# Bigraded structures and the depth of Blow-up algebras

Gemma Colomé Nin
(joint work with J.Elias)

Departament d'Àlgebra i Geometria
Universitat de Barcelona

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# Notations

$(R, \mathfrak{m})$ local Cohen-Macaulay ring, $\dim R = d > 0$, $k = R/\mathfrak{m}$ infinite.
$I$ $\mathfrak{m}-$primary ideal ($I^{-i} = 0$), $J \subseteq I$ minimal reduction of $I$, $r_J(I)$ reduction number.

Blow-up algebras:

- Rees algebra of $I$: $\mathcal{R}(I) = \bigoplus_{n\geq 0} I^n t^n \subset R[t]$
- Associated graded ring to $I$: $gr_I(R) = \bigoplus_{n\geq 0} I^n/I^{n+1}$

$$0 \longrightarrow I\mathcal{R}(I) \longrightarrow \mathcal{R}(I) \longrightarrow \frac{\mathcal{R}(I)}{I\mathcal{R}(I)} = gr_I(R) \longrightarrow 0$$

In this case: $\dim \mathcal{R}(I) = d+1, \quad \dim gr_I(R) = d$

# Conjectures

Consider the following integers

$$\Delta(I,J)=\sum_{p\geq 0} l_R\left(\frac{I^{p+1}\cap J}{I^pJ}\right)=\sum_{p\geq 0}\Delta_p(I,J)$$

$$\Lambda(I,J)=\sum_{p\geq 0} l_R\left(\frac{I^{p+1}}{I^pJ}\right)=\sum_{p\geq 0}\Lambda_p(I,J)$$

Valabrega-Valla'78: $\Delta = 0 \Leftrightarrow gr_I(R)$ is CM.

$$\bigoplus_{p\geq 0}\frac{I^{p+1}\cap J}{I^pJ}$$ Valabrega-Valla module

## Conjecture (Guerrieri'94)

$$\mathrm{depth}(gr_I(R)) \geq d - \Delta(I, J)$$

Guerrieri: $\Delta(I, J) = 1$, partial cases for $\Delta(I, J) = 2$

Wang: $\Delta(I, J) = 2$

Guerrieri-Rossi: partial results for $\Delta(I, J) = 3$

Wang: partial results for $\Delta(I, J) = 4$

Wang: counterexample for $\Delta(I, J) = 5$.

### Question (Guerrieri-Huneke'93)

$\Delta_p(I, J) \leq 1$, $p \geq 0 \Rightarrow \mathrm{depth}(gr_I(R)) \geq d - 1$?

Wang'02: counterexample. If $R$ is regular?

C-Elias: $\mathrm{depth}(gr_I(R)) \geq d - 2$

Huckaba-Marley'95: $e_1(I) \leq \Lambda(I, J)$, and there is equality if and only if $\operatorname{depth}(gr_I(R)) \geq d - 1$.

We define $\delta(I, J) = \Lambda(I, J) - e_1(I) \geq 0$.

Wang'00: $\delta(I, J) \leq \Delta(I, J)$. Guerrieri's conjecture is implied by:

## Conjecture (Wang'00)

$$\operatorname{depth}(gr_I(R)) \geq d - 1 - \delta(I, J).$$

## Conjecture (Wang'00)

$$\mathrm{depth}(gr_I(R)) \geq d - 1 - \delta(I, J).$$

Huckaba-Marley'95: $\delta(I, J) = 0$

Wang'00, Polini'00: $\delta(I, J) = 1$

Rossi-Guerrieri'99: partial cases for $\delta(I, J) = 2$ with $R/I$ Gorenstein

Wang'01: counterexample for $d = 6$ and $\delta(I, J) = 5$.

In our work we decompose $\delta(I,J)$ as a finite sum

$$\delta(I,J) = \sum_{p\geq 0} \delta_p(I,J)$$

with $0 \leq \delta_p(I,J) \leq \Delta_p(I,J)$.

## Theorem

*If* $\bar{\delta} = \max\{\delta_p(I,J) \mid p \geq 0\} \leq 1$, *then*

$$\operatorname{depth} \mathcal{R}(I) \geq d - \bar{\delta} \qquad \operatorname{depth} gr_I(R) \geq d - 1 - \bar{\delta}$$

IDEA: we will define a non-standard bigraded module $\Sigma^{I,J}$ such that

$$\Delta_p(I,J) \geq \delta_p(I,J) = \Lambda_p(I,J) - e_0(\Sigma^{I,J}_{[p]})$$

and

$$\Delta(I,J) \geq \delta(I,J) = \Lambda(I,J) - e_1(I)$$

# Bigraded structures

$(R, \mathfrak{m})$ Cohen-Macaulay local ring.
$I = (b_1, \ldots, b_\mu)$ $\mathfrak{m}$-primary ideal of $R$.
$J = (a_1, \ldots, a_d)$ minimal reduction of $I$.

Associated graded ring of $\mathcal{R}(I)$ with respect to the homogeneous ideal $Jt\mathcal{R}(I) = \bigoplus_{n \geq 0} JI^{n-1}t^n$:

$$gr_{Jt}(\mathcal{R}(I)) = \bigoplus_{j \geq 0} \frac{(Jt\mathcal{R}(I))^j}{(Jt\mathcal{R}(I))^{j+1}} U^j$$

This ring has a natural bigraded structure:
The piece of degree $j = 0$ is:

$$\frac{\mathcal{R}(I)}{Jt\mathcal{R}(I)} = \bigoplus_{i\geq 0} \frac{I^i}{I^{i-1}J}t^i$$

which is a homomorphic image of the graded ring $R[V_1, \ldots, V_\mu]$ by the degree one $R$-algebra homogeneous morphism

$$\sigma : R[V_1, \ldots, V_\mu] \longrightarrow \frac{\mathcal{R}(I)}{Jt\mathcal{R}(I)} = \bigoplus_{i\geq 0} \frac{I^i}{I^{i-1}J}t^i$$

defined by $\sigma(V_i) = b_i t \in \frac{I}{J}t$.
$R[V_1, \ldots, V_\mu]$ is endowed with the standard graduation.

Consider the bigraded ring $B := R[V_1, \ldots, V_\mu; T_1, \ldots, T_d]$ with $\deg(V_i) = (1,0)$ and $\deg(T_i) = (1,1)$.
There exists an exact sequence of bigraded $B$-rings

$$0 \longrightarrow K^{I,J} \longrightarrow C^{I,J} := \frac{\mathcal{R}(I)}{Jt\mathcal{R}(I)}[T_1, \ldots, T_d] \xrightarrow{\pi} gr_{Jt}(\mathcal{R}(I)) \longrightarrow 0$$

with $\pi(T_i) = a_i tU$, $i = 1, \ldots, d$.

$K^{I,J}$ is the ideal of initial forms of $Jt\mathcal{R}(I)$.

Notice that

$$gr_{Jt}(\mathcal{R}(I))_{(i+j,j)} = \frac{I^i J^j}{I^{i-1} J^{j+1}} t^{i+j} U^j$$

$$C^{I,J}_{(i+j,j)} = \frac{I^i}{J I^{i-1}} t^i [T_1, \ldots, T_d]_j$$

and

$$gr_{Jt}(\mathcal{R}(I)) = \bigoplus_{i,j \geq 0} \frac{I^i J^j}{I^{i-1} J^{j+1}} t^{i+j} U^j$$

# Diagonals

$M$ bigraded $B-$module, $p \in \mathbb{Z}$

$$M_{[p]} = \bigoplus_{m-n=p+1} M_{(m,n)} \qquad R[T_1, \ldots, T_d] - \text{module}$$

$$M_{\geq p} = \bigoplus_{n \geq p} M_{[n]} \qquad B - \text{submodule of } M$$

In our case, $K_{[p]}$, $C^{I,J}_{[p]} = \frac{I^{p+1}}{JI^p} t^{p+1}[T_1, \ldots, T_d]$ and

$$gr_{Jt}(\mathcal{R}(I))_{[p]} = \bigoplus_{i \geq 1} \frac{J^i I^{p+1}}{J^{i+1} I^p} t^{p+1+i} U^i$$

are $\mathcal{R}(J)-$modules, and they vanish for $p \leq -2$ and $p \geq r_J(I)$.

$$gr_{Jt}(\mathcal{R}(I))_{[-1]} \cong \mathcal{R}(J)$$

We define the following bigraded $B-$modules

Bigraded Sally module:

$$\Sigma^{I,J} := \bigoplus_{p\geq 0} gr_{Jt}(\mathcal{R}(I))_{[p]}$$

$$\mathcal{M}^{I,J} := \bigoplus_{p\geq 0} C^{I,J}_{[p]} = \bigoplus_{p\geq 0} \frac{I^{p+1}}{JI^p} t^{p+1}[T_1,\ldots,T_d]$$

$$K^{I,J} := \bigoplus_{p\geq 0} K_{[p]}$$

We have the following isomorphism of $\mathcal{R}(J)-$modules

$$gr_{Jt}(\mathcal{R}(I)) \cong \mathcal{R}(J) \oplus \Sigma^{I,J}$$

Since $\Sigma^{I,J}$ and $\mathcal{M}^{I,J}$ are vanished by $J$, there is the exact sequence of bigraded $A = (R/J)[V_1, \dots, V_\mu, T_1, \dots, T_d]-$modules

$$0 \longrightarrow K^{I,J} \longrightarrow \mathcal{M}^{I,J} \longrightarrow \Sigma^{I,J} \longrightarrow 0$$

and for each $p \geq 0$ there is the exact sequence of $(R/J)[T_1, \dots, T_d]-$modules (diagonals)

$$0 \longrightarrow K^{I,J}_{[p]} \longrightarrow \mathcal{M}^{I,J}_{[p]} \longrightarrow \Sigma^{I,J}_{[p]} \longrightarrow 0$$

(in this case we can use the classical Hilbert function)

# Bigraded Hilbert function

(Roberts'00 with $C$ a field)
Let $M$ be a finitely generated bigraded module over a bigraded algebra

$$R = C[X_1, \dots, X_r, Y_1, \dots, Y_s, T_1, \dots, T_u]$$

with $\deg(X_i) = (1,0)$, $\deg(Y_i) = (0,1)$, $\deg(T_i) = (1,1)$ and $C$ an Artin ring.

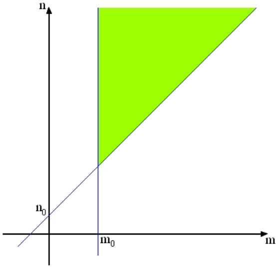
Hilbert function of $M$:

$$h_M(m,n) = \sum_{j \leq n} l_R(M_{(m,j)})$$

There exist a polynomial $p_M(m,n)$ and integers $m_0, n_0$ such that

$$h_M(m,n) = p_M(m,n)$$

for $m \geq m_0$ and $n \geq m + n_0$.




Note: If there are no generators of degree $(0,1)$, the polynomial doesn't depend on $n$

$$p_M(m,n) = p_M(m)$$

We are interested in computing the Hilbert function of $\Sigma^{I,J}$. Notice that

$$\Sigma^{I,J}_{(m,*)} = \bigoplus_{j=0}^{m-1} \frac{I^{m-j}J^j}{I^{m-1-j}J^{j+1}} t^m U^j$$

Considering the length as an $R-$module,

$$\begin{aligned} l_R(\Sigma^{I,J}_{(m,*)}) &= \sum_{j=0}^{m-1} l_R\left(\frac{I^{m-j}J^j}{I^{m-1-j}J^{j+1}}\right) \\ &= l_R\left(\frac{I^m}{IJ^{m-1}}\right) + l_R\left(\frac{IJ^{m-1}}{J^m}\right) \\ &= l_R(S_J(I)_{m-1}) + l_R\left(\frac{IJ^{m-1}}{J^m}\right) \end{aligned}$$

where $S_J(I)$ is the Sally module of $I$ with respect to $J$. ?

For $m \geq m_0, n \geq m + n_0$, for some $m_0, n_0 \geq 0$,

$$\begin{aligned} h_{\Sigma^{I,J}}(m,n) &= h_{S_J(I)}(m-1) + l_R\left(\frac{IJ^{m-1}}{J^m}\right) \\ &= h_{S_J(I)}(m-1) + l_R(I/J)\binom{m-1+d-1}{d-1} \end{aligned}$$

and hence

$$p_{\Sigma^{I,J}}(m,n) = p_{\Sigma^{I,J}}(m) = \sum_{i=0}^{d-1}(-1)^i e_{i+1}(I)\binom{m-1+d-i-1}{d-i-1}$$

We deduce:

$$e_0(\Sigma^{I,J}) = e_1(I)$$

For $\mathcal{M}^{I,J} = \bigoplus_{p\geq 0} \frac{I^{p+1}}{JI^p} t^{p+1}[T_1, \ldots, T_d]$ (CM, $d-\dim$), for $m \geq m_0 \geq r_J(I), n \geq m + n_0$ (for some integers $m_0, n_0$)

$$p_{\mathcal{M}^{I,J}}(m,n) = p_{\mathcal{M}^{I,J}}(m) = \sum_{i\geq 1} l_R\left(\frac{I^i}{I^{i-1}J}\right)\binom{m-i+d-1}{d-1}$$

and

$$e_0(\mathcal{M}^{I,J}) = \sum_{i\geq 1} l_R\left(\frac{I^i}{I^{i-1}J}\right) = \sum_{p\geq 0} l_R\left(\frac{I^{p+1}}{I^pJ}\right) = \Lambda(I,J)$$

### Proposition

*The following conditions hold:*

(i) $\deg(p_{\mathcal{M}^{I,J}}) = d - 1$ *and* $e_0(\mathcal{M}^{I,J}) = \Lambda(I, J)$.

(ii) *If* $\Sigma^{I,J} = 0$ *then* $gr_I(R)$ *is a Cohen-Macaulay ring . If* $\Sigma^{I,J} \neq 0$ *then* $\deg(p_{\Sigma^{I,J}}) = d - 1$ *and* $e_0(\Sigma^{I,J}) = e_1(I)$.

(iii) $e_0(K^{I,J}) = \delta(I, J) = \Lambda(I, J) - e_1(I)$*; if* $K^{I,J} \neq 0$ *then* $\deg(p_{K^{I,J}}) = d - 1$. *In particular,*

$$\Lambda(I, J) \geq e_1(I).$$

## Proposition

(i) $\forall p \geq 0$

$$e_0(\Sigma^{I,J}_{[p]}) = l_R\left(\frac{I^{p+1}}{JI^p}\right) - e_0(K^{I,J}_{[p]}) \geq 0,$$

$$e_1(I) = \sum_{p\geq 0} e_0(\Sigma^{I,J}_{[p]}) = \sum_{p\geq 0}\left(l_R\left(\frac{I^{p+1}}{JI^p}\right) - e_0(K^{I,J}_{[p]})\right).$$

(ii) $\forall p \geq 0$

$$l_R\left(\frac{I^{p+1}\cap J}{JI^p}\right) \geq e_0(K^{I,J}_{[p]}),$$

$$\delta(I,J) = e_0(K^{I,J}) = \sum_{p\geq 0} e_0(K^{I,J}_{[p]}) \geq 0.$$

$\delta_p(I,J) = e_0(K^{I,J}_{[p]})$
$\Lambda_p(I,J) = l_R(I^{p+1}/JI^p)$
$\Delta_p(I,J) = l_R(I^{p+1} \cap J/JI^p)$

For all $p \geq 0$

$$\Delta_p(I,J) \geq \delta_p(I,J) = \Lambda_p(I,J) - e_0(\Sigma^{I,J}_{[p]}) = e_0(K^{I,J}_{[p]}) \geq 0$$

and adding with respect to $p$, we recover the known

$$\Delta(I,J) \geq \delta(I,J) = \Lambda(I,J) - e_1(I) \geq 0$$

# Depth of Blow-up algebras

## Conjecture (Wang)

$$\operatorname{depth} gr_I(R) \geq d - 1 - \delta(I, J)$$

True for $\delta(I, J) = 0, 1$.

We have decomposed, as finite sum,

$$\delta(I, J) = \sum_{p \geq 0} \delta_p(I, J) = \sum_{p \geq 0} e_0(K_{[p]}^{I,J})$$

We consider the hypotheses over $\delta_p(I, J)$ instead of over $\delta(I, J)$ to refine this conjecture.

Consider $\overline{\delta} = \max\{\delta_p(I, J) \mid p \geq 0\}$

**Theorem**

*If* $\overline{\delta} \leq 1$, *then*

$$\operatorname{depth} \mathcal{R}(I) \geq d - \overline{\delta}$$

$$\operatorname{depth} gr_I(R) \geq d - 1 - \overline{\delta}$$

# Sketch of the Proof

We use several times the Depth Formulas proved by Huckaba and Marley:

**Theorem (Huckaba-Marley'94)**

$(R, \mathfrak{m})$ *CM local ring,* $\dim R = d > 0$, $I$ $\mathfrak{m}-$*primary ideal. Then*

$$\operatorname{depth} \mathcal{R}(I) \geq \operatorname{depth} gr_I(R)$$

*If* $gr_I(R)$ *is not CM,*

$$\operatorname{depth} \mathcal{R}(I) = \operatorname{depth} gr_I(R) + 1$$

Case $\overline{\delta} = 0$:
Then $K^{I,J} = 0$, and hence $\mathcal{M}^{I,J} \cong \Sigma^{I,J}$ as $A-$modules. Since $\mathcal{M}^{I,J}$ is $d-$dimensional CM, then $\operatorname{depth} \Sigma^{I,J} = d$, and by depth counting on

$$0 \longrightarrow \Sigma^{I,J} \longrightarrow gr_{Jt}(\mathcal{R}(I)) \longrightarrow \mathcal{R}(J) \longrightarrow 0$$

we get that $\operatorname{depth} gr_{Jt}(\mathcal{R}(I)) \geq d$.

Then $\operatorname{depth} \mathcal{R}(I) \geq d = d - \overline{\delta}$.

- If $gr_I(R)$ is CM, $\operatorname{depth} gr_I(R) = d \geq d - \overline{\delta} - 1$.
- If $gr_I(R)$ is not CM, by [HM94], $\operatorname{depth} gr_I(R) = \operatorname{depth} \mathcal{R}(I) - 1 \geq d - \overline{\delta} - 1$.

Case $\overline{\delta} = 1$:
Cases $d = 1, 2$ are true by [HM94].
Case $d \geq 3$: Since $K_{[p]}^{I,J} \subset \frac{I^{p+1}}{JI^p}[T_1, \ldots, T_d] = \mathcal{M}_{[p]}^{I,J}$, if $\delta_p = 1$, then $K_{[p]}^{I,J}$ is a rank one torsion-free $k[T_1, \ldots, T_d]-$module.

Now, by Theorem,

$$\operatorname{depth} gr_{Jt}(\mathcal{R}(I)) \geq d - 1$$

and so,

$$\operatorname{depth} \mathcal{R}(I) \geq d - 1 = d - \overline{\delta}$$

Now, by [HM94],

- If $gr_I(R)$ is CM, then $\operatorname{depth} gr_I(R) = d > d - 1 - \overline{\delta}$
- If $gr_I(R)$ is not CM, then $\operatorname{depth} gr_I(R) = \operatorname{depth} \mathcal{R}(I) - 1 \geq d - 1 - \overline{\delta}$

□

## Theorem

*If $d \geq 3$, $K^{I,J} \neq 0$ and for all $p$, $K^{I,J}_{[p]} = 0$ or $K^{I,J}_{[p]}$ is a rank one torsion-free $k[T_1, \ldots, T_d]-$module, then*

$$\operatorname{depth} gr_{Jt}(\mathcal{R}(I)) \geq d - 1$$

# Sketch of the Proof

Let $p_1 < \cdots < p_n$ be the integers such that $K^{I,J}_{[p_i]} \neq 0$.

For all $p$, there is the exact sequence,

$$0 \longrightarrow K^{I,J}_{[p]} \longrightarrow \mathcal{M}^{I,J}_{[p]} \longrightarrow \Sigma^{I,J}_{[p]} \longrightarrow 0 \qquad (*)$$

If $p \neq p_1, \ldots, p_n$, then $\Sigma^{I,J}_{[p]} \cong \mathcal{M}^{I,J}_{[p]} = \frac{I^{p+1}}{JI^p}[T_1, \ldots, T_d]$ $(**)$.

If $p = p_1, \ldots, p_n$, $(\delta_{p_i} = 1)$ $K^{I,J}_{[p_i]}$ is an ideal of $D = k[T_1, \ldots, T_d]$.

Since $\operatorname{depth} S_J(I) \geq 1$ (Polini'00), and by depth counting on $(*), (**)$ and in the exact sequences (Vaz-Pinto)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma^{I,J}_{[1]} & \longrightarrow & C_1 = S_J(I) & \longrightarrow & C_2 & \longrightarrow & 0 \\
0 & \longrightarrow & \Sigma^{I,J}_{[2]} & \longrightarrow & C_2 & \longrightarrow & C_3 & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & \Sigma^{I,J}_{[r-2]} & \longrightarrow & C_{r-2} & \longrightarrow & \Sigma^{I,J}_{[r-1]} & \longrightarrow & 0
\end{array}$$

with $C_p = \bigoplus_{n \geq 0} \frac{I^{n+p}}{J^n I^p}$, and since $D$ is factorial, we can prove that in fact, $K^{I,J}_{[p_i]}$ is a principal ideal, and so $\operatorname{depth} K^{I,J}_{[p_i]} = d$. By depth counting again in $(*)$, $\forall p \geq 0$,

$$\operatorname{depth} \Sigma^{I,J}_{[p]} \geq d - 1.$$

Now, by depth counting on

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma^{I,J}_{[0]} & \longrightarrow & \Sigma^{I,J}_{\geq 0} = \Sigma^{I,J} & \longrightarrow & \Sigma^{I,J}_{\geq 1} & \longrightarrow & 0 \\
0 & \longrightarrow & \Sigma^{I,J}_{[1]} & \longrightarrow & \Sigma^{I,J}_{\geq 1} & \longrightarrow & \Sigma^{I,J}_{\geq 2} & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & \Sigma^{I,J}_{[r-2]} & \longrightarrow & \Sigma^{I,J}_{\geq r-2} & \longrightarrow & \Sigma^{I,J}_{[r-1]} & \longrightarrow & 0
\end{array}
$$

we get that

$$\operatorname{depth} \Sigma^{I,J} \geq d-1.$$

Finally, by depth counting on

$$0 \longrightarrow \Sigma^{I,J} \longrightarrow gr_{Jt}(\mathcal{R}(I)) \longrightarrow \mathcal{R}(J) \longrightarrow 0$$

since $\operatorname{depth} \mathcal{R}(J) = d+1$ and $\operatorname{depth} \Sigma^{I,J} \geq d-1$ then

$$\operatorname{depth} gr_{Jt}(\mathcal{R}(I)) \geq d-1$$

□

## Question (Guerrieri-Huneke'93)

$\Delta_p(I,J) \leq 1,\, p \geq 0 \Rightarrow \text{depth}(gr_I(R)) \geq d-1?$

(Wang: Negative answer)

As a corollary, we obtain:

## Proposition

$$\Delta_p(I,J) \leq 1, \forall p \geq 0 \Rightarrow \text{depth}\, gr_I(R) \geq d-2$$

Proof: Since $0 \leq \delta_p \leq \Delta_p \leq 1$, then $\overline{\delta} \leq 1$ and hence

$$\text{depth}\, gr_I(R) \geq d-1-\overline{\delta} \geq d-2$$

$\square$

# BIGR ÀCIES / AZIE !

# Sally module

(Vasconcelos'94, Vaz Pinto'95)

The Sally module $S_J(I)$ of $I$ with respect to $J$, is defined by the exact sequence of $\mathcal{R}(J)-$modules:

$$0 \rightarrow I\mathcal{R}(J) \hookrightarrow I\mathcal{R}(I) \rightarrow S_J(I) = \bigoplus_{n \geq 0} \frac{I^{n+1}}{J^n I} \rightarrow 0$$

Hilbert function of the Sally module:

$$h_{S_J(I)}(n) = l_R(I^{n+1}/J^n I)$$

If $S_J(I) \neq 0$, then $\dim\ S_J(I) = d$, and the Hilbert polynomial is

$$p_{S_J(I)}(n) = \sum_{i=0}^{d-1} (-1)^i s_i \binom{n+d-i-1}{d-i-1}$$

If $R$ is CM:
$e_0(I) = l_R(R/J)$
$e_1(I) = s_0 + l_R(I/J)$
$e_i(I) = s_{i-1}$, for $i = 2, \ldots, d$.