, we have found by means of an example an important characteristic in our setup. Adding a link to a network can be detrimental for an organization. Communication capabilities can not be assigned indiscriminately through the different members of an organization. The intuition for this is simple. In the example we have just analyzed, adding a link increases the ability of agent 1 to infer the information available to other agents but diminishes the abilities of agents 2 and 3 to infer the information available to agent 1. If the coordination motive is sufficiently large it is better for agent 1 to fully coordinate with two agents than partially coordinate with three.

This result is remarkable because it highlights how small variations in the network structure can generate highly negative informational externalities for the structure as whole. The literature on information aggregation about learning from others and word-of-mouth communication has usually assumed a dynamic decision process in which agents learn information from the actions others have previously exerted (see for example the seminal work of Bikhchandani *et al.*, 1992, and chapter 6 in Vives, 2008, for a complete survey of the literature and further references), and the potentially negative externalities this dynamic information transmissions process can generate, such as herding and informational cascades. In our case, in a static framework with both payoff and informational externalities, we obtain that increasing the information available in an organization by adding new connections to the network of relations, can also generate a negative effect. Of course, it increases the precision of available information but it can hurt in its interplay with the necessary inference of others information when payoff externalities are sufficiently large, i.e. when coordination among the different members of the organization is important.

On optimal communication networks Proposition 7 suggests that the complete network is the optimal communication network. But, as Marshak and Radner (1972) warn: "Ideally, one would like to compare information structures on the basis of net value of information, namely gross value of information minus the cost of both the information and the associated best decision function. Therefore, any comparison between the gross values of two information structures is meaningful only in the context of some assumption about the relative costs of the two structures" (p. 224).

In our model, information structures are fully determined by the underlying communication network. Abstracting away from cognitive decision costs, we assume in what follows that communication links are costly (to set up and maintain). More precisely, we fix to some value $\gamma \in \{n-1, ..., n(n-1)/2\}$ the total number of communication links available, and we solve for the following optimal network design problem:

$$\max_{g} \{U^{*}\left(r, \mathbf{\Sigma}^{1}(g)\right) : \sum_{i,j} g_{ij} \leq 2\gamma\}. \tag{16}$$

This is a finite optimization problem, that admits at least one solution. For instance, when $\gamma =$

$n(n-1)/2$, the complete network g^N solves (16). More generally, the optimal network corresponds to the geometric arrangement of the available communication resources that best accounts for the weighted external and internal concern of all players.

Although a full characterization of the solutions to (16) is not available, the next discussion clarifies the forces at play.

Suppose first that α takes a very small value. Then, private signals allow for a very accurate estimation of the true value of θ, and the more so do communication reports. The external target concern and its associated welfare loss are then of secondary importance. Instead, the internal coordination concern becomes more (relatively) demanding. Therefore, the optimal network architecture is the one that primarily solves the coordination problem, namely, a regular or distributed network.[14]

Suppose now that α takes a large value. Then, private signals are very noisy. More inaccurate predictions of the underlying exogenous parameter increase the welfare loss from the external target concern. Suppose further that r is very small. Then, the external target concern is the driver for the optimal network solving (16). In this case, two opposite forces are at work. On one hand, a distributed or regular network homogenizes the forecasts about θ across players, but sets a common upper bound $2\gamma/n$ on the information sources available to every player. On the other hand, a centralized or irregular network leads to heterogeneous forecast about θ. But, by allocating most of the available information sources to a handful of players, it allows those ones to make a more accurate predictions that in the dispersed network.

The optimal network is the one that solves this trade-off optimally.

Formally, given a network g, let $c_i(g) = \sum_{j=1}^{n} c_{ij}(g)$, the number of two-link away contacts of player i in the network. Then, $c_i(g)/g_i$ is the ratio of two-link away contacts per direct link. It measures the range of indirect contacts. A high (resp. low) value of the ratio $c_i(g)/g_i$ corresponds to a long-range (resp. a short-span) of indirect contacts.

Consider for instance a star encompassing n players, where player 1 is the hub and players $2, ..., n$ are the spokes. Through a spoke-to-hub link, a spoke gains indirect access to every other spoke in the star. Instead, the spoke-to-hub link does not add any indirect contact to the hub but for the spoke itself. A spoke-to-hub link thus warrants a long-range of indirect contacts to the spoke but a short-span to the hub. Formally, $c_2(g)/g_2 = n/2 > 2(n-1)/n = c_1(g)/g_1$.

Define now:

$$S(g) = \sum_{i=1}^{n} \frac{c_i(g)}{g_i}.$$

This is the aggregate span or range of indirect contacts per network link. This span index increases with the number of super-connectors in the network, that give access to a wide range of indirect contacts to the nodes appending to them. It also increases with the number of triangles in the

[14] Or, to be more precise, the most regular network available given the link resource constraint γ.

network. In other words, a high (resp. low) span index is tantamount to an irregular (resp. distributed) and clustered (resp. open-knitted) network geometry.

We have the following result.

Proposition 8 *Fix γ. There exists $\underline{\alpha} > 0$ and $0 \leq \overline{r} < 1$ such that, for all $\alpha \geq \underline{\alpha}$ and $r \leq \overline{r}$, the optimal network that solves (16) is $g^* \in \arg\max_g\{S(g) : \sum_{i,j} g_{ij} \leq 2\gamma\}$;*

Fix the total number of network links available. Different geometric arrangements of these links lead to different network structures.

When the internal coordination problem is not too overwhelming and the signals about the exogenous target value are sufficiently noisy, the optimal geometric arrangement maximizes the aggregate network span. Namely, optimal networks are clustered and display an irregular distribution of connectivities.

We illustrate this point below with two examples.

Example 1. Suppose first that $\gamma = n-1$. This is the minimal number of links required to get a connected network, where each player is indirectly linked to every other one. Minimally connected networks are known as trees. The line and the star are two examples of trees.

Remark 3 *For sufficiently high α and low r, the optimal tree is the star.*

In words, the tree with maximal span is the most irregular one (for the distribution of connectivities).

In a star, spoke players play the same strategy, but the hub a different one. This mismatch induces a coordination loss. When r is small, this coordination loss is not much of a concern, and the optimal geometry is the one that reduces the welfare loss induced by external target concern.

The hub of a star makes as accurate as possible a prediction about the exogenous target value than any other node in any alternative tree configuration. This is because the hub has access trough the communication channels available to him to every other private signal in the population. On the contrary, the prediction made by spokes is as inaccurate as possible −they only have access to one additional source of information. When private signals are sufficiently noisy, the gain in accuracy for the hub compensates for the loss in accuracy for the spokes, and the star is the optimal tree.

Example 2. Consider the two network architectures in Figure 5 with $n = 4$ players and $\gamma = 4$ links, that we name the kite and the wheel.

Figure 5.

One can readily check that $S(g)$ is worth 38/3 for the kite, and 12 for the wheel. According to Proposition 8, the kite is the optimal network (yielding higher aggregate payoffs) for high enough values of α and low enough values of r. In this case, since we are assuming small enough values for r we know that the kite, that equals g' in Figure 3, performs better than g of Figure 3, and we do not need to do any computation for this: because of the remark in Example 1, in the case of four agents and three available links the optimal configuration is a star. Hence, the star performs better than g in Figure 3 for high enough values of α and low enough values of r. Since, the kite, g' in Figure 3, is a closure of the star, we conclude applying Proposition 7 that g' is better than any other network when there are four links available, and in particular it is better than g in Figure 3 for this range of parameters.

Hence, the order between g and g' is reversed when the coordination concern is small. The tension between information externalities derived from network structure and payoffs externalities vanishes when the latter is small enough.

More than one communication round Suppose that similarly to the above communication process, agents connected through the network communicate now in pairs for a fixed number of rounds. At each round, agents average the stream of signals previously received from their network contacts, and communicate this average signal back to them. However, from round two onwards, this simple heuristic that treats signals in the stream as *de facto* mutually independent, fails to adjust properly for redundant information from a common third-party. In fact, DeMarzo *et al.* (2003) motivate this information gathering scheme on the basis of bounded rationality in the form of persuasion bias, and show that the converging beliefs dynamics for this simple rule over-weight the private information of more "central" agents in the communication network. For this reason, asymptotic beliefs are not correct when the underlying network is irregular. Here we analyze how persuasion bias in information gathering affects welfare and which are the optimal network structures in this case.

In this case the communication report of agent i at period t, y_i^t depends on the communication reports from the previous period of his neighbours in the network of communication. In particular it is equal to the following weighted average:

$$y_i^t = \frac{1}{g_i} \sum_j g_{ij} y_j^{t-1} \qquad \text{for } t > 1$$

Hence, when players communicate repeatedly with their network interlocutors in the network g, and average the incoming stream of signals before sending it back to them, the resulting communication reports after t completed rounds of communication are $\mathbf{y}^t = \overline{\mathbf{G}}^t \mathbf{y}^{t-1}$. We denote this communication process by $\mathbb{P}^t(g)$.

The covariance between the output signals x_i^t and x_j^t is then readily computed:

$$\sigma_{ij}^t(g) = \phi_\varepsilon \sum_{k=1}^n \overline{g}_{ik}^{[t]} \overline{g}_{jk}^{[t]}, \text{ for all } i, j,$$

where $\overline{\mathbf{G}}^t = \left[\overline{g}_{ij}^{[t]}\right]$. In matrix notation, we have $\mathbf{\Sigma}^{t+1}(g) = \phi_\varepsilon \overline{\mathbf{G}}^{t+1} \overline{\mathbf{G}}^{t+1\prime} = \phi_\varepsilon \overline{\mathbf{G}} \mathbf{\Sigma}^t(g) \overline{\mathbf{G}}'$. The equilibrium actions can then be readily computed from Theorem 1.

In particular, suppose that players communicate an infinite number of rounds, $t \to +\infty$. Writing $\overline{\mathbf{G}}^{t+1} = \overline{\mathbf{G}}\overline{\mathbf{G}}^t$, one can view $\overline{\mathbf{G}}$ as the Markov transition matrix for the row probability vectors $\left(\overline{g}_{i1}^{[t]}, ..., \overline{g}_{in}^{[t]}\right)$ of the row-normalized matrix $\overline{\mathbf{G}}$. We thus have $\lim_{t \to +\infty} \overline{g}_j^{[t]} = \overline{g}_j^\infty$, where $\overline{\mathbf{g}}_j^\infty$ is the unique invariant distribution of the irreducible and aperiodic Markov process with transitions $\overline{\mathbf{G}}$. In turn, the fact that all row vectors of are identical implies that long-run beliefs for $\mathbb{P}^\infty(g)$ are common to all players, that is:

$$x_i^\infty = x^\infty = \overline{g}_1^\infty x_1 + \cdots + \overline{g}_n^\infty x_n, \text{ for all } i,$$

a weighted sum of private input signals.

We compute the weights. With an un-directed network we have $g_i \overline{g}_{ij}^{[t]} = \overline{g}_{ji}^{[t]} g_j$ from simple algebra, from which we obtain $g_i \overline{g}_j^\infty = \overline{g}_i^\infty g_j$ at the limit, and thus $\overline{g}_i^\infty = g_i / (g_1 + \cdots + g_n)$. Because averages of incoming signal streams at each communication round do not discount properly for redundant information from common sources, better connected players in the communication network end up credited with a higher weight in the emergent long-run consensual beliefs.

When all players share the same beliefs, the knowledge index hits its upper bound of one. Aggregate long-run payoffs then take the following simple form:

$$U^*(r, \mathbf{\Sigma}^\infty(g)) = n(1-r)\phi_\theta \left[\frac{1}{1 + \alpha \frac{g_1^2 + \cdots + g_n^2}{(g_1 + \cdots + g_n)^2}} - 1\right].$$

Fix the total supply of links, $(g_1 + \cdots + g_n)/2$. Then, these payoffs are maximal when $g_1^2 + \cdots + g_n^2$ is minimal, namely, on a regular network.

Therefore, we obtain in this case an unambiguous result about optimal network structures: in the presence of persuasion bias effects regular networks are always as regular as possible. this is in clear contrast with the result obtained for the one-period communication process, in which asymmetry can be optimal. Aggregate payoffs are thus higher for the wheel than for the kite in Figure 3.

8 Conclusion

The field of organizational theory has already provided a few examples of socio-metric analyses of a firm's inner network structure.[15] This literature has highlighted some agent's network characteristics, such as his degree, as a valuable proxy for the correct targeting of the relevant actors inside and organization. Our analysis shows that simple network statistics might not provide enough information for this. Instead, the knowledge index we introduce seems to do a better job on that inquiry. It encompasses in a subtle, and computable, form the strategic effects derived from unequal positions of different agents in the network. It simplifies and drives the analysis, that differs from the existing in the previous literature, and it is robust to alternative quadratic payoffs. The consequences of the interplay between other payoff profiles that show strategic complementarities/subtitutabilities and the informational externalities induced by communication is an open question that we plan to pursue in future research.

Our study of communication in networks entails both positive and normative implications. We have shown that adding new links, i.e. communication capabilities, have an ambiguous effect. Our results shed light on which links are detrimental and which are beneficial from the organization's point of view. Furthermore, we have developed an analysis about the global optimal network geometry of the organization's communication layout. Altogether, this opens the door to network target policies and it establishes well-defined network interventions to foster organizational effectiveness. Of course, besides normally distributed signals and quadratic payoffs, this analysis is not done without loss of generality.

We have assumed an static framework. This has allowed us to extract a number of conclusions about the structure of communication lines within the organization and it shows up a possible tension between information and payoff externalities routed at the geometry of the network structure, since more information, in the form of extended communication capabilities, can be detrimental in some circumstances. One possible extension would be to study a dynamic model in which agents' knowledge nourishes both from direct communication and also from the observation of others' actions. In this setup, information transmission through social networks could counterbalance potential herding effects generated by the previous stream of actions.

We have also assumed that agents know the exact shape of the whole network structure. This is crucial in our analysis. For example, knowing the degree distribution does not suffice for an agent to infer others' information. If knowledge about the network structure is limited, two different dimensions of incomplete information (the task to perform and the geometry of the network) are present and possibly overlap. It would be interesting to get a better understanding of how these two dimensions interfere with each other in the strategic decision process.[16]

[15] See Krackhardt and Hanson (1993), Ahuja and Carley (1999) and Cummings and Cross (2003).

[16] For more about games with incomplete information about network structure see Galeotti et al. (2007) and references therein.

9 Proofs

From (2) we deduce that $(\theta, y_1, ..., y_n)$ follows the following multinormal distribution:

$$(\theta, y_1, ..., y_n) \sim N(\theta_0 \mathbf{1}_{n+1}, \begin{bmatrix} \phi_\theta & \phi_\theta \mathbf{1}_n^t \\ \phi_\theta \mathbf{1}_n & \mathbf{\Sigma} \end{bmatrix}),$$

where $\mathbf{1}_{n+1}$ is the $(n+1)-$dimensional vector of all ones. The previous expression leads to:

$$(y_1, ..., y_n) \sim N\left(\theta_0 \mathbf{1}_n, \mathbf{\Sigma} + \phi_\theta \mathbf{J}_n\right), \tag{17}$$

where $\mathbf{J}_n$ is the $n-$square matrix of all ones.

Proof of Proposition 1: Let $\mathbf{M} = [\mathbf{I}_n - \rho\mathbf{\Omega}]^{-1}$. Noticing that $\mathbf{M}\cdot[\mathbf{I}_n - \rho\mathbf{\Omega}] = \mathbf{I}_n$ and differentiating with respect to ρ leads to:

$$\frac{\partial}{\partial\rho}\mathbf{M} = \mathbf{\Omega}\cdot\mathbf{M}^2,$$

and thus

$$\frac{\partial}{\partial r}\mathbf{M} = \frac{1}{n-1}\mathbf{\Omega}\cdot\mathbf{M}^2. \tag{18}$$

From $\mathbf{k} = (1-r)\,\mathbf{M}\cdot\mathbf{1}_n$ and (18) we obtain:

$$\frac{\partial}{\partial r}\mathbf{k} = -\frac{1}{1-r}\mathbf{k} + \frac{1-r}{n-1}\mathbf{\Omega}\mathbf{M}\mathbf{k},$$

and thus, since $[\mathbf{I}_n - \rho\mathbf{\Omega}]\,\mathbf{\Omega} = \mathbf{\Omega}\,[\mathbf{I}_n - \rho\mathbf{\Omega}]$ and $[\mathbf{I}_n - \rho\mathbf{\Omega}]\,\mathbf{M} = \mathbf{I}_n$,

$$\begin{aligned} [\mathbf{I}_n - \rho\mathbf{\Omega}]\,\frac{\partial}{\partial r}\mathbf{k} &= -\frac{1}{1-r}\,[\mathbf{I}_n - \rho\mathbf{\Omega}]\,\mathbf{k} + \frac{1-r}{n-1}\,[\mathbf{I}_n - \rho\mathbf{\Omega}]\,\mathbf{\Omega}\mathbf{M}\mathbf{k} \\ &= -\mathbf{1}_n + \frac{1-r}{n-1}\mathbf{\Omega}\mathbf{k}. \end{aligned}$$

We know that $\mathbf{0}_n \leq \mathbf{k} \leq \mathbf{1}_n$, and that $(1/(n-1))\,\mathbf{\Omega}\mathbf{1}_n \leq \mathbf{1}_n$ (from the fact that $0 \leq \omega_{ij} \leq 1$ for all i, j). Therefore, $((1-r)/(n-1))\,\mathbf{\Omega}\mathbf{k} \leq \mathbf{1}_n$, that is,

$$[\mathbf{I}_n - \rho\mathbf{\Omega}]\,\frac{\partial}{\partial r}\mathbf{k} \leq \mathbf{0}. \tag{19}$$

We show that $\partial k_i/\partial r \leq 0$**,** for all i. Suppose not. Let $i^* \in \arg\max\{\partial k_i/\partial r : i = 1, ..., n\}$. By assumption, $\partial k_{i^*}/\partial r > \mathbf{0}$. We have:

$$\rho\mathbf{\Omega}\frac{\partial}{\partial r}\mathbf{k} \leq \frac{\partial k_{i^*}}{\partial r}\rho\mathbf{\Omega}\mathbf{1}_n \leq r\frac{\partial k_{i^*}}{\partial r}\mathbf{1}_n.$$

The i^*th coordinate of the left-hand side of (19) is thus bounded from below by:

$$(1-r)\,\frac{\partial k_{i^*}}{\partial r} > 0,$$

which contradicts (19). ■

Proof of Theorem 1: We look for a linear equilibrium strategy, $\tilde{a}_i^*(\widehat{y}_i) = \alpha_i + \beta_i \widehat{y}_i$. Plugging back into (4), we obtain a linear system of equations with unknowns $\{\alpha_i, \beta_i\}_{i=1,\dots,n}$:

$$\alpha_i + \beta_i \widehat{y}_i = (1-r) E_i[\theta] + \rho \sum_{j \neq i} \left(\alpha_j + \beta_j E_i[y_j]\right), \text{ for all } \widehat{y}_i \in \mathbb{R} \text{ and } i = 1, ..., n. \tag{20}$$

We first show that (10) is indeed a Bayes-Nash equilibrium. Plugging back (3) into (20) we get the following equilibrium conditions:

$$\alpha_i = (1-r)(1-f_i)\theta_0 + \rho \sum_{j \neq i} \alpha_j + \rho \sum_{j \neq i} \beta_j (1-\omega_{ji})\theta_0, \text{ for all } i = 1, ..., n, \tag{21}$$

$$\beta_i = (1-r) f_i + \rho \sum_{j \neq i} \beta_j \omega_{ji}, \text{ for all } i = 1, ..., n. \tag{22}$$

We start by solving the system (22) of equations that relates the β parameters. Notice that $f_j \omega_{ji} = f_i \omega_{ij}$, for all i,j. Let $\gamma_i = \beta_i / f_i$. Dividing equation i by f_i in (22) leads to:

$$\gamma_i = (1-r) + \rho \sum_{j \neq i} \omega_{ij} \gamma_j, \text{ for all } i = 1, ..., n.$$

Notice also that $c(i,j) \leq g_j$, and thus $\omega_{ij} \leq 1$, for all i, j. Denote by $\mu(\mathbf{A})$ the largest eigenvalue of a matrix $\mathbf{A}$. Then, $\mu(\mathbf{\Omega}) \leq \mu(\mathbf{I}_n - \mathbf{J}_n) = n-1$. Therefore $\rho\mu(\mathbf{\Omega}) < 1$ is equivalent to $r < 1$. From Theorem 1 in Ballester *et al.* (2006) we deduce that $\beta_i = f_i k_i(r, \mathbf{\Sigma})$.

Adding for each $i = 1, \ldots, n$ the corresponding condition in (22) times θ_0 to the corresponding condition in (21) leads to:

$$\alpha_i - \rho \sum_{j \neq i} \alpha_j = (1-r)\theta_0 - \beta_i \theta_0 + \rho \sum_{j \neq i} \beta_j \theta_0, \text{ for all } i = 1, ..., n, \tag{23}$$

All the conditions in (23) can be gathered in matrix form as:

$$(\mathbf{I} - \rho(\mathbf{J} - \mathbf{I})) \cdot \boldsymbol{\alpha} = (1-r)\theta_0 \cdot \mathbf{1} - \theta_0 (\mathbf{I} - \rho(\mathbf{J} - \mathbf{I})) \cdot \boldsymbol{\beta}$$

Solving for $\boldsymbol{\alpha}$ we obtain

$$\boldsymbol{\alpha} = (1-r)\theta_0 \cdot (\mathbf{I} - \rho(\mathbf{J} - \mathbf{I}))^{-1} \cdot \mathbf{1} - \theta_0 \boldsymbol{\beta}$$

Some algebra shows that all diagonal entries in the matrix $(\mathbf{I} - \rho(\mathbf{J} - \mathbf{I}))^{-1}$ are equal to $(1-(n-2)\rho) / [(1+\rho)(1-(n-1)\rho)]$ while all its out-diagonal entries are equal to $\rho / [(1+\rho)(1-(n-1)\rho)]$. Hence, each entry of the vector $(\mathbf{I} - \rho(\mathbf{J} - \mathbf{I}))^{-1} \cdot \mathbf{1}$ is equal to $(1-(n-1)\rho)^{-1} = (1-r)^{-1}$. Therefore

$$\alpha_i = (1 - f_i k_i(r))\theta_0, \text{ for all } i = 1, ..., n.$$

This implies, as we wanted to show, that

$$\tilde{a}_i^* (\widehat{y}_i) = \alpha_i + \beta_i \widehat{y}_i = (1 - f_i k_i(r)) \theta_0 + f_i k_i(r) \widehat{y}_i = (1 - k_i(r)) \theta_0 + k_i(r) E_i [\theta], \text{ for all } i = 1, ..., n.$$

We now show that the linear equilibrium identified in (10) is the unique equilibrium. We proceed in two steps.

First, define the following payoff function:

$$V (a_1, ..., a_n) = -(1 - r) \sum_{i=1}^{n} (a_i - \theta)^2 - \frac{1}{2} \rho \sum_{i=1}^{n} \sum_{j \neq i} (a_i - a_j)^2 . \tag{24}$$

It is readily checked that $\partial E [u_i (a) | y_i] / \partial a_i = \partial E [V (a) | y_i] / \partial a_i$, for all action profile and information. In words, $V (\cdot)$ is a potential for the game payoffs u_i.[17] Therefore, the Bayesian Nash equilibria coincide with the team person-by-person maximal decisions for the team objective function (24) and the information structure (2).

Second, Theorem 4 in Radner (1962) gives a sufficient condition for uniqueness of the optimal and the person-by-person maximal team decision functions (which, then, necessarily coincide) when team payoffs are quadratic as in (24). This condition boils down to the $n-$square matrix of cross derivatives $\mathbf{Q} = [q_{ij}] = [\partial^2 V (a) / \partial a_i \partial a_j]$ being negative definite. We have:

$$q_{ij} = \begin{cases} -1, \text{ if } i = j \\ \rho, \text{ if } i \neq j \end{cases} .$$

We compute the determinant of $\mathbf{Q}$:

$$\begin{aligned} \det (\mathbf{Q}) &= (-1)^n \begin{vmatrix} 1 & -\rho & \cdots & -\rho \\ -\rho & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\rho \\ -\rho & \cdots & -\rho & 1 \end{vmatrix} = (-1)^n (1 - r) \begin{vmatrix} 1 & -\rho & \cdots & -\rho \\ 1 & \ddots & \ddots & \vdots \\ \vdots & -\rho & \ddots & -\rho \\ 1 & & -\rho & 1 \end{vmatrix} \\ &= (-1)^n (1 - r) \begin{vmatrix} 1 & -\rho & \cdots & -\rho \\ 0 & 1 + \rho & 0 & \\ \vdots & \cdots & \cdots & -\rho \\ 0 & \cdots & 0 & 1 + \rho \end{vmatrix} = (-1)^n (1 - r) (1 + \rho)^{n-1} \end{aligned}$$

The first equality is obtained by adding up all the columns to the first one, and then factorizing by the common term $1 - r$. The second equality is obtained by substracting the first row to every other row. We are left with an upper triangular matrix; the determinant is just the product of the diagonal terms.

[17] See Ui (2004) for a formal definition and a general existence result of a (Bayesian) potential for Bayesian quadratic games with symmetric cross-derivatives of payoffs.

The determinant of $\mathbf{Q}$ has thus the same sign than $(-1)^n$. Mutatis mutandis, we deduce that the minors of order p of $\mathbf{Q}$ have the sign of $(-1)^p$. The matrix $\mathbf{Q}$ is thus definite negative, and this concludes the proof. ■

Proof of Proposition 2: Developing the square terms and summing over all is gives:

$$\sum_{i=1}^{n} u_i\left(\tilde{a}^*(y)\right) = -(1-r)\left[n\theta^2 + \sum_{i=1}^{n} \tilde{a}_i^{*2} - 2\theta \sum_{i=1}^{n} \tilde{a}_i^*\right] - \rho\left[2(n-1)\sum_{i=1}^{n} \tilde{a}_i^{*2} - 2\sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{a}_i^* \tilde{a}_j^*\right]. \tag{25}$$

Then, noticing that $E[\theta] = 0$ and $E\left[\theta^2\right] = \phi_0$, we obtain the following identities:

$$\begin{aligned} E_\theta E_{(y_1,\ldots,y_n)|\theta}\left[\tilde{a}_i^*|\theta\right] &= \phi_\theta f_i k_i \\ E_\theta E_{(y_1,\ldots,y_n)|\theta}\left[\tilde{a}_i^{*2}|\theta\right] &= \phi_\theta f_i k_i^2 \\ E_\theta E_{(y_1,\ldots,y_n)|\theta}\left[\tilde{a}_i^* \tilde{a}_j^*|\theta\right] &= \phi_\theta f_i \omega_{ij} k_i k_j,\ i \neq j \end{aligned}$$

Using (25) and the previous identities, we obtain:

$$\begin{aligned} U^*(r) &= -(1-r)\, n\phi_\theta + 2\rho\phi_\theta \sum_{i=1}^{n}\sum_{j=1}^{n} f_i \omega_{ij} k_i k_j \\ &\quad - (1+r)\,\phi_\theta \sum_{i=1}^{n} f_i k_i^2 + 2\phi_\theta \sum_{i=1}^{n} f_i k_i. \end{aligned}$$

Next, notice that:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f_i \omega_{ij} k_i k_j = (\mathbf{Fk})^t\, \mathbf{\Omega k} = \mathbf{k}^t \mathbf{F \Omega k} = \frac{1}{\rho} \mathbf{k}^t \mathbf{F}\left[\mathbf{k} - \mathbf{1}_n\right].$$

Plugging back into the previous expression for $U^*(r)$ and simplifying gives the desired result. ■

Proof of Proposition 3:[18] Aggregate payoffs are equal to

$$u(a_1,\ldots,a_n) = -(1-r)\sum_i (a_i - \theta)^2 - \frac{r}{n-1}\sum_i \sum_{j\neq i} (a_i - a_j)^2 \tag{26}$$

In the socially optimal action profile all agents maximize the common utility function $u(a_1,\ldots,a_n)$. Disregarding the terms that do not depend on a_i, we get that agent i maximizes

$$\hat{u}_i(a_1,\ldots,a_n) = -(1-r)(a_i - \theta)^2 - \frac{2r}{n-1}\sum_{j\neq i} (a_i - a_j)^2$$

With an adequate rescaling, this utility function is equivalent to

$$\bar{u}_i(a_1,\ldots,a_n) = -(1-\bar{r})(a_i - \theta)^2 - \frac{\bar{r}}{n-1}\sum_{j\neq i} (a_i - a_j)^2 \qquad \text{for all } i$$

[18] We thank an anonymous referee for suggesting this elegant proof.

with $\bar{r} = 2r/(1+r)$, that correspond to the utility functions of the initial bayesian game, and only differ in the level of coordination. Therefore socially optimal actions are equal to the equilibrium actions with the coordination concern equal to $2r/(1+r)$. ■

Proof of Theorem 2: It is clear that $U^*(r)$ increases with $\mathbf{F}$ and with $\mathbf{k}$ (for the component-wise ordering). It is also clear that $\mathbf{F}$ increases with $\mathbf{\Sigma}$. We show that $\mathbf{k}(r)$ increases with $\mathbf{\Omega}$. Let $\mathbf{\Omega}' \geq \mathbf{\Omega}$. We write $\mathbf{\Omega}' = \mathbf{\Omega} + \mathbf{D}$, with $d_{ij} \geq 0$ for all i, j. Then,

$$[\mathbf{I}_n - \rho\mathbf{\Omega}]\,\mathbf{k} = (1-r)\,\mathbf{1} = \left[\mathbf{I}_n - \rho\mathbf{\Omega}'\right]\mathbf{k}' = [\mathbf{I}_n - \rho\mathbf{\Omega}]\,\mathbf{k}' - \rho\mathbf{D}\mathbf{k}',$$

from which we obtain:

$$\mathbf{k}' - \mathbf{k} = \rho\,[\mathbf{I}_n - \rho\mathbf{\Omega}]^{-1}\,\mathbf{D}\mathbf{k}',$$

and thus $\mathbf{k}' - \mathbf{k} \geq \mathbf{0}$ from the fact that $[\mathbf{I}_n - \rho\mathbf{\Omega}]^{-1}$ is a non-negative matrix when the spectral index condition holds (Debreu and Herstein, 1953). ■

Proof of Proposition 4: The proof uses the comparative statics of the knowledge index with respect to $\mathbf{\Omega}$ established in the proof of Theorem 2, and the following calculations:

$$\begin{aligned}\frac{\partial\omega_{ji}}{\partial\sigma_{ii}} &= \frac{\partial}{\partial\sigma_{ii}}\left(\frac{\phi_\theta + \gamma_{ij}\sqrt{\sigma_{ii}\sigma_{jj}}}{\phi_\theta + \sigma_{ii}}\right) = \frac{1}{(\phi_\theta + \sigma_{ii})^2}\left[\frac{1}{2}\gamma_{ij}\sqrt{\frac{\sigma_{jj}}{\sigma_{ii}}}\,(\phi_\theta + \sigma_{ii}) - \left(\phi_\theta + \gamma_{ij}\sqrt{\sigma_{ii}\sigma_{jj}}\right)\right] \\ &= \frac{1}{(\phi_\theta + \sigma_{ii})^2}\left[\frac{1}{2}\phi_\theta\gamma_{ij}\sqrt{\frac{\sigma_{jj}}{\sigma_{ii}}} - \frac{1}{2}\gamma_{ij}\sqrt{\sigma_{ii}\sigma_{jj}} - \phi_\theta\right]\end{aligned}$$

Let

$$H(x) = \frac{1}{2}x\gamma_{ij}\sqrt{\frac{\sigma_{jj}}{\sigma_{ii}}} - \frac{1}{2}\gamma_{ij}\sqrt{\sigma_{ii}\sigma_{jj}} - x.$$

A sufficient condition for $\partial\omega_{ji}/\partial\sigma_{ii} \leq 0$ is that $H(x) \leq 0$ for all $x \geq 0$. Notice that $H(0) \leq 0$. So, it suffices to check that $H'(x) \leq 0$ for all $x \geq 0$, that is:

$$\gamma_{ij}\sqrt{\frac{\sigma_{jj}}{\sigma_{ii}}} \leq 2 \Leftrightarrow \sigma_{ij} \leq 2\sigma_{ii}, \text{ for all } i, j,$$

which is true under the conditions of Theorem 1. ■

Proof of Proposition 5: Let $\mathbf{\Omega}(\lambda)$ be a matrix identical to $\mathbf{\Omega}$ but for cells i, j and k, l where the new coefficients are, respectively, $(1-\lambda)\,\omega_{ij} + \lambda\omega_{kl}$ and $(1-\lambda)\,\omega_{kl} + \lambda\omega_{ij}$, with $0 \leq \lambda \leq 1$. Notice that $\mathbf{\Omega}(0) = \mathbf{\Omega}$. We have:

$$\frac{\partial}{\partial\lambda}\mathbf{M} = \rho\mathbf{M}\left(\frac{\partial}{\partial\lambda}\mathbf{\Omega}\right)\mathbf{M},$$

and thus

$$\frac{\partial}{\partial\lambda}\mathbf{k} = \rho\mathbf{M}\left(\frac{\partial}{\partial\lambda}\mathbf{\Omega}\right)\mathbf{k}.$$

Notice that $\partial\mathbf{\Omega}/\partial\lambda$ is a matrix with all cells equal to zero but for cells i,j and k,l equal, respectively, to $-(\omega_{ij}-\omega_{kl})$ and $\omega_{ij}-\omega_{kl}$. When $i=k$, we have:

$$\frac{\partial}{\partial\lambda}\mathbf{k}=\rho\left(\omega_{ij}-\omega_{il}\right)\left(k_l-k_j\right)\begin{bmatrix} m_{1i} \\ \vdots \\ m_{ni} \end{bmatrix},$$

and thus the sign of $\partial\mathbf{k}/\partial\lambda$ is that of $\left(\omega_{ij}-\omega_{il}\right)\left(k_l-k_j\right)$. The result follows from the fact that swapping is equivalent to setting $\lambda=1$. ■

Proof of Proposition 6: Recall that equilibrium payoffs are

$$U^*(r)=\phi_0\left(1-r\right)\left[\mathbf{k}^t\mathbf{F}\mathbf{k}-n\right].$$

Therefore:

$$\frac{1}{\phi_\theta}\frac{\partial}{\partial r}U^*(r,\mathbf{\Sigma})=n-\mathbf{k}^t\mathbf{F}\mathbf{k}+(1-r)\left(\frac{\partial}{\partial r}\mathbf{k}\right)^t\mathbf{F}\mathbf{k}. \tag{27}$$

Notice that $n-\mathbf{k}^t\mathbf{F}\mathbf{k}\geq 0$ while $\left(\partial\mathbf{k}/\partial r\right)^t\mathbf{F}\mathbf{k}\leq 0$, so that (27) is the sum of a positive and a negative term, with ambiguous overall sign. However, $\mathbf{\Omega}$, $\mathbf{M}$, and $\mathbf{k}$ all take values in a compact set when $\mathbf{\Sigma}\in\mathbb{R}^{n\times n}$, with $\sigma_{ij}\leq\sigma_{ii}$, for all i,j. By (19), so does $\partial\mathbf{k}/\partial r$. Also, $\lim_{(\sigma_{11},\ldots,\sigma_{nn})\uparrow+\infty}\mathbf{F}(\mathbf{\Sigma})=\mathbf{0}_{n\times n}$. Therefore,

$$\lim_{(\sigma_{11},\ldots,\sigma_{nn})\uparrow+\infty}\frac{1}{\phi_\theta}\frac{\partial}{\partial r}U^*=n.$$

■

Proof of Proposition 7: Let $ij\notin g$. We have:

$$\omega_{ij}\left(g\right)=\frac{g_j}{\alpha+g_j}\left[1+\alpha\frac{c_{ij}\left(g\right)}{g_ig_j}\right],$$

and

$$\omega_{ij}\left(g+ij\right)=\frac{g_j+1}{\alpha+g_j+1}\left[1+\alpha\frac{c_{ij}\left(g\right)+2}{\left(g_i+1\right)\left(g_j+1\right)}\right].$$

Tedious algebra then gives:

$$\omega_{ij}\left(g+ij\right)-\omega_{ij}\left(g\right)=\frac{\alpha}{g_i\left(g_i+1\right)\left(\alpha+g_j\right)\left(\alpha+g_j+1\right)}\left[g_i\left(g_i+1+2g_j+\alpha\right)-c_{ij}\left(g\right)\left(g_i+1+g_j+\alpha\right)\right].$$

Noticing that $c_{ij}\left(g\right)\leq g_i$, we can conclude that the term in brackets is non-negative, and thus:

$$\omega_{ij}\left(g+ij\right)\geq\omega_{ij}\left(g\right).$$

Let now x,y such that $i,j\notin\{x,y\}$. Then, clearly, $\omega_{xy}\left(g+ij\right)=\omega_{xy}\left(g\right)$.

Finally, let $k\neq j$. We have:

$$\omega_{ik}\left(g\right)=\frac{g_k}{\alpha+g_k}\left[1+\alpha\frac{c_{ik}\left(g\right)}{g_ig_k}\right],$$

and

$$\omega_{ik}\left(g+ij\right)=\frac{g_k}{\alpha+g_k}\left[1+\alpha\frac{c_{ik}\left(g+ij\right)}{\left(g_i+1\right)g_k}\right].$$

We distinguish two cases.

First, $g_{jk}=1$. Then, $c_{ik}\left(g+ij\right)=c_{ij}\left(g\right)+1$, and we can easily check that $\omega_{ik}\left(g+ij\right)\geq\omega_{ik}\left(g\right)$.

Second, $g_{jk}=0$. Then, $c_{ik}\left(g+ij\right)=c_{ik}\left(g\right)$, and thus $\omega_{ik}\left(g+ij\right)<\omega_{ik}\left(g\right)$, for all $\alpha>0$.

Therefore, let g' obtained from g by adding some links, that is, $g\subset g'$. Then, if any new link between two new partners i and j is such that $g_{ik}=1$ implies that $g_{jk}=1$, for all k, and for all pair of newly linked partners i,j, then $\mathbf{\Omega}\left(g'\right)\geq\mathbf{\Omega}\left(g\right)$. Otherwise, the inequality need not hold. ■

Proof of Proposition 8: At a first order in r, we have:

$$k_i^2(r,\mathbf{\Sigma}\left(g\right))=1-2r+2\rho\sum_{j=1}^{n}\omega_{ij}\left(g\right)+o\left(r\right). \tag{28}$$

In particular, if g is a regular network with degree d we have $\sum_{j=1}^{n}\omega_{ij}\left(g\right)=\varpi\left(d\right)$.

For large values of α, (28) can be rewritten as:

$$k_i^2(r,\mathbf{\Sigma}\left(g\right))=1-2r+2\rho\frac{c_i\left(g\right)}{g_i}+O\left(1/\alpha\right)+o(r).$$

Denoting by $f_i\left(d\right)$ the forecast coefficient in a regular network of degree d, we have:

$$\frac{f_i\left(g\right)}{f_i\left(d\right)}=1+O(1/\alpha).$$

Denoting by $U^*\left(d\right)$ the ex ante aggregate equilibrium payoffs in a regular network, we have:

$$\frac{U^*(r,\mathbf{\Sigma}\left(g\right))}{U^*\left(d\right)}=\frac{1-2r+2\frac{\rho}{n}\sum_{i=1}^{n}\frac{c_i(g)}{g_i}}{1-2r+2\rho\varpi\left(d\right)}+O(1/\alpha)+o(r),$$

and the result follows. ■

Proof of Remark 3: We establish the result by induction on the size n of the population.

When $n=2,3$ the only trees are the star, so the result holds trivially in these cases.

When $n=4$, the only two possible trees are the star and the line. Straight algebra gives $S\left(g_{n=4}^{star}\right)=23/2>31/3=S\left(g_{n=4}^{wheel}\right)$. So the result also holds for $n=4$. Notice that, strictly speaking, the induction argument does not require to check the validity of the result for $n=4$ given that it already holds for $n=2,3$. However, the induction argument that follows is established for $n\geq5$, and so the case $n=4$ needs to be worked out separately.

Suppose thus that $S\left(g_n^{star}\right)\in\arg\max\{S\left(g\right):g\text{ a tree on }n\text{ players}\}$. Straight algebra leads to:

$$S\left(g_n^{star}\right)=\frac{n\left(n-1\right)}{2}+\frac{2\left(n-1\right)}{n}.$$

Consider an arbitrary tree g_{n+1} on $n+1$ players. Let $i \in \{1, ..., n+1\}$ such that $g_{ij,n+1} = 1$ for some unique $j \neq i$. In words, the link ij is a loose-end of the tree g_{n+1}. Notice that g_{n+1} being a tree implies that at least two such loose-ends exist.

Given that ij is a loose-end, g_{n+1}^{-ij} (the network deduced from g_{n+1} by eliminating the link ij) is a tree on n players. Noticing that $c_k(g_{n+1}) - c_k\left(g_{n+1}^{-ij}\right) = g_{kj}$, for all $i \neq j \neq k$, we have:

$$S(g_{n+1}) = S\left(g_{n+1}^{-ij}\right) + \frac{c_i(g_{n+1})}{g_{i,n+1}} + \frac{c_j(g_{n+1})}{g_{j,n+1}} - \frac{c_j\left(g_{n+1}^{-ij}\right)}{g_{j,n+1}-1} + \sum_{j\neq k\neq i} \frac{g_{kj,n+1}}{g_{k,n+1}}.$$

Notice that $c_j\left(g_{n+1}^{-ij}\right) = c_j(g_{n+1}) - 2$ and that $c_i(g_{n+1})/g_{i,n+1} = g_{j,n+1}/2$. Also, by the induction hypothesis, $S\left(g_{n+1}^{-ij}\right) \leq S\left(g_n^{star}\right)$. Therefore, $S(g_{n+1}) \leq S\left(g_{n+1}^{star}\right)$ is equivalent to:

$$\frac{g_{j,n+1}}{2} + \frac{2}{g_{j,n+1}-1} - \frac{c_j(g_{n+1})}{g_{j,n+1}(g_{j,n+1}-1)} + \sum_{j\neq k\neq i} \frac{g_{kj,n+1}}{g_{k,n+1}} \leq S\left(g_{n+1}^{star}\right) - S\left(g_n^{star}\right) = n + \frac{2}{n(n+1)}.$$

Notice that $g_{k,n+1} \geq 2$, for all k, and thus $\sum_{j\neq k\neq i} g_{kj,n+1}/g_{k,n+1} \leq (n-1)/2$. It thus suffices to show that:

$$\frac{g_{j,n+1}}{2} + \frac{2}{g_{j,n+1}-1} + \frac{n-1}{2} \leq n + \frac{2}{n(n+1)}.$$

It is easily checked that

$$\frac{g_{j,n+1}}{2} + \frac{2}{g_{j,n+1}-1} \leq \frac{n}{2} + \frac{2}{n-1}, \text{ for all } n \geq 3.$$

We are thus left to show that

$$\frac{2}{n-1} \leq \frac{1}{2} + \frac{2}{n(n+1)},$$

which is true for all $n \geq 5$. ■

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