

Illusory Correlation as the Outcome of Experience Sampling

Jerker Denrell (denrell@gsb.stanford.edu)

Graduate School of Business, Stanford University, 518 Memorial Way Stanford, CA 94305-5015 USA

Gaël Le Mens (glemens@stanford.edu)

Graduate School of Business, Stanford University, 518 Memorial Way Stanford, CA 94305-5015 USA

Abstract

Individuals are typically more likely to repeatedly select alternatives they have a positive impression of. This paper shows that this sequential sampling feature of the information acquisition process might lead to the emergence of illusory correlation between attributes of multi-attribute alternatives. This suggests an alternative explanation for illusory correlation that does not rely on biased information processing. The model also shows that illusory correlation can emerge even when the attributes are independently distributed and the distributions are not skewed.

Keywords: Learning; Experience; Sampling; Illusory Correlation.

Introduction

Much research on illusory correlation has focused on how an individual's processing of information might produce biases in her assessment of the correlation between features or attributes of objects or alternatives. Previous explanations have proposed that prior expectations (e.g. Chapman & Chapman, 1967), the differential distinctiveness of positive versus negative stimuli (e.g. Allan, 1993) and the greater distinctiveness of infrequent events (e.g. Hamilton & Gifford, 1996) distort the encoding and recall of information used to estimate the correlation between features (Fiedler, 2000). Some researchers have also proposed that illusory correlation might emerge from skewed distributions of the features in the population (Fiedler, 1991, 2000).

Most of these prior approaches assume that individuals have access to information about two or more dimensions and try to explain why the perceived correlation, based on this sample, would diverge from the true one. In reality, information is not always immediately available. Individuals may have to sample the information sequentially. For example, in learning about the attributes of others individuals may obtain information only by interacting with others. In such contexts, sampling is often endogenous: prior observations usually influence the probability of future sampling (Denrell, 2005). For example, an individual may not want to continue to interact with others unless he or she believes that the interaction will be productive or interesting. In this paper, we show that when decision makers sequentially sample information, the resulting sample selection bias might produce illusory correlation even when features are independently distributed in the population and information is correctly processed.

Suppose, for example, that you are learning about the competence and creativity of individuals in the population. Suppose, in addition, that you only want to interact with individuals that you believe are either competent or creative. Be-

ing mindful of not wasting your time, you keep a record of all your interactions with each individual. Your estimate of the competence and creativity of a given individual is a weighted average of all your past interactions with that individual. Even if competence and creativity are independently distributed in the population, we show that you will come to believe that those two traits are positively correlated.

The key to this result is that you may stop interacting with another individual B depending on your assessment of attributes of that individual. Suppose you mistakenly believe that B is incompetent (B is actually competent). If you believe that B is not creative, you are unlikely to interact with B again and thus your belief about B 's incompetence will tend to persist. If, on the contrary, you believe that B is creative, you are likely to interact with B again. Such an interaction provides you with an opportunity to correct your mistaken belief about B 's competence. Because of regression to the mean, your belief about B 's competence is likely to become more positive after updating. The illusory positive correlation between competence and creativity emerges as a consequence of this asymmetry in opportunities for correction of mistaken estimates.

In the following, we characterize when illusory correlation might develop through experiential learning, when it is positive and when it is negative. We show that whether the perceived correlation is positive or negative depends on how the decision maker forms a combined evaluation of the object based on beliefs about its dimensions. A positive illusory correlation emerges when evaluations are compensatory (an object is evaluated positively if the average of the two features is positive) or disjunctive (an object is evaluated positively if one of two features is positive). A negative illusory correlation can emerge when the evaluation of an object is conjunctive (an object is evaluated positively only if both features is positive). We also consider what happens when the number of attributes learnt about is higher than two. Because our model relies on a mechanism different from existing accounts of illusory correlation, it predicts that illusory correlation might emerge in settings not encompassed by previous theories.

Model

To illustrate how illusory correlations can emerge as a result of experience sampling, we develop a model in which an individual learns the values of two attributes of an alternative. Let X denote the value of the first attribute and Y denote the value of the second attribute. We assume that the belief about X (resp. Y) is a function of past observations of that attribute.

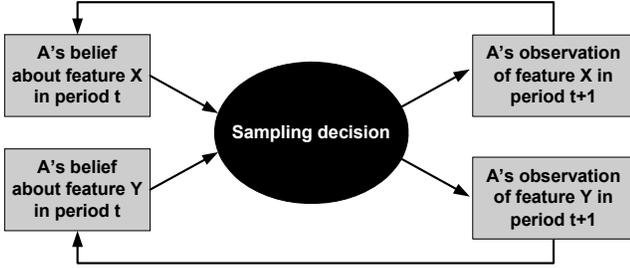


Figure 1: Structure of the model. In each period, the decision maker, A , updates her beliefs about the values of features X and Y on the basis of her new observations, if any. The sampling decision depends on the current beliefs about the values of the features .

That is, observations of Y (resp. X) do not affect the beliefs about X (resp. Y). The structure of the model is represented on Figure 1. It works as follows:

1. In each period, the decision maker can choose to sample the alternative or to abstain. If the decision maker samples the alternative in period t , he or she gets to observe the two attributes. The observations in period t , x_t and y_t , are iid draws from the random variables X and Y with respective densities $f_1(\cdot)$ and $f_2(\cdot)$.
2. The decision maker forms beliefs about X and Y based on his or her experiences. We assume that this process of belief formation can be modeled as a sequential revision of current beliefs in view of new evidence. Specifically, the revised belief of decision maker about the random variable X after having sampled in period $t + 1$, denoted \hat{x}_{t+1} is $\hat{x}_{t+1} = (1 - b)\hat{x}_t + bx_{t+1}$, where $b \in (0, 1)$. If the decision maker does not sample, the belief does not change: $\hat{x}_{t+1} = \hat{x}_t$. While such a weighted average model of belief updating might seem simplistic, Denrell (2005) has shown that it provides a reasonable approximation to actual belief updating in several experimental studies. In addition, it can be derived from more realistic connectionist models (e.g. Busemeyer & Myung, 1992).
3. The probability of sampling the alternative in period $t + 1$ is denoted $Q(\hat{x}_t, \hat{y}_t)$. This probability is assumed to be a function of the values of \hat{x}_t and \hat{y}_t . We will assume that it is an increasing function of both arguments. The idea is that the decision maker values the attributes positively. Sometimes, we will assume that $Q(\hat{x}_t, \hat{y}_t) = f(u(\hat{x}_t, \hat{y}_t))$, where $u(\hat{x}_t, \hat{y}_t)$ is a function measuring the value of the alternative given the beliefs about the dimensions.

In summary, based on past observations the decision maker learns about the two attributes. The decision maker is more likely to sample the alternative if she has a positive belief about the two dimensions. In the following, we demonstrate how this learning process leads to an illusory correlations.

The Stationary Density of Beliefs

Due to the sample bias inherent in the above process, the beliefs of the decision maker about the two attributes do not converge to their true distributions. Specifically, let \hat{X} and \hat{Y} be the asymptotic beliefs about the values of the two attributes, as $t \rightarrow \infty$. Denrell & Le Mens (2007) showed that the joint distribution of the asymptotic beliefs is

$$h(\hat{x}, \hat{y}) = \frac{\frac{1}{Q(\hat{x}, \hat{y})} g_1(\hat{x}) g_2(\hat{y})}{\int \int \frac{1}{Q(\hat{x}, \hat{y})} g_1(\hat{x}) g_2(\hat{y}) d\hat{x} d\hat{y}} \quad (1)$$

where $g_i(z)$ is the density that the belief about dimension i would converge to (asymptotically) if the alternative were sampled in every period. For example, if $f_1(\cdot)$ and $f_2(\cdot)$ are both Normal densities with mean zero and variance σ^2 , then $g_i(\cdot)$ is a Normal density with mean zero and variance $\sigma^2 b / (2 - b)$.

To show that the asymptotic joint density has the form $h(\hat{x}, \hat{y})$, note that the process can be viewed as a (two-dimensional) semi-Markov process. Transitions are Markovian but only occur if the alternative is sampled. The stationary distribution of a semi-Markov process is a weighted density of the stationary density of the underlying Markov chain (here $g(\hat{x})$), where the weight is the average time spent in a state before a transition occurs, which here is $\frac{1}{Q(\hat{x}, \hat{y})}$.

It is easy to generalize this idea to more than 2 dimensions. In general, for N dimensions, the stationary joint densities of the beliefs is

$$h(\hat{x}_1, \dots, \hat{x}_N) = \frac{\frac{1}{Q(\hat{x}_1, \dots, \hat{x}_N)} g(\hat{x}_1) \dots g(\hat{x}_N)}{\int \frac{1}{Q(x_1, \dots, x_N)} g(x_1) \dots g(x_N) dx_1 \dots dx_N} \quad (2)$$

When Are Beliefs Positively or Negatively Correlated When $N = 2$?

In this section, we specify the conditions under which, in the context of the model introduced above, the asymptotic beliefs about the values of the two attributes are positively correlated or negatively correlated. We first report a general condition on the sampling probability Q , and we illustrates how this specializes when Q is more precisely specified.

A General Condition for Illusory Correlation

When $Q(\cdot, \cdot)$ is doubly continuously differentiable, it is possible to specify a condition on $Q(x, y)$ that, when met, implies \hat{X} and \hat{Y} are positively correlated. Similarly, it is possible to specify a condition on $Q(x, y)$ that implies \hat{X} and \hat{Y} are negatively correlated. These are formulated in the following proposition:

Proposition 1 *i) If $\partial^2 \ln[Q(x, y)] / (\partial x \partial y) < 0$ for all $x, y \in \mathbb{R}$, then x and y are positively correlated. ii) If $\partial^2 \ln[Q(x, y)] / (\partial x \partial y) > 0$ for all $x, y \in \mathbb{R}$, then x and y are negatively correlated.*

Proof. This formal result has been proven by Denrell & Le Mens (2007) in another context. Their proof uses the fact that

$$\frac{\partial^2 \ln h(x, y)}{\partial x \partial y} = - \frac{\partial^2 \ln Q(x, y)}{\partial x \partial y}$$

and uses some results by Karlin & Rinott (1980) on the relation between $\frac{\partial^2 \ln h(x, y)}{\partial x \partial y}$ and the correlation of X and Y (see the Appendix) ■

Note that whenever $\frac{\partial^2 Q(x, y)}{\partial x \partial y} \leq 0$ for all $x, y \in \mathbb{R}$, the correlation between x and y is positive. To see why, note that

$$\frac{\partial^2 \ln Q(x, y)}{\partial x \partial y} = \frac{Q(x, y) \frac{\partial^2 Q(x, y)}{\partial x \partial y} - \frac{\partial Q(x, y)}{\partial x} \frac{\partial Q(x, y)}{\partial y}}{Q(x, y)^2} \quad (3)$$

and $\frac{\partial Q(x, y)}{\partial x} \frac{\partial Q(x, y)}{\partial y} > 0$ because we assumed that $Q(x, y)$ is increasing in both x and y .

Attributes That Are Substitutes

It is possible to use the conditions specified in Proposition 1 to examine what happens when the attributes are substitutes to each others. To do so, let's assume that the probability of sampling depends on beliefs and a 'utility' function as follows:

1. The probability of sampling in period t is $Q(\hat{x}_t, \hat{y}_t) = P[u(\hat{x}_t, \hat{y}_t)]$.
2. $u(\hat{x}_t, \hat{y}_t)$ is a quantity that measures the combined value of the alternative, given the beliefs about the two attributes. $u(\hat{x}_t, \hat{y}_t)$ is increasing in both \hat{x}_t and \hat{y}_t .
3. The probability of sampling depends on the 'combined utility' through the logistic choice function: $P(u) = 1/(1 + \exp[-su])$.

When $\frac{\partial^2 u(x, y)}{\partial x \partial y} < 0$, the sensitivity of the utility to an attribute decreases as the value of the other attribute increases. In that sense, the attributes are substitutes to each others. The following proposition states that when this is the case, the correlation between \hat{X} and \hat{Y} is positive.

Proposition 2 *If $Q(x, y) = 1/(1 + \exp[-su(x, y)])$ and $\frac{\partial^2 u(x, y)}{\partial x \partial y} \leq 0$ for all $x, y \in \mathbb{R}$, then the correlation between \hat{X} and \hat{Y} is positive.*

Proof. With these assumptions on Q , $\partial^2 \ln Q(x, y)/\partial x \partial y$ is equal to

$$\frac{s(1 + e^{su(x, y)}) \frac{\partial^2 u(x, y)}{\partial x \partial y} - e^{su(x, y)} s^2 \frac{\partial u(x, y)}{\partial x} \frac{\partial u(x, y)}{\partial y}}{(1 + e^{su(x, y)})^2}. \quad (4)$$

By symmetry, $\frac{\partial u(x, y)}{\partial x} = \frac{\partial u(x, y)}{\partial y}$ and $\frac{\partial u(x, y)}{\partial x} \frac{\partial u(x, y)}{\partial y} > 0$.

Thus, $\partial^2 \ln Q(x, y)/\partial x \partial y$ is negative whenever $\frac{\partial^2 u(x, y)}{\partial x \partial y} \leq 0$. ■

Note that when the attributes make independent contributions to the total value ($\frac{\partial^2 u(x, y)}{\partial x \partial y} = 0$), a positive illusory correlation also emerges.

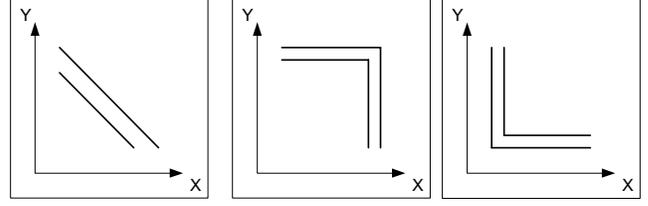


Figure 2: Iso-utility curves for quasi-linear mean utility. The left quadrant corresponds to $p = 1$, the middle quadrant to $p = +\infty$ and the right quadrant to $p = -\infty$.

With this specification, it is not possible to say much about attributes that are complements (when $\frac{\partial^2 u(x, y)}{\partial x \partial y} > 0$). To address this, we further specify the probability of sampling in the next subsection.

Conjunctive and Disjunctive Evaluation Functions

Here we make assumptions about the form of the 'utility function'. The structure is exactly as above, but the utility function, $u(\hat{x}_t, \hat{y}_t)$, is the 'root power mean' or the 'quasi-linear' mean: $u(\hat{x}_t, \hat{y}_t) = u_p(\hat{x}_t, \hat{y}_t) = [0.5\hat{x}_t^p + 0.5\hat{y}_t^p]^{1/p}$ where $p \in \mathbb{R}$ and $0 < \hat{x}_t < a, 0 < \hat{y}_t < a$ for some $a > 0$ (i.e. we assume positive and bounded random variables). This function has been used in previous literature (Dawes, 1964; Einhorn, 1970) to represent both conjunctive, compensatory, and disjunctive utility functions. It has the following properties:

- For all $p \in \mathbb{R}$, $\min(x, y) \leq u_p(x, y) \leq \max(x, y)$.
- If $p = 1$, then it is the arithmetic average: $u_1(x, y) = 0.5x + 0.5y$. This corresponds to a situation where the utility function is compensatory.
- If $p > 1$, then it puts more weight on the maximum of x and y . Thus, when $p > 1$, the utility function is 'compensatory': the two attributes tend to substitute for each other. Moreover, it can be shown that $\lim_{p \rightarrow +\infty} u_p(x, y) = \max(x, y)$. This corresponds to a situation where the utility function is disjunctive.
- If $p < 1$, then it puts more weight on the minimum of x and y . Thus, when $p < 1$, the utility function is 'non-compensatory'. Moreover, it can be shown that $\lim_{p \rightarrow -\infty} u_p(x, y) = \min(x, y)$. This corresponds to a situation where the utility function is conjunctive.

Iso-utility curves corresponding to cases where $p = 1$, $p = +\infty$ and $p = -\infty$ are shown on Figure 2.

More generally, this utility function enables us to think about the cases in which the attributes are complements or substitutes. Note first that

$$\frac{\partial^2 u_p(x, y)}{\partial x \partial y} = \frac{1}{4} (1 - p) x^{p-1} y^{p-1} [0.5x^p + 0.5y^p]^{\frac{1}{p}-2} \quad (5)$$

Thus, we have: $p = 1$ iff $\frac{\partial^2 u_p(x, y)}{\partial x \partial y} = 0$, $p < 1$ iff $\frac{\partial^2 u_p(x, y)}{\partial x \partial y} > 0$ and $p > 1$ iff $\frac{\partial^2 u_p(x, y)}{\partial x \partial y} < 0$.

Given these assumptions, when is there a positive or negative correlation?

Proposition 3 *If $Q(x, y) = 1/(1 + \exp[-su(x, y)])$ and $u(x, y) = u_p(x, y) - c = [0.5x^p + 0.5y^p]^{1/p} - c$ where $c \in \mathbb{R}$, the correlation is positive whenever $p \geq 1$. Moreover, there is a value of $p^* < 1$ such that the correlation becomes negative for $p < p^*$.*

Proof. We have

$$\frac{\partial^2 \ln Q(x, y)}{\partial x \partial y} = -k [e^{r-sc}(r + p - 1) + p - 1], \quad (6)$$

where $r = s \left(\frac{x^p + y^p}{2} \right)^{\frac{1}{p}}$ and

$$k = \frac{rx^{p-1}y^{p-1}}{(1 + e^{r-sc})^2(x^p + y^p)^2} > 0.$$

Now, whenever $p \geq 1$, then $e^{r-sc}(r + p - 1) + p - 1 > 0$ and the correlation is positive by Proposition 1 *i*).

Next, we show that there is a value of $p = p^* < 1$, such that $\frac{\partial^2 \ln Q(x, y)}{\partial x \partial y} > 0$ for all $p < p^*$. Note that $r \in [0, sa]$. By taking p low enough with $p < 0$, $e^{r-sc}(r + p - 1) + p - 1$ can be made negative because r is bounded. Hence, $\frac{\partial^2 \ln Q(x, y)}{\partial x \partial y}$ is positive for p low enough and the correlation is negative by Proposition 1 *ii*). ■

Thus, whether we get a positive or negative correlation depends on how the two attributes interact. If the attributes are substitutes ($\frac{\partial^2 u(x, y)}{\partial x \partial y} \leq 0$), a positive illusory correlation emerges. When the attributes are *strong* complements (p is low enough), a negative illusory correlation emerges. Note that the assumption of complementarity ($\frac{\partial^2 u(x, y)}{\partial x \partial y} > 0$) is not enough to guarantee the emergence of a negative illusory correlation.

The picture is clearer when one consider the extremes cases of disjunctive and conjunctive sampling rules. When the rule is disjunctive, ($u(x, y) = \max(x, y)$), a positive illusory correlation emerges. When the rule is conjunctive ($u(x, y) = \min(x, y)$), a negative illusory correlation emerges. Finally, note that in the special case where the utility is the arithmetic average of the attribute values ($p = 1$), there is a positive illusory correlation (this is a special case of compensatory utility function).

Whereas the above theoretical results are asymptotic, simulations show that illusory correlations tend to emerge pretty quickly. Figure 3 displays the size of the illusory correlation after 10 and 50 periods, as a function of the parameter p . The amplitude of the illusory correlation can be substantial, even after only 10 periods. As discussed above, when p is equal to or higher than 1, the correlation is positive. Also, the higher p (the more compensatory the utility function), the higher the correlation. Similarly, the correlation is negative for low values of p .

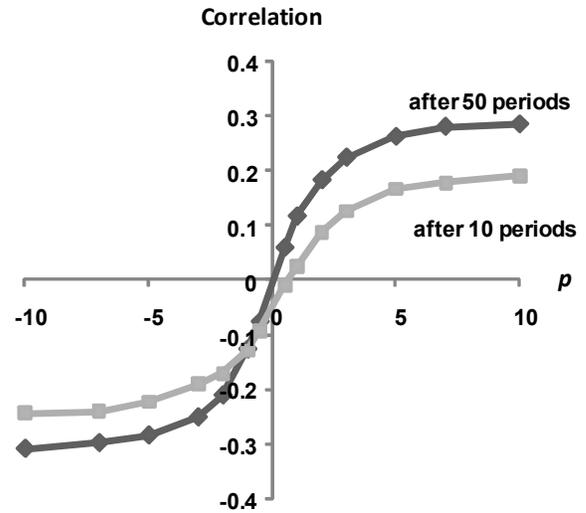


Figure 3: Amplitude of the illusory correlation after 10 periods and 50 periods. Based on 100 000 simulations with x and y uniformly distributed between 0 and 1, $s = 10$, $b = 0.5$ and $c = 0.5$.

Why Do Illusory Correlations Emerge Through Experience Sampling?

In the model analyzed here, illusory correlation emerges because the decision maker bases his or her decision to select an alternative based on his or her beliefs about the values of *both* attributes. In particular, alternatives that are believed to have low values on both attributes are less likely to be selected. Several researchers have demonstrated that in such contexts where beliefs are formed from experience and experiences are random, there is a negativity bias: negative experiences or observations tend to have persistent effects (March, 1996; Fazio, Eiser & Shook, 2004). The reason is that there is an asymmetry in terms of opportunities for correction of estimation errors. To see why, note that an alternative which receives negative observations or experiences is less likely to be selected again, and potential underestimations of the value of that alternative are unlikely to be corrected. When the value of an alternative is overestimated, it is likely to be selected again. When this happens, the decision maker has an opportunity to correct his or her error of overestimation.

Here, the value of an alternative is an increasing function of both attributes. To fix things, suppose that the estimate of the first attribute, \hat{x}_t , is low. If the estimate of the second attribute is also low, then the alternative is unlikely to be selected again, and both estimates are likely to remain low. If, on the contrary, the estimate of the second alternative is high, then the alternative is likely to be selected again. When this happens, the estimate of the first attribute is more likely to increase than to decrease. This is a consequence of regression to the mean.

If the estimate of the first attribute is initially high, the effect of the second attribute works in the opposite direction: a high estimate of the value of the second attribute is more likely to induce a diminution of the estimate of the value of the first attribute.

Whether the illusory correlation is positive or negative therefore depends on which of those two effects is stronger. The relative strengths of the effects depends on $\frac{\partial^2 \ln Q(x,y)}{\partial x \partial y}$, the cross derivative of the probability of sampling. When $\frac{\partial^2 \ln Q(x,y)}{\partial x \partial y} > 0$, then the effect of \hat{y}_t on \hat{x}_t is stronger when \hat{x}_t is low. This corresponds to the situation where a high value of \hat{y}_t tends to ‘pull’ \hat{x}_t up. Because the effect of \hat{x}_t on \hat{y}_t works in a symmetric way, this implies, overall, a positive correlation between the estimates of both attributes.

When $\frac{\partial^2 \ln Q(x,y)}{\partial x \partial y} < 0$, then the effect of \hat{y}_t on \hat{x}_t is stronger when \hat{x}_t is high. This corresponds to the situation where a high value of \hat{y}_t tends to ‘push’ \hat{x}_t down. Overall, this implies that the correlation between the estimates of both attributes will become negative.

Discussion

The model developed in this paper makes only minimal assumptions about the learning mechanism and how information is stored in memory. It is possible to show that, when the belief about the value of an attribute is any weighted average of past observations, the result still holds (as in Denrell and Le Mens, 2007). This suggests that almost any reinforcement learning algorithm would lead to the emergence of illusory correlation, provided that observations of an attribute are contingent on the beliefs about the other attribute.

It is possible to look at the task facing the decision maker in terms of the trade-off between exploration and exploitation. Here, we assume that the decision maker explores, but only to some extent. When he or she believes that the alternative is not attractive, the probability of sampling is reduced. The amplitude of the effect is a function to the sensitivity of the probability of sampling on the beliefs, or in other words, of the propensity to explore in spite of negative beliefs: When the sensitivity is low (i.e. the amount of exploration is high), the size of the illusory correlation is low.

Our results illustrate that it is not necessary that information processing be biased for illusory correlation to emerge. While Fiedler (1991, 2000) also proposed a model where illusory correlation can emerge when information is integrated in an unbiased way, his model assumes that the distributions of the attributes are skewed. Also, Fiedler’s model relies crucially on the fact that the number of observations of each attributes is bounded. Here, we show that illusory correlation can emerge even when the distributions of the attributes are not skewed and if the decision maker makes a very large number of observations.

Rather than challenging existing explanations of illusory correlation, our model suggests that illusory correlation might emerge in contexts where existing theories are not applicable: if the decision maker has to search for information himself or

herself, illusory correlation can emerge even if the assumptions of the other theories are not met.

Positive Illusory Correlation When the Number of Attributes Is $N > 2$.

In this section, we extend the above analysis to settings where the alternatives have more than two attributes. Suppose there are N attributes and that the attributes are independently distributed in the population. Let X_1, \dots, X_N be N independent random variables representing the values of the N attributes.

1. The probability of sampling is $Q(\hat{x}_1, \dots, \hat{x}_N) = P[u(\hat{x}_1, \dots, \hat{x}_N)]$.
2. $u(x_1, \dots, x_N)$ is a function that measures the combined value of the alternative, given the beliefs about the N dimensions. We now assume that $u(x_1, \dots, x_N) = (\frac{1}{N} \sum_{i=1}^N x_i^p)^{1/p} - c$, and $0 < x_i < a$ for all i where $a > 0$.
3. The probability of sampling depends on the ‘combined utility’ through the logistic choice function: $P(u) = 1/(1 + \exp[-su])$.

The function $u_p(x_1, \dots, x_N)$ has many of the same properties as when $n = 2$. It is now possible to state a result about the pairwise correlations between the asymptotic beliefs about the N attributes.

Proposition 4 *Suppose*

$$u(x_1, \dots, x_N) = \left(\frac{1}{N} \sum_{i=1}^N x_i^p \right)^{1/p} - c$$

and

$$Q(x_1, \dots, x_N) = 1/(1 + \exp[-su(x_1, \dots, x_N)]).$$

Then the pairwise correlations between the asymptotic beliefs about the values of the attributes are all positive whenever $p \geq 1$.

Proof. The proof uses results on the relation between joint density and pairwise positive correlation due to Karlin & Rinott (1980). According to Claim 4 in the Appendix, it is sufficient to show that for all j and k , $\partial^2 \ln h(x_1, \dots, x_N) / (\partial x_j \partial x_k) > 0$, where all x_i , $i \neq j, k$, are considered fixed. This is equivalent to $\partial^2 \ln Q(x_1, \dots, x_N) / \partial x_j \partial x_k < 0$. After some algebra, we get:

$$\frac{\partial^2 \ln Q(x_1, \dots, x_N)}{\partial x_j \partial x_k} = -k \left[e^{r-sc} (r + p - 1) + p - 1 \right]$$

where $r = s \left(\frac{1}{N} \sum_{i=1}^N x_i^p \right)^{1/p}$ and $k = r x_j^{p-1} x_k^{p-1} / \left((1 + e^{r-sc}) \sum_{i=1}^N x_i^p \right)^2 > 0$. If $p \geq 1$,

then $\frac{\partial^2 \ln Q(x_1, \dots, x_N)}{\partial x_j \partial x_k} < 0$. Claim 4 in the Appendix implies that all pairwise correlations between asymptotic beliefs are positive. ■

When the number of attributes is $N > 2$, it is much harder to study negative correlation. The reason is that there does not seem to exist a simple condition on the joint density that would imply pairwise negative correlations.

Appendix: On Correlations and Densities

When the Number of Attributes is $N = 2$

Here we state a condition on the joint density function, which is sufficient for a positive correlation.

Definition 1 (Karlin & Rinott, 1980) A non-negative function $f(x, y)$ is totally positive of order 2 (TP_2) if $f(x^*, y^*)f(x, y) \geq f(x^*, y)f(x, y^*)$ for all $x^* > x$ and $y^* > y$.

Claim 1 (Karlin & Rinott, 1980) Two random variables, with joint density $f(x, y)$ are positively correlated, i.e., $E(XY) - E(X)E(Y) > 0$, if $f(x, y)$ is TP_2 .

If $f(x, y)$ is continuously differentiable, there is an easy way to check whether the density is TP_2 . In this case, a non-negative function $f(x, y)$ is totally positive of order 2 (TP_2) iff $\frac{\partial^2 \ln f(x, y)}{\partial x \partial y} > 0$.

Here are conditions on the joint density sufficient for a negative correlation

Definition 2 (Karlin & Rinott, 1980) A non-negative function $f(x, y)$ is reverse rule of order 2 (RR_2) if $f(x^*, y^*)f(x, y) \leq f(x^*, y)f(x, y^*)$ for all $x^* > x$ and $y^* > y$.

Claim 2 (Karlin & Rinott, 1980) Two random variables, with joint density $f(x, y)$ are negatively correlated, i.e., $E(XY) - E(X)E(Y) < 0$, if $f(x, y)$ is RR_2 .

If $f(x, y)$ is continuously differentiable, there is an easy way to check whether the density is RR_2 . In this case, a non-negative function $f(x, y)$ is reverse rule of order 2 (RR_2) iff $\frac{\partial^2 \ln f(x, y)}{\partial x \partial y} < 0$.

When the Number of Attributes is $N \geq 2$

Here we state a condition on the joint density of $N \geq 2$ random variables that is sufficient for the correlation to be positive. In the following, we set: $x = (x_1, \dots, x_N)$ and $y = (y_1, \dots, y_N)$. We start with the following definition:

Definition 3 (Karlin & Rinott, 1980) A non-negative function $f(x_1, \dots, x_N)$ is multivariate totally positive of order 2 (MTP_2) iff $f(x \vee y)f(x \wedge y) \geq f(x)f(y)$ where, $x \vee y = (\max(x_1, y_1), \dots, \max(x_N, y_N))$ and $x \wedge y = (\min(x_1, y_1), \dots, \min(x_N, y_N))$.

This definition is a generalization of the TP_2 condition and it is equivalent to the definition of TP_2 if $N = 2$.

The following implication is useful: if a joint density is MTP_2 then it is MTP_2 also for any subset of two random variables. Remember that MTP_2 for $N = 2$ is equivalent to TP_2 . Thus, we have the following:

Claim 3 If a joint density is MTP_2 then any two random variables are pairwise positively correlated.

If the joint density is continuously differentiable then this condition is equivalent to $\partial^2 \ln f(x_1, \dots, x_N) / \partial x_m \partial x_s > 0$ for all pairs m and s ($m \neq s$). So we have

Claim 4 Consider a joint density $f(x_1, \dots, x_N)$ of a set of random variables, x_1, \dots, x_N . Whenever $\partial^2 \ln f(x_1, \dots, x_N) / \partial x_m \partial x_s > 0$ for all pairs m and s ($m \neq s$), then any two random variables x_i and x_j are positively correlated.

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