Multigraded Structures and the Depth of Blow-up Algebras

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GOALS OF THE THESIS

- Understanding cohomological properties of non-standard multigraded modules
- Applying results obtained to multigraded blow-up algebras
- Estimating the depth of blow-up algebras by means of bigraded structures

Outline

- ▷ Multigraded structures
- ▷ Asymptotic depth of multigraded modules
- ▷ Veronese multigraded modules
- ▷ Bigraded structures and the depth of blow-up algebras

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Notation:

$\underline{n} = (n_1, \ldots, n_r) \in \mathbb{Z}^r$ and $|\underline{n}| = |n_1| + \cdots + |n_r|$.

$\underline{n} + \underline{m} = (n_1 + m_1, \ldots, n_r + m_r)$ and $\underline{n} \cdot \underline{m} = (n_1 \cdot m_1, \ldots, n_r \cdot m_r)$.

$\underline{m} \geq \underline{n}$ if and only if $m_i \geq n_i$ for all $i = 1, \ldots, r$.

A $\mathbb{Z}^r$-graded ring S is a ring endowed with a direct sum decomposition $S = \bigoplus_{\underline{n} \in \mathbb{Z}^r} S_{\underline{n}}$, such that $S_{\underline{m}} S_{\underline{n}} \subset S_{\underline{m}+\underline{n}}$ for any $\underline{m}, \underline{n} \in \mathbb{Z}^r$.

A $\mathbb{Z}^r$-graded S-module M is an S-module with a direct sum decomposition $M = \bigoplus_{\underline{n} \in \mathbb{Z}^r} M_{\underline{n}}$ such that $S_{\underline{m}} M_{\underline{n}} \subset M_{\underline{m}+\underline{n}}$ for any $\underline{m}, \underline{n} \in \mathbb{Z}^r$.

Case of study

Let S be a Noetherian $\mathbb{Z}^r$-graded ring generated over a local ring $(S_{\underline{0}}, \mathfrak{m})$ by elements

$$\{g_i^1, \dots, g_i^{\mu_i}\}_{i=1,\dots,r},$$

with g_i^j of degree $\gamma_i = (\gamma_1^i, \dots, \gamma_i^i, 0, \dots, 0) \in \mathbb{N}^r$ with $\gamma_i^i \neq 0$ and $i = 1, \dots, r$, $j = 1, \dots, \mu_i$.

We denote $\mathcal{M} = \mathfrak{m} \oplus \bigoplus_{\underline{n} \neq \underline{0}} S_{\underline{n}}$ the homogeneous maximal ideal of S, and $S_+ = \bigoplus_{\underline{n} \neq \underline{0}} S_{\underline{n}}$.

For $i = 1, \ldots, r$, let I_i be the ideal of S generated by the homogeneous components of S of multidegree $(b_1, \ldots, b_i, 0, \ldots, 0)$ with $b_i \neq 0$. We define the irrelevant ideal of S as $S_{++} = I_1 \cdots I_r$.

Let $\mathrm{Proj}^r(S)$ be the set of all relevant homogeneous prime ideals on S, i.e. the set of all homogeneous prime ideals $\mathfrak{p}$ in S such that $\mathfrak{p} \not\supset S_{++}$.

Given a finitely generated $\mathbb{Z}^r$-graded S-module M, we define the homogeneous support of M as

$$Supp_{++}(M) = \{\mathfrak{p} \in \mathrm{Proj}^r(S) \mid M_{\mathfrak{p}} \neq 0\}.$$

Following the definition in Verma-Katz-Mandal'94 for the standard bigraded case, we define:
Relevant dimension of S:

$$\operatorname{rel.dim}(S) = \begin{cases} r-1 & \text{if } \operatorname{Proj}^r(S) = \emptyset \\ \max\{\dim(S/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Proj}^r(S)\} & \text{if } \operatorname{Proj}^r(S) \neq \emptyset. \end{cases}$$

It holds that $\dim(\operatorname{Proj}^r(S)) = \operatorname{rel.dim}(S) - r$.

Relevant dimension of a module M:

$$\operatorname{rel.dim}(M) = \begin{cases} r-1 & \text{if } Supp_{++}(M) = \emptyset \\ \max\{\dim(S/\mathfrak{p}) \mid \mathfrak{p} \in Supp_{++}(M)\} & \text{if } Supp_{++}(M) \neq \emptyset. \end{cases}$$

It holds that $\dim(Supp_{++}(M)) = \operatorname{rel.dim}(M) - r$.

Let $(R, \mathfrak{m})$ be a Noetherian local ring and $I_1, \ldots, I_r$ be ideals of R.

The multigraded Rees algebra associated to $I_1, \ldots, I_r$ is defined by

$$\mathcal{R}(I_1, \ldots, I_r) = \bigoplus_{\underline{n} \in \mathbb{N}^r} I_1^{n_1} t_1^{n_1} \cdots I_r^{n_r} t_r^{n_r} \subset R[t_1, \ldots, t_r],$$

and for $k = 1, \ldots, r$, the k-th associated multigraded ring of $I_1, \ldots, I_r$ in R is

$$gr_{I_1,\ldots,I_r;I_k}(R) = \bigoplus_{\underline{n} \in \mathbb{N}^r} \frac{I_1^{n_1} \cdots I_k^{n_k} \cdots I_r^{n_r}}{I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}} = \frac{\mathcal{R}(I_1, \ldots, I_r)}{I_k \mathcal{R}(I_1, \ldots, I_r)}.$$

For $k = 1, \ldots, r$, we define the k-th extended multigraded Rees algebra by

$$\mathcal{R}_k^*(I_1, \ldots, I_r) = \bigoplus_{\substack{n_k \in \mathbb{Z} \\ (n_1 \ldots, \widehat{n_k}, \ldots, n_r) \in \mathbb{N}^{r-1}}} I_1^{n_1} t_1^{n_1} \cdots I_r^{n_r} t_r^{n_r} \subset R[t_1, \ldots, t_r, t_k^{-1}].$$

The Hilbert function of M is defined by

$$\begin{array}{cccc} h_M : & \mathbb{Z}^r & \longrightarrow & \mathbb{Z} \\ & \underline{n} & \mapsto & \text{length}_{S_{\underline{0}}}(M_{\underline{n}}). \end{array}$$

- Non-standard graded case: Bruns-Herzog'93, Dichi-Sangaré'99,...
- Standard multigraded case: Herrmann-Hyry-Ribbe-Tang'97, Verma-Katz-Mandal'94, Roberts'98,...
- Non-standard multigraded case: Lavila'99, Roberts'98, Hoang-Trung'03, Fields'00,...

Given $\underline{\beta} \in \mathbb{N}^r$ and $\gamma_1, \ldots, \gamma_r \in \mathbb{N}^r$ linearly independent vectors, we define the cone with vertex $\underline{\beta}$ with respect to $\gamma_1, \ldots, \gamma_r$ as

$$C_{\underline{\beta}} := \left\{ \underline{\alpha} \in \mathbb{N}^r \mid \underline{\alpha} = \underline{\beta} + \sum_{i=1}^{r} \lambda_i \gamma_i, \quad \lambda_i \in \mathbb{R}_{\geq 0} \right\}.$$

Given a cone $C_{\underline{\beta}}$ with vertex $\underline{\beta} \in \mathbb{N}^r$ with respect to $\gamma_1, \ldots, \gamma_r$, we define the basic cell $\Pi_{\underline{\beta}}$ as

$$\Pi_{\underline{\beta}} := \left\{ \underline{\alpha} \in \mathbb{N}^r \mid \underline{\alpha} = \underline{\beta} + \sum_{i=1}^{r} m_i \gamma_i, \quad 0 \leq m_i < 1 \right\}.$$

For any element $\underline{\alpha} \in C_{\underline{\beta}} \subset \mathbb{N}^r$, there is a unique representative of $\underline{\alpha}$ modulo $\gamma_1, \ldots, \gamma_r$ in $\Pi_{\underline{\beta}}$.

A function $f : \mathbb{N}^r \to \mathbb{Z}$ is a quasi-polynomial function of polynomial degree d on $\underline{\beta}, \gamma_1, \ldots, \gamma_r$ if there exist periodic functions, for $\underline{\alpha} \in \mathbb{N}^r$ and $|\underline{\alpha}| \leq d$,

$$c_{\underline{\alpha}} : \mathbb{N}^r \to \mathbb{Z}$$

with respect to $\gamma_1, \ldots, \gamma_r$ such that for $\underline{n} \in C_{\underline{\beta}}$

$$f(\underline{n}) = \sum_{|\underline{\alpha}| \leq d} c_{\underline{\alpha}}(\underline{n}) \underline{n}^{\underline{\alpha}}$$

and $f(\underline{n}) = 0$ when $\underline{n} \notin C_{\underline{\beta}}$, and there is some $\underline{\alpha} \in \mathbb{N}^r$ with $|\underline{\alpha}| = d$ such that $c_{\underline{\alpha}} \neq 0$. We call an expression $\sum_{|\underline{\alpha}| \leq d} c_{\underline{\alpha}}(\underline{n}) \underline{n}^{\underline{\alpha}}$ a quasi-polynomial.

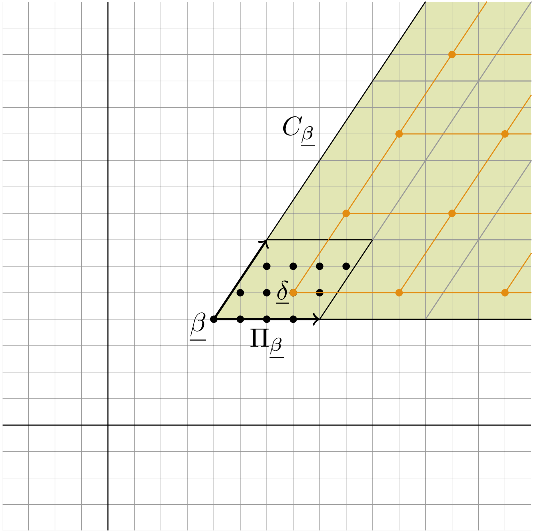
This definition of a quasi-polynomial $P(\underline{n}) = \sum_{|\underline{\alpha}| \leq d} c_{\underline{\alpha}}(\underline{n}) \underline{n}^{\underline{\alpha}}$ is equivalent to giving a collection of polynomials of total degree $\leq d$

$$f_{\underline{\delta}}(\underline{n}) = \sum_{\underline{\alpha} \in \mathbb{N}^r} c_{\underline{\alpha}}(\underline{\delta}) \underline{n}^{\underline{\alpha}} \in \mathbb{Z}[\underline{n}]$$

for each $\underline{\delta} \in \Pi_{\underline{\beta}}$, $\underline{n} = \underline{\delta} + \langle \gamma_1, \ldots, \gamma_r \rangle_{\mathbb{N}}$.

Multigraded structures

Quasi-polynomial functions



Proposition

Let S be an $\mathbb{N}^r$-graded ring, where $S_{\underline{0}}$ is an Artinian local ring, and S is generated over $S_{\underline{0}}$ by elements $g_1^1, \ldots, g_1^{\mu_1}, \ldots, g_r^1, \ldots, g_r^{\mu_r}$ with g_i^j of multidegree $\gamma_i = (\gamma_1^i, \ldots, \gamma_i^i, 0, \ldots, 0) \in \mathbb{N}^r$ with $\gamma_i^i \neq 0$, for all $i = 1, \ldots, r$ and $j = 1, \ldots, \mu_i$. Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. Then there exist a quasi-polynomial P_M of polynomial degree $\mathrm{rel.\,dim}(M) - r$ and a cone $C_{\underline{\beta}} \subset \mathbb{N}^r$, such that

$$h_M(\underline{n}) = P_M(\underline{n})$$

for any $\underline{n} \in C_{\underline{\beta}}$.

Proposition (Grothendieck-Serre formula)

Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. Then for all $\underline{n} \in \mathbb{Z}^r$,

$$h_M(\underline{n}) - P_M(\underline{n}) = \sum_{i \geq 0} (-1)^i \operatorname{length}_{S_{\underline{0}}}(H^i_{S_{++}}(M)_{\underline{n}}).$$

Let $I_1, \ldots, I_r$ be $\mathfrak{m}$-primary ideals of the Noetherian local ring $(R, \mathfrak{m})$. For $k = 1, \ldots, r$, we denote

$$f_k(\underline{n}) = \operatorname{length}_R\left(\frac{R}{I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}}\right).$$

There exists a $\underline{\beta}_k \in \mathbb{N}^r$, such that for $\underline{n} \geq \underline{\beta}_k$, $f_k(\underline{n}) = p_k(\underline{n}) \in \mathbb{Z}[n_1, \ldots, n_r]$.

For an element $\underline{\delta} \in \mathbb{N}^r$, we define $\mathcal{H}^k_{\underline{\delta}}$ as the set of elements $\underline{n} \in \mathbb{Z}^r$ such that $(n_1, \ldots, n_{k-1}, n_{k+1}, \ldots, n_r) \geq (\delta_1, \ldots, \delta_{k-1}, \delta_{k+1}, \ldots, \delta_r)$ and $n_k \in \mathbb{Z}$.

Theorem

There exists an element $\underline{\delta} \in \mathbb{N}^r$ such that for all $\underline{n} \in \mathcal{H}^k_{\underline{\delta}}$

$$p_k(\underline{n}) - f_k(\underline{n}) = \sum_{i \geq 0} (-1)^i \operatorname{length}_R(H^i_{\mathcal{R}_{++}}(\mathcal{R}^*_k)_{\underline{n}+e_k}).$$

▷ Multigraded structures

▷ Asymptotic depth of multigraded modules

▷ Veronese multigraded modules

▷ Bigraded structures and the depth of blow-up algebras

- Burch'72: $(R, \mathfrak{m})$ Noetherian local ring, I ideal,

$$l(I) \leq \dim(R) - \min_{n \geq 1}\{\mathrm{depth}(R/I^n)\}$$

with $l(I) = \dim(\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I))$.

- Brodmann'79: M f.g. R-module. Then, $Ass(M/I^nM)$ is stable for $n \gg 0$ and hence $\mathrm{depth}(M/I^nM)$ is constant for $n \gg 0$. Moreover,

$$l(I, M) \leq \dim(M) - \lim_{n \to +\infty} \mathrm{depth}(M/I^nM)$$

with $l(I, M) = \dim(\bigoplus_{n \geq 0} I^nM/\mathfrak{m}I^nM)$.

- Herzog-Hibi'05: E graded module over a standard graded algebra. They prove that $\mathrm{depth}(E_n)$ is constant for $n \gg 0$ via the Hilbert polynomial of Koszul homology modules, instead of the associated primes. Moreover,

$$\dim(E/\mathfrak{m}E) \leq \dim(E) - \lim_{n \to +\infty} \mathrm{depth}(E_n).$$

Asymptotic depth of multigraded modules

Motivation

- Branco Correia-Zarzuela'06: $E \subsetneq G \cong R^e$, R-modules, $e > 0$, $\mathcal{R}_G(E) = \bigoplus_{n\geq 0} E_n$ and $\mathcal{R}_G(G) = \bigoplus_{n\geq 0} G_n$. Then $\operatorname{depth}(G_n/E_n)$ is constant for $n \gg 0$, using associated primes. Moreover,

$$l_G(E) \leq \dim(R) + e - 1 - \min_{n\geq 1}\{\operatorname{depth}(G_n/E_n)\},$$

  where $l_G(E) = \dim(\mathcal{R}_G(E)/\mathfrak{m}\mathcal{R}_G(E))$.
- Hayasaka'06: $A \subset B$ standard multigraded rings with $A_{\underline{0}} = B_{\underline{0}} = R$ a local ring. M f.g. multigraded B-module, $N \subset M$ f.g. multigraded A-submodule. Then $Ass(M_{\underline{n}}/N_{\underline{n}})$ is stable for $\underline{n} \gg \underline{0}$ and $\operatorname{depth}(M_{\underline{n}}/N_{\underline{n}})$ is asymptotically constant. Moreover,

$$s(A) \leq s(B) + \dim(R) - \operatorname{depth}(A, B),$$

  where $\operatorname{depth}(A, B)$ is the asymptotic depth of $B_{\underline{n}}/A_{\underline{n}}$ and $s(G) = \dim \operatorname{Proj}^r(G/\mathfrak{m}G) + 1$.

What happens in the non-standard multigraded case?

Asymptotic depth of multigraded modules

Theorem

Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. There exist an element $\underline{\beta} \in \mathbb{N}^r$ and an integer $\rho \in \mathbb{N}$ such that

$$\operatorname{depth}(M_{\underline{n}}) \geq \rho,$$

for all $\underline{n} \in C_{\underline{\beta}}$ with $M_{\underline{n}} \neq 0$, and

$$\operatorname{depth}(M_{\underline{n}}) = \rho,$$

for some $\underline{\delta} \in \Pi_{\underline{\beta}}$ and for all $\underline{n} \in \{\underline{\delta} + \sum_{i=1}^{r} \lambda_i \gamma_i \mid \lambda_i \in \mathbb{N}\} \subset C_{\underline{\beta}}$.

KEY POINT: the quasi-polynomial behavior of the Hilbert function of the Koszul homology modules of M with respect to a system of generators $x_1, \ldots, x_n$ of the maximal ideal $\mathfrak{m}$ of $S_{\underline{0}}$.

Asymptotic depth of multigraded modules

When the quasi-polynomial is, in fact, a polynomial, we can assure constant depth in all the cone:

- **Proposition**: If S is an algebra generated over $S_{\underline{0}}$ by elements of degrees $(1, 0, \ldots, 0)$, $(*, 1, 0, \ldots, 0)$, $\ldots$, $(*, *, *, \ldots, 1) \in \mathbb{N}^r$, then $\mathrm{depth}(M_{\underline{n}}) = \rho$ for $\underline{n} \in C_{\underline{\beta}}$.

- **Corollary**: If S is a standard algebra, then $\mathrm{depth}(M_{\underline{n}}) = \rho$ for $\underline{n} \geq \underline{\beta}$.

For ideals $I_1, \ldots, I_r$ of a Noetherian local ring $(R, \mathfrak{m})$, $\mathcal{R}(I_1, \ldots, I_r)$ and $gr_{I_1,\ldots,I_r;I_k}(R)$, for $k = 1, \ldots, r$, are finitely generated standard $\mathbb{Z}^r$-graded $\mathcal{R}(I_1, \ldots, I_r)$-modules, and each homogeneous component is a finitely generated R-module.

Proposition

There exist an element $\underline{\beta} \in \mathbb{N}^r$ and an integer $\delta \in \mathbb{N}$ such that for all $\underline{n} \geq \underline{\beta}$

$$\operatorname{depth}(I_1^{n_1} \cdots I_r^{n_r}) = \delta + 1$$

and

$$\operatorname{depth}\left(\frac{I_1^{n_1} \cdots I_r^{n_r}}{I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}}\right) = \delta$$

for all $k = 1, \ldots, r$.

We are interested in the depth of $R/I_1^{n_1}\cdots I_r^{n_r}$ for $\underline{n}$ large enough. In this case, we can take advantage of the constant asymptotic depth of these last two modules and the relation with $R/I_1^{n_1}\cdots I_r^{n_r}$ by means of some short exact sequences of R-modules where we can use the depth counting techniques.

Theorem

There exist an element $\underline{\varepsilon} \in \mathbb{N}^r$ and an integer $\rho \in \mathbb{N}$ such that

$$\operatorname{depth}\left(\frac{R}{I_1^{n_1}\cdots I_r^{n_r}}\right) = \rho \leq \delta$$

for all $\underline{n} \geq \underline{\varepsilon}$. Moreover, if there exists an $\underline{n} \geq \underline{\beta}$ such that $\operatorname{depth}\left(\frac{R}{I_1^{n_1}\cdots I_r^{n_r}}\right) \geq \delta$, then $\rho = \delta$.

We bound the asymptotic depth of the modules $R/I_1^{n_1}\cdots I_r^{n_r}$.

Proposition

Let $\rho \in \mathbb{N}$ be the asymptotic depth of $R/I_1^{n_1}\cdots I_r^{n_r}$. Then,

$$\rho \leq \dim(R) - \dim \operatorname{Proj}^r\left(\frac{\mathcal{R}(I_1,\ldots,I_r)}{\mathfrak{m}\mathcal{R}(I_1,\ldots,I_r)}\right).$$

Veronese multigraded modules

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Veronese multigraded modules

Motivation

- Multigraded blow-up algebras defined by powers of ideals

$$\mathcal{R}(I_1,\ldots,I_r)^{(\underline{a})} = \mathcal{R}(I_1^{a_1},\ldots,I_r^{a_r})$$

  $\underline{a} = (a_1,\ldots,a_r)$.
- Herrmann-Hyry-Ribbe'93: Cohen-Macaulay and Gorenstein properties of multigraded Rees algebras of powers of ideals.
- Elias'04: If R is quotient of a regular local ring,

$$\operatorname{depth}(\mathcal{R}(I^n)) = constant \qquad \operatorname{depth}(gr_{I^n}(R)) = constant$$

  for $n \gg 0$.

What happens to non-standard multigraded modules?

The Veronese transform of S with respect to $\underline{a} \in \mathbb{N}^{*r}$, or $(\underline{a})$-Veronese, is the subring of S

$$S^{(\underline{a})} = \bigoplus_{\underline{n} \in \mathbb{N}^r} S_{\phi_{\underline{a}}(\underline{n})},$$

where $\phi_{\underline{a}}(\underline{n}) = \sum_{i=1}^{r} (n_i a_i)\gamma_i$.

Given a $\mathbb{Z}^r$-graded S-module M, the Veronese transform of M with respect to $\underline{a} \in \mathbb{N}^{*r}$, $\underline{b} \in \mathbb{N}^r$, or $(\underline{a}, \underline{b})$-Veronese, is the $S^{(\underline{a})}$-module

$$M^{(\underline{a},\underline{b})} = \bigoplus_{\underline{n} \in \mathbb{Z}^r} M_{\phi_{\underline{a}}(\underline{n})+\underline{b}}.$$

Proposition

Let M be a f.g. $\mathbb{Z}^r$-graded S-module. Then, for all $i \geq 0$ and $\underline{a} \in \mathbb{N}^{*r}$, $\underline{b} \in \mathbb{N}^r$,

$$(H^i_{\mathcal{M}}(M))^{(\underline{a},\underline{b})} \cong H^i_{\mathcal{M}^{(\underline{a})}}(M^{(\underline{a},\underline{b})}).$$

Veronese asymptotic depth of M:

$vad(M^{(*)}) = \max\{\text{depth}(M^{(\underline{a})}) \mid \underline{a} \in \mathbb{N}^{*r}\}$

$vad(M^{(*,*)}) = \max\{\text{depth}(M^{(\underline{a},\underline{b})}) \mid \underline{a}, \underline{b} \in \mathbb{N}^{*r}\}$

Proposition

Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. Let $s = vad(M^{(*)})$. There exists $\underline{a} = (a_1, \dots, a_r) \in \mathbb{N}^{*r}$ such that for all $\underline{b} \in \{(\lambda_1 a_1, \dots, \lambda_r a_r) \mid \lambda_i \in \mathbb{N}^*\}$

$$\text{depth}(M^{(\underline{b})}) = s.$$

In order to extend the result on the asymptotic depth of the Veronese modules to regions of $\mathbb{N}^r$, we have to study the vanishing of the local cohomology modules of a multigraded module M.

For a finitely generated $\mathbb{Z}^r$-graded S-module M, we define the generalized depth of M with respect to the homogeneous maximal ideal $\mathcal{M}$ of S as

$$\operatorname{gdepth}(M) = \max\{k \in \mathbb{N} \mid S_{++} \subset rad(Ann_S(H^i_{\mathcal{M}}(M))) \text{ for all } i < k\}.$$

We can prove the invariance of the generalized depth under Veronese transfoms:

Theorem

Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. If $S_{\underline{0}}$ is the quotient of a regular local ring, for all $\underline{a} \in \mathbb{N}^{*r}, \underline{b} \in \mathbb{N}^r$, it holds

$$\operatorname{gdepth}(M^{(\underline{a},\underline{b})}) = \operatorname{gdepth}(M).$$

We say that a $\mathbb{Z}^r$-graded S-module M is Γ-finitely graded if there exists a cone $C_{\underline{\beta}} \subset \mathbb{N}^r$ where $M_{\underline{n}} = 0$ for all $\underline{n} \in \mathbb{Z}^r$ with $\underline{n}^* = (|n_1|, \ldots, |n_r|) \in C_{\underline{\beta}}$. We define

$$\Gamma\text{-fg}(M) = \max\{k \geq 0 \mid H^i_{\mathcal{M}}(M) \text{ is } \Gamma\text{-finitely graded for all } i < k\}.$$

To assure that $H^k_{\mathcal{M}}(M)$ is Γ-finitely graded for all $k \geq 0$, if M is Γ-finitely graded, we need to restrict the graduation to the almost-standard case.

Almost-standard graduation: degrees $\gamma_1, \ldots, \gamma_r$ with $\gamma_i = (0, \ldots, 0, \gamma_i^i, 0, \ldots, 0)$ and $\gamma_i^i > 0$ for all $i = 1, \ldots, r$.

Theorem

Let S be an almost-standard multigraded ring. Let M be a finitely generated $\mathbb{Z}^r$-graded S-module, then

$$\Gamma\text{-fg}(M) = \text{gdepth}(M).$$

If $S_{\underline{0}}$ is the quotient of a regular local ring, $\Gamma\text{-fg}$ is invariant under Veronese.

Now, we have new tools to prove the theorem that assures constant depth for the $(\underline{a}, \underline{b})$-Veronese in a region of $\mathbb{N}^r \times \mathbb{N}^r$, instead of a net. However the restriction to the almost-standard case is still necessary.

Theorem

Let S be an almost-standard multigraded ring such that $S_{\underline{0}}$ is the quotient of a regular ring. Let M be a finitely generated $\mathbb{Z}^r$-graded S-module and let $s = vad(M^{(*,*)})$. Then, there exists $\underline{\beta} \in \mathbb{N}^r$ such that for all $\underline{b} \geq \underline{\beta}$ and for all $\underline{a} \in \mathbb{N}^r$ such that $a_i \geq (\beta_i + b_i)/\gamma_i^i$,

$$\operatorname{depth}(M^{(\underline{a},\underline{b})}) = s.$$

For $\mathbb{Z}$-graded modules, we obtain:

Proposition

Let S be a $\mathbb{Z}$-graded ring such that $S_{\underline{0}}$ is the quotient of a regular ring. Let M be a finitely generated graded S-module. Then $\operatorname{depth}(M^{(a)})$ is constant for $a \gg 0$.

Bigraded structures and the depth of blow-up algebras

- Multigraded structures
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One of the classical problems in commutative algebra is to estimate the depth of the associated graded ring $gr_I(R) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ and the Rees algebra $\mathcal{R}(I) = \bigoplus_{n\geq 0} I^n t^n$ for ideals I having good properties.

For a Cohen-Macaulay local ring $(R, \mathfrak{m})$ of dimension $d > 0$ with infinite residue field and an $\mathfrak{m}$-primary ideal I of R with minimal reduction J, we consider the following integers that appear in some conjectures:

$$\Delta(I,J) = \sum_{p\geq 0} \text{length}_R \left(\frac{I^{p+1} \cap J}{I^p J} \right) = \sum_{p\geq 0} \Delta_p(I,J),$$

$$\Lambda(I,J) = \sum_{p\geq 0} \text{length}_R \left(\frac{I^{p+1}}{J I^p} \right) = \sum_{p\geq 0} \Lambda_p(I,J).$$

Valabrega-Valla'78: $\Delta(I,J)=0 \Leftrightarrow gr_I(R)$ is Cohen-Macaulay.

$$\bigoplus_{p\geq 0} \frac{I^{p+1}\cap J}{I^pJ}$$ Valabrega-Valla module

Conjecture (Guerrieri'94)

$$\operatorname{depth}(gr_I(R)) \geq d-\Delta(I,J)$$

- Guerrieri'93: $\Delta(I,J)=1$, partial cases for $\Delta(I,J)=2$;
- Wang'00: $\Delta(I,J)=2$;
- Guerrieri-Rossi'99: partial results for $\Delta(I,J)=3$ and R/I Gorenstein;
- Wang'01: partial results for $\Delta(I,J)=4$;
- Wang'01: counterexample for $d=6$ and $\Delta(I,J)=5$.

Question (Guerrieri-Huneke'93)

$$\Delta_p(I,J) \leq 1, \text{ for all } p \geq 0 \quad \Rightarrow \quad \mathrm{depth}(gr_I(R)) \geq d-1?$$

Wang'02: counterexample. What if R is regular?

We prove:

Theorem

If $\Delta_p(I,J) \leq 1$, for all $p \geq 0$, then,

$$\mathrm{depth}(gr_I(R)) \geq d-2.$$

Huckaba-Marley'97: $e_1(I) \leq \Lambda(I,J)$, and the equality holds if and only if $\operatorname{depth}(gr_I(R)) \geq d-1$.

We define $\delta(I,J) = \Lambda(I,J) - e_1(I) \geq 0$.

Wang'00: $\delta(I,J) \leq \Delta(I,J)$. Guerrieri's conjecture is implied by:

Conjecture (Wang'00)

$$\operatorname{depth}(gr_I(R)) \geq d-1-\delta(I,J)$$

- Huckaba-Marley'97: $\delta(I,J) = 0$;
- Wang'00, Polini'00: $\delta(I,J) = 1$;
- Rossi-Guerrieri'99: partial cases for $\delta(I,J) = 2$ with R/I Gorenstein;
- Wang'01: counterexample for $d = 6$.

In the thesis we will decompose $\delta(I,J)$ as a finite sum

$$\delta(I,J) = \sum_{p \geq 0} \delta_p(I,J)$$

with $0 \leq \delta_p(I,J) \leq \Delta_p(I,J)$.

Theorem

If $\bar{\delta} = \max\{\delta_p(I,J) \mid p \geq 0\} \leq 1$, *then*

$$\operatorname{depth}(\mathcal{R}(I)) \geq d - \bar{\delta} \quad \textit{and} \quad \operatorname{depth}(gr_I(R)) \geq d - 1 - \bar{\delta}.$$

IDEA: we will define a non-standard bigraded module $\Sigma^{I,J}$ such that

$$\Delta_p(I,J) \geq \delta_p(I,J) = \Lambda_p(I,J) - e_0(\Sigma^{I,J}_{[p]})$$

and

$$\Delta(I,J) \geq \delta(I,J) = \Lambda(I,J) - e_1(I).$$

We consider the associated graded ring of $\mathcal{R}(I)$ with respect to the homogeneous ideal $Jt\mathcal{R}(I) = \bigoplus_{n\geq 0} JI^{n-1}t^n$

$$gr_{Jt}(\mathcal{R}(I)) = \bigoplus_{j\geq 0} \frac{(Jt\mathcal{R}(I))^j}{(Jt\mathcal{R}(I))^{j+1}} U^j.$$

This ring has a natural bigraded structure. If we consider the bigraded ring $B := R[V_1, \ldots, V_\mu; T_1, \ldots, T_d]$ with $\deg(V_i) = (1,0)$ and $\deg(T_i) = (1,1)$, then there exists an exact sequence of bigraded B-rings

$$0 \longrightarrow K^{I,J} \longrightarrow C^{I,J} \longrightarrow gr_{Jt}(\mathcal{R}(I)) \longrightarrow 0,$$

where $C^{I,J} := \frac{\mathcal{R}(I)}{Jt\mathcal{R}(I)}[T_1, \ldots, T_d]$ and $K^{I,J}$ is the ideal of initial forms of $Jt\mathcal{R}(I)$.

Diagonals: Given a bigraded B-module M and an integer $p \in \mathbb{Z}$, we define the $R[T_1, \ldots, T_d]$-module

$$M_{[p]} = \bigoplus_{m-n=p+1} M_{(m,n)}.$$

In our case, the modules $K^{I,J}_{[p]}$, $C^{I,J}_{[p]}$ and $gr_{Jt}(\mathcal{R}(I))_{[p]}$ are $\mathcal{R}(J)$-modules, and, eventually, they do not vanish for a finite set of indexes $p \in \mathbb{Z}$.

From now on, we will be interested in considering the non-negative diagonals of these modules and so, let us consider the following finitely generated bigraded B-modules:

$$\Sigma^{I,J} := \bigoplus_{p\geq 0} gr_{Jt}(\mathcal{R}(I))_{[p]} = \bigoplus_{p\geq 0}\bigoplus_{i\geq 0} \frac{J^i I^{p+1}}{J^{i+1} I^p} t^{p+1+i} U^i,$$

$$\mathcal{M}^{I,J} := \bigoplus_{p\geq 0} C_{[p]}^{I,J} = \bigoplus_{p\geq 0} \frac{I^{p+1}}{I^p J} t^{p+1} [T_1, \ldots, T_d],$$

and from now on, we consider the new B-module

$$K^{I,J} := \bigoplus_{p\geq 0} K_{[p]}^{I,J}.$$

We call $\Sigma^{I,J}$ the bigraded Sally module of I with respect J.
There exists a natural isomorphism of $\mathcal{R}(J)$-modules

$$gr_{Jt}(\mathcal{R}(I)) \cong \mathcal{R}(J) \oplus \Sigma^{I,J}.$$

Since the modules $\Sigma^{I,J}$ and $\mathcal{M}^{I,J}$ are annihilated by J, we have an exact sequence of bigraded $A = R/J[V_1, \ldots, V_\mu; T_1, \ldots, T_d]$-modules

$$0 \longrightarrow K^{I,J} \longrightarrow \mathcal{M}^{I,J} \longrightarrow \Sigma^{I,J} \longrightarrow 0.$$

By considering each diagonal, for all $p \geq 0$ we have an exact sequence of $R/J[T_1, \ldots, T_d]$-modules,

$$0 \longrightarrow K^{I,J}_{[p]} \longrightarrow \mathcal{M}^{I,J}_{[p]} = \frac{I^{p+1}}{JI^p}[T_1, \ldots, T_d] \longrightarrow \Sigma^{I,J}_{[p]} \longrightarrow 0,$$

which are, in fact, graded modules, and so we can consider for them the (classic) Hilbert function.

Let M be a finitely generated bigraded module over a bigraded algebra

$$R = C[X_1, \ldots, X_r, Y_1, \ldots, Y_s, T_1, \ldots, T_u]$$

with $\deg(X_i) = (1,0)$, $\deg(Y_i) = (0,1)$, $\deg(T_i) = (1,1)$ and C an Artin ring.

Cumulative Hilbert function of M:

$$h_M(m,n) = \sum_{j \leq n} \mathrm{length}_R(M_{(m,j)}).$$

There exist a polynomial $p_M(m,n)$ and integers m_0, n_0 such that

$$h_M(m,n) = p_M(m,n)$$

for $m \geq m_0$ and $n \geq m + n_0$.

Note: If there are no generators of degree $(0,1)$, the polynomial does not depend on n, $p_M(m,n) = p_M(m)$.

- $p_{\Sigma^{I,J}}(m) = \sum_{i=0}^{d-1} (-1)^i e_{i+1}(I) \binom{m-1+d-i-1}{d-i-1}$.

- $\deg(p_{\mathcal{M}^{I,J}}) = d-1$ and $e_0(\mathcal{M}^{I,J}) = \Lambda(I,J)$.

- If $\Sigma^{I,J} = 0$ then $gr_I(R)$ is a Cohen-Macaulay ring.
  If $\Sigma^{I,J} \neq 0$ then $\deg(p_{\Sigma^{I,J}}) = d-1$ and $e_0(\Sigma^{I,J}) = e_1(I)$.

- $e_0(K^{I,J}) = \delta(I,J)$. If $K^{I,J} \neq 0$ then $\deg(p_{K^{I,J}}) = d-1$.
  In particular, $\Lambda(I,J) \geq e_1(I)$.

- For all $p \geq 0$, it holds

$$e_0(\Sigma_{[p]}^{I,J}) = \Lambda_p(I,J) - e_0(K_{[p]}^{I,J}) \geq 0$$

and

$$e_1(I) = \sum_{p \geq 0} (\Lambda_p(I,J) - e_0(K_{[p]}^{I,J})).$$

- For all $p \geq 0$, it holds

$$\Delta_p(I,J) \geq e_0(K_{[p]}^{I,J})$$

and

$$\delta(I,J) = e_0(K^{I,J}) = \sum_{p \geq 0} e_0(K_{[p]}^{I,J}) \geq 0.$$

We define

$$\delta_p(I,J) = e_0(K^{I,J}_{[p]}).$$

We prove a refined version of Wang's Conjecture by considering some special configurations of the set $\{\delta_p(I,J)\}_{p\geq 0}$ instead of $\delta(I,J) = \sum_{p\geq 0} \delta_p(I,J)$.
Let us consider

$$\overline{\delta}(I,J) = \max\{\delta_p(I,J) \mid p \geq 0\}.$$

Theorem

Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d>0$. Let I be an $\mathfrak{m}$-primary ideal of R and J a minimal reduction of I. If $\overline{\delta}(I,J)\leq 1$, then

$$\operatorname{depth}(\mathcal{R}(I)) \geq d - \overline{\delta}(I,J)$$

and

$$\operatorname{depth}(gr_I(R)) \geq d - 1 - \overline{\delta}(I,J).$$

Note that for $\delta(I,J) = 0,1$ we recover the known cases of Wang's Conjecture.

For the proof we need the following important results. In particular, we need to study the depth of the associated bigraded ring $gr_{Jt}(\mathcal{R}(I))$.

Theorem

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 3$. Let I be an $\mathfrak{m}$-primary ideal of R and let J be a minimal reduction of I. Let us assume that $K^{I,J} \neq 0$, and either $K^{I,J}_{[p]} = 0$ or $K^{I,J}_{[p]}$ is a rank one torsion free $\mathbf{k}[T_1, \ldots, T_d]$-module for $p \geq 0$. Then,

$$\operatorname{depth}(gr_{Jt}(\mathcal{R}(I))) \geq d - 1.$$

Lemma

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$. Let I be an $\mathfrak{m}$-primary ideal of R with minimal reduction J. If $\delta_p(I, J) = 1$ then $K^{I,J}_{[p]}$ is a rank one torsion free $\mathbf{k}[T_1, \ldots, T_d]$-module.

Finally, we are able to give an answer to the question of Guerrieri and Huneke mentioned before.

Theorem

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$. Let I be an $\mathfrak{m}$-primary ideal of R and J a minimal reduction of I. If $\Delta_p(I, J) \leq 1$ for all $p \geq 1$, then

$$\operatorname{depth}(gr_I(R)) \geq d - 2.$$

This result follows from the theorem since $\delta_p(I, J) \leq \Delta_p(I, J) \leq 1$.

Publications and preprints:

- G. Colomé-Nin and J. Elias, *Bigraded structures and the depth of blow-up algebras*, Proceedings of the Royal Society of Edinburgh, 136A, 1175-1194, 2006.
- G. Colomé-Nin and J. Elias, *Cohomological properties of non-standard multigraded modules*, Journal of Algebra, to appear.
- G. Colomé-Nin and J. Elias, *On the asymptotic depth of multigraded modules*, preprint, 2008.

Moltes gràcies!