Bigraded structures and the depth of blow-up algebras

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Let $(R, \mathbf{m}, \mathbf{k})$ be a d-dimensional Cohen-Macaulay local ring. Let I be an $\mathbf{m}-$primary ideal of R with minimal reduction J.

Problem: to estimate the depth of the associated graded ring $gr_I(R) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ and the Rees algebra $\mathcal{R}(I) = \bigoplus_{n\geq 0} I^n t^n$ for ideals I having good properties.
Attached to the pair I, J we can consider the integers

$$\Delta(I,J) = \sum_{p\geq 1} length_R\left(\frac{I^{p+1}\cap J}{I^pJ}\right) \quad , \quad \Lambda(I,J) = \sum_{p\geq 0} length_R\left(\frac{I^{p+1}}{JI^p}\right)$$

Related conjectures on the depth of $gr_I(R)$:
Conjecture (Guerrieri, [2]) $depth(gr_I(R)) \geq d - \Delta(I,J)$.
($\Delta(I,J) = 0$ Valabrega-Valla, $\Delta(I,J) = 1$ Guerrieri, $\Delta(I,J) = 2$ Wang.)

We consider the non-negative integer $\delta(I,J) = \Lambda(I,J) - e_1(I) \geq 0$ (Huckaba-Marley). Wang showed that $\delta(I,J) \leq \Delta(I,J)$ and that Guerrieri's conjecture is implied by the following one,
Conjecture (Wang, [8]) $depth(gr_I(R)) \geq d - 1 - \delta(I,J)$.
($\delta(I,J) = 0$ Huckaba-Marley, $\delta(I,J) = 1$ Wang and Polini gave a simpler proof, $\delta(I,J) = 2$ Guerrieri-Rossi in the Gorenstein case. Counterexample of Wang for $\delta(I,J) = 3$.)

These conjectures aren't true in general case.

Our main result is to prove a refined version of Wang's conjecture, Theorem 2.2. We naturally decompose the integer $\delta(I,J) = \sum_{p\geq 0} \delta_p(I,J)$ as a finite sum of non-negative integers $\delta_p(I,J)$, with $\Delta_p(I,J) \geq \delta_p(I,J) \geq 0$. Let us consider the maximum, say $\bar{\delta}(I,J)$, of the integers $\delta_p(I,J)$ for $p \geq 0$.

Theorem 2.2. *Assume that $\bar{\delta}(I,J) \leq 1$.*
Then $depth(\mathcal{R}(I)) \geq d - \bar{\delta}(I,J)$ and $depth(gr_I(R)) \geq d - 1 - \bar{\delta}(I,J)$.

The aim of this work is to introduce a non-standard bigraded module $\Sigma^{I,J}$ in order to study the depth of the associated graded ring $gr_I(R)$ and the Rees algebra $\mathcal{R}(I)$ of I. This module can be considered as a refinement of the Sally module previously introduced by W. Vasconcelos. A secondary purpose is to present a unified framework where several results and objects appearing in the papers on the above conjectures can be studied. The key tool of this paper is the Hilbert function of non-standard bigraded modules.

1 Bigraded Sally module

Let $I = (b_1, \dots, b_\mu)$ be an $\mathbf{m}$-primary ideal of a Cohen-Macaulay local ring R and $J = (a_1, \dots, a_d)$ a minimal reduction of I. The associated graded ring of $\mathcal{R}(I)$ with respect to the homogeneous ideal $Jt\mathcal{R}(I) = \bigoplus_{n\geq 0} JI^{n-1}t^n$ is

$$gr_{Jt}(\mathcal{R}(I)) = \bigoplus_{j\geq 0} \frac{(Jt\mathcal{R}(I))^j}{(Jt\mathcal{R}(I))^{j+1}} U^j.$$

It has a natural bigraded structure.

Now, given a bigraded $A-$module M, $A = R/J[V_1, \dots, V_\mu; T_1, \dots, T_d]$, we can consider the Hilbert function of M defined by

$$h_M(m,n) = \sum_{0\leq j\leq n} length_A(M_{(m,j)})$$

As $deg(V_i) = (1,0)$ and $deg(T_j) = (1,1)$, there exist integers $f_i(M) \in \mathbb{Z}$, $i \geq 0$, and an integer $c \geq 0$, such that the polynomial

$$p_M(m) = \sum_{i=0}^{c-1} f_i(M)\binom{m+c-i}{c-i},$$

verifies $h_M(m,n) = p_M(m)$ for all $m \geq m_0$ and $n \geq n_0 + m$ for some integers $m_0, n_0 \geq 0$. This situation holds for the $A-$modules $\Sigma^{I,J}$, $\mathcal{M}^{I,J}$, and $K^{I,J}$.

Proposition 1.3

$$p_{\Sigma^{I,J}}(m) = \sum_{i=0}^{d-1}(-1)^i e_{i+1}(I)\binom{m-1+d-i-1}{d-i-1}$$

And the following conditions hold:

(i) $e_0(\mathcal{M}^{I,J}) = \Lambda(I,J)$

(ii) *If $\Sigma^{I,J} = 0$, then $gr_I(R)$ is Cohen-Macaulay.*
If $\Sigma^{I,J} \neq 0$, then $e_0(\Sigma^{I,J}) = \sum_{p\geq 0} e_0(\Sigma^{I,J}_{[p]}) = e_1(I)$
and $e_0(\Sigma^{I,J}_{[p]}) = length_R(\frac{I^{p+1}}{JI^p}) - e_0(K^{I,J}_{[p]})$

(iii) $e_0(K^{I,J}) = \sum_{p\geq 0} e_0(K^{I,J}_{[p]}) = \delta(I,J)$

(iv) $length_R(\frac{I^{p+1}\cap J}{JI^p}) \geq e_0(K^{I,J}_{[p]})$

We have $\Delta(I,J) \geq \delta(I,J) = \Lambda(I,J) - e_1(I) \geq 0$. In the next result we show that these inequalities can we deduced from some "local" inequalities. For all $p \geq 0$ we define the following the integers

$$\Delta_p(I,J) = length_R\left(\frac{I^{p+1}\cap J}{JI^p}\right) \quad , \quad \delta_p(I,J) = e_0(K^{I,J}_{[p]}) \quad \text{and} \quad \Lambda_p(I,J) = length_R\left(\frac{I^{p+1}}{JI^p}\right)$$

From the last result we deduce:

Proposition 1.4 *For all $p \geq 0$ the following inequalities hold*

$$\Delta_p(I,J) \geq \delta_p(I,J) = \Lambda_p(I,J) - e_0(\Sigma^{I,J}_{[p]}) \geq 0.$$

Summing up these inequalities with respect p we get

$$\Delta(I,J) \geq \delta(I,J) = \Lambda(I,J) - e_1(I) \geq 0.$$

Consider the bigraded ring $B := R[V_1, \dots, V_\mu; T_1, \dots, T_d]$ with $deg(V_i) = (1,0)$ and $deg(T_i) = (1,1)$. There exists an exact sequence of bigraded B-rings

$$0 \longrightarrow K^{I,J} \longrightarrow C^{I,J} := \frac{\mathcal{R}(I)}{Jt\mathcal{R}(I)}[T_1, \dots, T_d] \stackrel{\pi}{\longrightarrow} gr_{Jt}(\mathcal{R}(I)) \longrightarrow 0$$

with $\pi(T_i) = a_i tU$, $i = 1, \dots, d$; $K^{I,J}$ is the ideal of initial forms of $Jt\mathcal{R}(I)$.

Given a $B-$bigraded module M and an integer $p \in \mathbb{Z}$, $M_{[p]}$ is the additive sub-group of M defined by the direct sum of the pieces $M_{(m,n)}$ such that $m - n = p + 1$. $M_{\geq p} = \bigoplus_{n\geq p} M_{[n]}$ is a sub-B-module of M, and we can consider the exact sequence of $R[T_1, \dots, T_d]$-modules

$$0 \longrightarrow M_{[p]} \longrightarrow M_{\geq p} \longrightarrow M_{\geq p+1} \longrightarrow 0.$$

In the case of our modules $K^{I,J}_{[p]}$, $C^{I,J}_{[p]}$ and $gr_{Jt}(\mathcal{R}(I))_{[p]}$, they are $\mathcal{R}(J)-$modules.
Let us consider the following bigraded finitely generated B-modules:

$$\Sigma^{I,J} = \bigoplus_{p\geq 0} gr_{Jt}(\mathcal{R}(I))_{[p]}$$

$$\mathcal{M}^{I,J} = \bigoplus_{p\geq 0} C^{I,J}_{[p]} \cong \bigoplus_{p\geq 0} \tfrac{I^{p+1}}{I^pJ}\, t^{p+1}[T_1, \dots, T_d]$$

$$K^{I,J} = \bigoplus_{p\geq 0} K^{I,J}_{[p]}$$

There exists a natural isomorphism of $\mathcal{R}(J)-$modules $gr_{Jt}(\mathcal{R}(I)) \cong \mathcal{R}(J) \oplus \Sigma^{I,J}$.
There exists the following exact sequence of $A = B \otimes_R R/J \cong R/J[V_1, \dots, V_\mu; T_1, \dots, T_d]-$bigraded modules

$$0 \longrightarrow K^{I,J} \longrightarrow \mathcal{M}^{I,J} \longrightarrow \Sigma^{I,J} \longrightarrow 0.$$

For all $p \geq 0$ we have an exact sequence of $R/J[T_1, \dots, T_d]-$modules

$$0 \longrightarrow K^{I,J}_{[p]} \longrightarrow \mathcal{M}^{I,J}_{[p]} = \frac{I^{p+1}}{JI^p}[T_1, \dots, T_d] \longrightarrow \Sigma^{I,J}_{[p]} \longrightarrow 0,$$

so we can consider the (classic) Hilbert function of $\Sigma^{I,J}_{[p]}$, $\mathcal{M}^{I,J}_{[p]}$ and $K^{I,J}_{[p]}$ with respect the variables $T_1, \dots, T_d$.

Definition 1.1 *$\Sigma^{I,J}$ is the bigraded Sally module of I with respect J.*

Remark 1.2

$$length_R(\Sigma^{I,J}_{(m+1,*)}) = length_R(S_J(I)_m) + length_R\left(\frac{IJ^m}{J^{m+1}}\right)$$

where $\Sigma^{I,J}_{(m+1,*)} \cong \bigoplus_{j=0}^{j=m} \frac{I^{m+1-j}J^j}{I^{m-j}J^{j+1}} t^{m+1}U^j$ and $S_J(I)_m$ is the degree m piece of the Sally module $S_J(I)$.

2 On the depth of the blow-up algebras

The aim of this section is to prove a refined version of Wang's conjecture by considering some special configurations of the set $\{\delta_p(I,J)\}_{p\geq 0}$ instead of $\delta = \sum_{p\geq 0} \delta_p(I,J)$, Theorem 2.2. As a by-product we recover the known cases of Wang's conjecture, Corollary 2.5. Let us consider $\bar{\delta}(I,J)$ the maximum of the integers $\delta_p(I,J)$ for $p \geq 0$.

Theorem 2.1 *Assume that $d \geq 3$. Let us assume that $K^{I,J} \neq 0$, and either $K^{I,J}_{[p]} = 0$ or $K^{I,J}_{[p]}$ is a rank one torsion free $\mathbf{k}[T_1, \dots, T_d]$-module for $p \geq 0$. Then*

$$depth(gr_{Jt}(\mathcal{R}(I))) \geq d - 1.$$

Theorem 2.2 *Assume that $\bar{\delta}(I,J) \leq 1$. Then*

$$depth(\mathcal{R}(I)) \geq d - \bar{\delta}(I,J) \quad \text{and} \quad depth(gr_I(R)) \geq d - 1 - \bar{\delta}(I,J)$$

Remark 2.3 An example of Wang in [9] shows that the last result is sharp in the sense that we cannot expect to have $depth(gr_I(R)) \geq d - 1$ provided $\bar{\delta}(I,J) = 1$. Precisely, it is a counterexample for the question formulated by Guerrieri in [1]. She asked if it were true that $depth(gr_I(R)) \geq d - 1$ for an $\mathbf{m}-$primary ideal I in a $d-$dimensional Cohen-Macaulay ring provided that $\Delta_p(I,J) \leq 1\ \forall p \geq 1$. Wang reformulate the question in the regular case. Relating to this, we are able to improve the bound for the Cohen-Macaulay case:

Proposition 2.4 *Assuming that $\Delta_p(I,J) \leq 1$ for all $p \geq 1$ we have that*

$$depth(gr_I(R)) \geq d - 2.$$

Now we prove the Conjecture of Wang in the known cases, [8], as a corollary of the previous results.

Corollary 2.5

$$depth(gr_I(R)) \geq d - 1 - \delta(I,J)$$

for $\delta(I,J) = 0, 1$.

References

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