Enlargement of filtrations with random times for processes with jumps*

Arturo Kohatsu-Higa and Makoto Yamazato
Universitat Pompeu Fabra and University of the Ryukyus

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Abstract

We treat an extension of Jacod's theorem for initial enlargement of filtrations with respect to random times. In Jacod's theorem the main condition requires the absolute continuity of the conditional distribution of the random time with respect to a non-random measure. Examples appearing in the theory on insider trading require extensions of this theorem where the reference measure can be random. In this article we consider such an extension which leads to an extra term in the semimartingale decomposition in the enlarged filtration. Furthermore we consider a slightly modified enlargement which allows for the semimartingale to have finite moments of the bounded variation part of the semimartingale decomposition depending on the modification considered. Various examples for Lévy processes are treated.

1 Introduction

In a recent article, Corcuera et. al. [5] have introduced a framework to study the behavior of an insider for markets driven by a Wiener process where the additional utility obtained is finite and furthermore the market does not allow for arbitrage. To explain this in detail, suppose that we have a stock market with one asset. Suppose that we have two agents, one which has the information contained in the price itself up to the current time. The other agent is an insider. That is, it possesses information regarding future movements of the stock price (such as the value at some time, the maximum value up to some time, the time at which the maximum will be taken, etc.). The goal is then to characterize the dynamics of the underlying for the insider and to quantify its advantage.

Mathematical models that deal with this situation in a stochastic analysis framework have been considered by Karatzas-Pikovski [16], Imkeller[12], [1], [13], Pontier [9], Grorud [8], Baudoin [2] between others. In all these models, if the information (or sometimes called signal in filtering theory) is a clear signal then the extra utility of the insider up to the revelation time of the signal is infinite. This result is on one side, mathematically evident: That is, the extra utility of the insider is infinite due to the degenerative behavior of the semimartingale decomposition of the Wiener process in the enlarged filtration. On the other hand, this issue restricted the practical interest of these results to the detection of unlawful insiders. The direction taken in Corcuera et. al. is to try to introduce insiders in the market so as to avoid this degenerative behavior but still letting the insider have information about the future. In particular, a model where insiders have dynamical information about an event in future time is provided. In this model the utility of insiders is finite and there is no arbitrage.

To be more specific, suppose that the information of the insider is composed by the signal plus some independent noise that disappears as the revelation time approaches. Then the filtration is being enlarged continuously as time evolves. Then the authors prove that the semimartingale

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decomposition of the driving Wiener process is a projection of the semimartingale decomposition in Jacod's theorem. Finally they apply it to the study of the logarithmic utility of the insider. It is proven that if the rate at which the independent noise disappears is slow enough then the market does not allow arbitrage and logarithmic utility of the insider is finite. This is related to the fact that the Wiener process becomes a square integrable semimartingale in the enlarged filtration.

There is obviously another practical reason to study this type of models. Insiders usually only have an idea of what the future information is. Therefore this modelling is closer to reality than assuming that the insider knows with probability one the value of a certain random variable. Therefore one needs to study "progressive" enlargement of filtration problems.

We are interested in extending the previous application to random times for jump processes. The problem can then be divided into two parts:

First, is it possible to do the enlargement of filtration for random times with an additional perturbing noise? The response to this question is that even in the simple case of random times without any perturbation, Jacod's theorem is not applicable. For example, for a simple Poisson process let T_n denote the time of the n-th jump we have that

$$P(T_n \geq x/\mathcal{F}_t) = 1\{x \leq T_n \leq t\} + 1\{T_n > t\}\int_{x-t}^{\infty} \frac{u^{n-1-N_t}e^{-u}}{(n-1-N_t)!}du.$$

Therefore the conditional law of the random variable T_n is not absolutely continous with respect to a fixed measure but to a random one. This case can not be handled by Jacod's Theorem as stated. In the financial application this corresponds to the insider that knows the time of the n-th jump of an stock price of size bigger than a certain number (see example 21). We propose in Section 4 a reformulation of Jacod's theorem to deal with random times in the framework of jump processes.

In the case that a perturbation of the random time is considered then the question is if an appropiate deformation of the information can give a semimartingale in the enlarged filtration where some moments (particularly the second) of the bounded variation part of the semimartingale decomposition are finite.

The second part deals with the implications of these semimartingale decompositions on markets with insiders driven by Lévy processes. This will be discussed in another article.

In this article we propose an extension of Jacod's theorem in order to apply it in situations where the additional random variable is a random time. Furthermore we extend our results to consider the "progressive" enlargement of filtration mentioned before. Although the definition of progressive enlagement that can be found in the literature does not formally include the situation described here, we preferred to keep using this terminology for the situation described here.

We present our result in two steps. First, in Section 2, we reconsider a new viewpoint of Jacod's theorem which is a slight improvement as it considers a reference measure which is not necessarily deterministic (see the remark after definition 2). This becomes important later when considering the same result for random times. Our formulation is related to the integration by parts formula of Malliavin Calculus. This is also explicitely stated in Theorem 6 and an example is given. Then we also consider the "progressive" enlargement of filtration which is of use in applications of insider modelling. The important issue being the rate of degeneration of the additional drift in the semimartingale decomposition in the enlarged filtration. The point of view taken in this Section, first appeared for the Wiener case in a preliminary version of [5].

Then in Section 3 we use the point of view started in Section 2 to consider an extension of Jacod's Theorem for random times. We then applied to various examples of interest in insider modelling.

2 Expansion of filtration with respect to semimartingales

Let $Z = \{Z_t, 0 \leq t \leq T\}$ be a d dimensional semimartingale defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Here, $(\mathcal{F}_t)_{t\in[0,T]} \equiv \left(\mathcal{F}_t^Z\right)_{t\in[0,T]}$ is the filtration generated by the process Z. We will assume through the article unless stated otherwise that Z satisfies

$$\sup_{t\in[0,T]} E|Z_t| < \infty. \tag{1}$$

Assume that the additional information until time t is given by a family of d dimensional random variables $\{I_s \ , s \le t\}$ which we sometimes call the signal. Suppose that these random variables have the following structure:

$$I_t = G(X, Y_t),$$

where X is an $\mathcal{F}_T^Z$-measurable random variable on $\mathbb{R}^d$, the process $Y = \{Y_t, 0 \le t \le T\}$ is a stochastic process on $\mathbb{R}^d$ adapted to a filtration $\mathcal{H} \supseteq \mathcal{F}$, such that for any integrable random variable $\psi \in \mathcal{F}_T^Y$ and any $s \le T$

$$E(\psi|\mathcal{F}_s^Z \vee \sigma(X)) = E(\psi|X). \tag{2}$$

$G : \mathbb{R}^{2d} \to \mathbb{R}^d$ is a given measurable function. We define $\mathcal{G}_t$ as the smallest filtration, satisfying the usual conditions that contains the filtration $\mathcal{F}_t^Z \vee \sigma(I_s, s \le t)$ (see [22, Section II.67]). Condition (2) states the $X-$conditional independence of Y and Z. This is obviously satisfied if the noise process Y is independent of Z. But this noise process can still depend on X with (2) still satisfied. We assume that this condition is satisfied throughout the article.

Example 1 *Consider $Y_t = Z'(X-t)$ where Z' is a semimartingale independent of Z defined for the "time interval" $(-\infty, \infty)$. This example satisfies (2). In this case note that Y is not independent of Z. For more references on this example, see Proposition 24. Another example is $Y_t = f(X)Z'(T-t)$. In practice this example may represent the case where the level of noise in the information X is determined by the value of the information itself.*

For each $t \in [0, T]$, we denote by $P_t(\omega, dx)$ a regular version of the conditional law of a random variable X given the σ-field $\mathcal{F}_t$, abbreviating it by $P_t(dx)$ if its nature as a measure is emphasized. We can choose this version in such a way that the following conditions are satisfied:

1. For every Borel set B on $\mathbb{R}^d$, $\{P_t(B), t \in [0, T]\}$ is an $(\mathcal{F}_t)_{t\in[0,T]}$-progressively measurable process.
2. For every $(t, \omega) \in [0, T] \times \Omega$, $P_t(\omega, dx)$ is a probability measure on $\mathbb{R}^d$.
3. For any bounded and $(\mathcal{F}_t)_{t\in[0,T]}$-adapted process $h : \Omega \times [0, T] \to \mathbb{R}$ and for any bounded and measurable function $f : \mathbb{R}^d \to \mathbb{R}$, we have

$$E\left(f(X)\int_0^T h_t dt\right) = E\left(\int_0^T \int_{\mathbb{R}^d} f(x)P_t(dx)h_t dt\right).$$

In order to establish the general formula for the compensator, we require the random variable X to belong to a certain class $\mathcal{L}_1$ to be defined below.

Definition 2 *We say that an $\mathcal{F}_T$-measurable random variable X belongs to the class $\mathcal{L}_1$ if there exists a random kernel $P_t^{(1)}(\omega, dx)$ and some deterministic finite signed measure m such that*

1. *For every Borel set B in the real line, $\{P_t^{(1)}(B), t \in [0, T)\}$ is an $(\mathcal{F}_t)_{t\in[0,T)}$-progressively measurable process.*
2. *For every $(t, \omega) \in [0, T) \times \Omega$, $P_t^{(1)}(\omega, dx)$ is a signed measure on the real line.*
3. *For every $t \in [0, T)$, $E\int_0^t \left|P_s^{(1)}\right| |m|(ds) < \infty$, where $|\nu|$ denotes the total variation of the measure ν.*

4. For any bounded and $(\mathcal{F}_t)_{t\in[0,T]}$-adapted process $h : \Omega \times [0,T] \to \mathbb{R}$, for any bounded and measurable function $f : \mathbb{R}^d \to \mathbb{R}$, and for every $t \in [0,T)$, we have

$$E\left(\left(Z_t - Z_s\right) f(X) h_s\right) = E\left(\int_s^t \int_{\mathbb{R}^d} f(x) P_u^{(1)}(dx) m(du) h_s\right).$$

Remark 3 *1. Definition 2 will replace the main condition in Jacod's theorem (see [14]). That is, $P_u << P_X$ will be replaced by $P_u^{(1)} << P_u$. This introduces some advantages as the reference deterministic measure in Jacod's theorem (usually the Lebesgue measure) is not used. In fact, example 21 (n-th jump of the driving process of size bigger than a) is an example where $P_u << P_X$ is not satisfied but an extension of the next theorem will be applicable and therefore the semimartingale decomposition can be obtained.*
2. If we set $f \equiv 1$ in 4. above, then

$$E(Z_t - Z_s|\mathcal{F}_s) = E(\int_s^t P_u^{(1)}(\mathbb{R}^d) m(du)|\mathcal{F}_s).$$

i.e. $Z_t - \int_0^t P_u^{(1)}(\mathbb{R}^d) m(du)$ is an $\mathcal{F}-$ martingale.
3. If $P_t^{(1)}(dx)$ has a Radon-Nikodym derivative $\alpha_t(x)$ with respect to $P_t(dx)$, then

$$E(Z_t - Z_s|\mathcal{F}_s \vee \sigma(X)) = E(\int_s^t \alpha_u(X) m(du)|\mathcal{F}_s \vee \sigma(X)).$$

This implies that $Z_t - \int_0^t \alpha_u(X) m(du)$ is an $\mathcal{F} \vee \sigma(X)$-martingale.
4. One could also consider various extensions of the above definition to suit various other cases. For example, under natural conditions one can consider the case where m is a random signed measure adapted to $\mathcal{F}$. Also condition 2 in Definition 2 can be relaxed using a sequence of localizing stopping times. We preferred this version to simplify the presentation. For more about this, see Section 3.

Theorem 4 *Suppose that Z is a semimartingale satisfying (1) and X is an $\mathcal{F}_T$-measurable random variable in the class $\mathcal{L}_1$ satisfying condition (2). Assume that for almost all $(t,\omega) \in [0,T) \times \Omega$, the signed measure $P_t^{(1)}(dx)$ is absolutely continuous with respect to $P_t(dx)$, and set*

$$\alpha_t(x) = \frac{dP_t^{(1)}}{dP_t}(x).$$

We can choose a version of $\alpha_t(x)$ which is $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable, where $\mathcal{P}$ denotes the $\mathcal{F}_t$-progressive σ-field. Set

$$\beta_t = E(\alpha_t(X)|\mathcal{G}_t),$$

where we consider a progressively measurable version. Then $Z_t - \int_0^t \beta_s dm(s)$ is a martingale with respect to the filtration $(\mathcal{G}_t)_{t\in[0,T)}$.

Proof. First note that due to property 3 in definition 2, we have that $E\int_0^t |\beta_s| dm(s) < \infty$ for all $t \in [0,T)$. We can choose a version of $\alpha_t(x)$ which is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable and adapted process therefore it has a version that is progressively measurable, see Meyer (1966)[19], page 68. Let h be an adapted bounded process and f a bounded measurable function on $\mathbb{R}^{dn}$, and $s_1 \leq \cdots \leq s_n \leq s < t$. Set $F = f(I_{s_1}, \ldots, I_{s_n})$. Denote $Y = (Y_{s_1}, ..., Y_{s_n})$ and $P(y_{s_1}, ..., y_{s_n}|X)$ the conditional probability measure of $Y = (y_{s_1}, ..., y_{s_n})$ conditioned to $\sigma(X)$. Using (2), we have that

$P(y_{s_1},...,y_{s_n}|\mathcal{F}_s \vee \sigma(X)) = P(y_{s_1},...,y_{s_n}|X)$ for any $s \in [0,T)$. Then we have

$$\begin{aligned}
& E\left((Z_t - Z_s)Fh_s\right) \\
= \; & E\left(\int_{\mathbb{R}^d} (Z_t - Z_s) \;\; f(G(X, y_{s_1}), \ldots, G(X, y_{s_n}))dP(y_{s_1},...,y_{s_n}|X)h_s\right) \\
= \; & E\left(\int_s^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(G(x, y_{s_1}), \ldots, G(x, y_{s_n}))dP(y_{s_1},...,y_{s_n}|X=x)P_u^{(1)}(dx)m(du)h_s\right) \\
= \; & E\left(\int_s^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(G(x, y_{s_1}), \ldots, G(x, y_{s_n}))\alpha_u(x)dP(y_{s_1},...,y_{s_n}|X=x)P_u(dx)m(du)h_s\right) \\
= \; & E\left(\int_s^t \int_{\mathbb{R}^d} f(G(X, y_{s_1}), \ldots, G(X, y_{s_n}))\alpha_u(X)dP(y_{s_1},...,y_{s_n}|X)m(du)h_s\right) \\
= \; & E\left(\int_s^t \alpha_u(X)m(du)Fh_s\right). \\
= \; & E\left(\int_s^t E(\alpha_u(X)|\mathcal{G}_u)m(du)Fh_s\right)
\end{aligned}$$

In the last step we have used that the measure m is deterministic and $F \in \mathcal{G}_s$. Therefore $Z_t - \int_0^t \beta_s dm(s)$ is a martingale in the filtration $(\mathcal{G}_t)_{t\in[0,T)}$. ■

One could have also weaken the integrability condition 2 in definition 2 to obtain a local martingale property. Then the proof follows through a localization argument combined with an argument similar to the proof of Theorem 20. This remark is valid for all theorems to follow.

Similarly, if $E\int_0^T |\beta_s|dm(s) < \infty$ then $Z_t - \int_0^t \beta_s dm(s)$ is a martingale in the filtration $(\mathcal{G}_t)_{t\in[0,T]}$. This will be used in some of the examples. In most examples, we will show that all the above conditions are satisfied, compute β and therefore giving a $\mathcal{G}$- martingale in $[0,T)$. Then we may finally discuss if we can close the martingale in the interval $[0,T]$ or if the local martingale property is satisfied without assuming the integrability condition (1).

The following Corollary is the parallel of Jacod's theorem in this setting. The reference measure is P^X. This theorem will not be applicable to the problems appearing in the next section.

Corollary 5 *Let Z be a martingale satisfying (1) and denote by P^X the law of the random variable $X \in \mathcal{F}_T$ satisfying (2). Assume that $P_t \ll P^X$ for any $t \in [0,T)$ a.s., let*

$$p_t(x)(\omega) = \frac{dP_t}{dP^X}(\omega, x).$$

Further assume the existence of a signed measure m such that

$$\alpha_t(x) = \frac{\frac{d<p.(x),Z>}{dm}(t)}{p_t(x)}$$

Suppose that $E\int_0^t |\alpha_s(X)|d|m|(s) < \infty$ for any $t \in [0,T)$. Then $X \in \mathcal{L}_1$ and $Z_t - \int_0^t E(\alpha_s(X)|\mathcal{G}_s)dm(s), 0 \le t < T$ is a martingale with respect to the filtration $(\mathcal{G}_t)_{t\in[0,T)}$.

Proof. It is easy to see that $p_t(x) = \frac{dP_t}{dP^X}(x)$, $t \in [0,T)$, is a martingale for P^X-a.a. x. Then, we have, using the definition of quadratic variation, for a bounded measurable function f and an $\mathcal{F}$ adapted bounded process h,

$$\begin{aligned}
E\left((Z_t - Z_s) f(X)h_s\right) &= E\left(\int_{\mathbb{R}^d} f(x)\frac{dP_t}{dP^X}(x)\left(Z_t - Z_s\right) \; P^X(dx)h_s\right) \\
&= E\left(\int_{\mathbb{R}^d} f(x)\int_s^t d<p.(x), Z>_u \; P^X(dx)h_s\right).
\end{aligned}$$

From here it follows that

$$P_t^{(1)}(dx) = \frac{d<p.(x), Z>}{dm}(t) \; P^X(dx).$$

Notice that $P(p_t(x)=0)=0$ for P^X-a.a. x. In fact,

$$
\begin{aligned}
\int P(p_t(x)=0)P^X(dx) &= E(\mathbf{1}_{\{p_t(X)=0\}}) = E(E(\mathbf{1}_{\{p_t(X)=0\}}|\mathcal{F}_t)) \\
&= E(\int_{\{p_t(x)=0\}} p_t(x)P^X(dx)) = 0.
\end{aligned}
$$

As a consequence, $\alpha_t(x)$ is well defined and it coincides with the Radon-Nikodym density of $P_t^{(1)}(dx)$ with respect to $P_t(dx)$. Therefore the assumptions of the previous proposition are satisfied and we obtain the conclusion. ■

The martingale condition of the above theorem was introduced just to simplify the proof. The general semimartingale case follows considering separately the bounded variation part and the martingale part. The following theorem gives an appropriate form to apply integration by parts formulae in the computation of β.

Theorem 6 *Suppose that X is an $\mathcal{F}_T$-measurable random variable satisfying (2) and Z a martingale satisfying (1) such that there exists a finite deterministic measure m with $\langle Z\rangle << m$ a.s. Furthermore, assume that there exists an $\mathcal{F}_T\otimes\mathcal{B}([0,T))$-measurable process $\xi=\{\xi_t, t\in[0,T)\}$ such that $E(|\xi_t|)<\infty$ and $E\int_0^t|\xi_s|d\langle Z\rangle_s<\infty$ for all $t\in[0,T)$, and that for any measurable and bounded function f we have*

$$f(X)=C_{f(X)}+\int_0^T E\left(f(X)\xi_t|\mathcal{F}_t\right)dZ_t \tag{3}$$

for almost all $(t,\omega)\in[0,T)\times\Omega$ and some constant $C_{f(X)}$. Then, X belongs to the class $\mathcal{L}_1$, the signed measure $P_t^{(1)}(dx)$ is absolutely continuous with respect to $P_t(dx)$ for almost all (t,ω), and the density $\alpha_t(x)=\frac{dP_t^{(1)}}{dP_t}(x)$ satisfies

$$\alpha_t(X)=E(\xi_t|\,\mathcal{F}_t\vee\sigma(X))$$

for almost all $(t,\omega)\in[0,T)\times\Omega$. Moreover, $Z_t-\int_0^t\beta_s d\langle Z\rangle_s$ is a $\mathcal{G}$ martingale for $t\in[0,T)$ where

$$\beta_t:=E(\alpha_t(X)|\mathcal{G}_t)=E(\xi_t|\mathcal{G}_t). \tag{4}$$

Proof. Any $\mathcal{F}_t\vee\sigma(X)$-measurable application from Ω to $\mathbb{R}^d$ can be written in the form $\alpha_t(\omega,X(\omega))$ for an appropriate $\mathcal{F}_t\otimes\mathcal{B}(\mathbb{R}^d)$-measurable application $\alpha_t(\omega,x):\Omega\times\mathbb{R}^d\to\mathbb{R}^d$. Let $\alpha_t(\omega,x)$ be the $\mathcal{F}_t\otimes\mathcal{B}(\mathbb{R}^d)$-measurable application that verifies

$$\alpha_t(X)=E(\xi_t|\,\mathcal{F}_t\vee\sigma(X)),$$

where we omit the explicit dependence on ω. As in Theorem 4 we can choose a version of $\alpha_t(x)$ which is $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable, where $\mathcal{P}$ denotes the progressive σ-field. Setting $P_t^{(1)}(dx)=\alpha_t(x)\frac{d\langle Z\rangle}{dm}(t)P_t(dx)$ we obtain that X belongs to the class $\mathcal{L}_1$. In fact, using (3) and properties of quadratic variation, one has

$$
\begin{aligned}
E\left((Z_t-Z_s)f(X)h_s\right) &= E\left(\int_s^t E\left(f(X)\xi_u|\,\mathcal{F}_u\right)d\langle Z\rangle_u h_s\right) \\
&= E\left(\int_s^t E(E\left(f(X)\xi_u|\mathcal{F}_u\vee\sigma(X)\right)|\,\mathcal{F}_u)\,d\langle Z\rangle_u h_s\right) \\
&= E\left(\int_s^t E(f(X)\alpha_u(X)|\,\mathcal{F}_u)d\langle Z\rangle_u h_s\right) \\
&= E\left(\int_s^t\int_{\mathbb{R}^d} f(x)\alpha_u(x)P_u(dx)\frac{d\langle Z\rangle}{dm}(u)dm(u)h_s\right).
\end{aligned}
$$

Finally, in order to show (4), take $B \in \mathcal{F}_t$, f a bounded measurable function on $\mathbb{R}^{dn}$, $s_1 \leq \cdots \leq s_n \leq t$ and set $F = f(I_{s_1}, \ldots, I_{s_n})$. Then, using the same notation and ideas as in the proof of Theorem 4, we have

$$\begin{aligned} & E\left(\xi_t \mathbf{1}_B F\right) \\ = \; & E\left(\xi_t \mathbf{1}_B f(G(X, Y_{s_1}), \ldots, G(X, Y_{s_n})\right) \\ = \; & E(E(\xi_t | \mathcal{F}_t \vee \sigma(X)) \mathbf{1}_B f(G(X, Y_{s_1}), \ldots, G(X, Y_{s_n})) \\ = \; & E\left(\alpha_t(X) \mathbf{1}_B f(G(X, Y_{s_1}), \ldots, G(X, Y_{s_n})\right) \\ = \; & E\left(\alpha_t(X) \mathbf{1}_B F\right). \end{aligned}$$

This implies (4). ■

In order to apply effectively the previous proposition, one needs an integration by parts (ibp) formula for the process Z. In the Wiener case, examples where developed in [17] For processes with jumps, such a formula is only available in certain cases. In the case that the Lévy measure behaves as a Lebesgue measure, Bichteller, Gravereaux and Jacod [3] have obtained an ibp formula. The case of Poisson driven sde's is treated by Carlen and Pardoux [4]. Just to show its application we consider the following example. For notation, see El-Khatib and Privault, [7].

Example 7 *In this example we consider the model*

$$dS_t = \alpha_t S_t dt + \sigma S_{t-} \beta_{N_t -} dN_t$$

$S_0 = x$. *Here* α_t *and* σ *are deterministic.* β_k, $k = 0, 1, \ldots$ *is a sequence of i.i.d. positive random variables independent of the Poisson process* N *and we consider* $X = \int_0^T S_u du$. *In this setting one has that for any* $f \in C_b^\infty$

$$f(X) = E(f(X)) + \int_0^T E(D_s f(X) / \mathcal{F}_s) d\widetilde{N}_s$$

Here $\widetilde{N}$ *denotes the compensated Poisson process and* D *stands for the stochastic derivative with respect to the jump times. In [7],* D_s *corresponds to* $w(u) = 1(s \leq u)$. *In this case we have, using the chain rule, that (see Proposition 5 in [7])*

$$D_s f(X) = f'(X) \int_s^T \sigma S_{u-} \beta_{N_u -} dN_u$$

Using a conditional form of the integration by parts formula introduced in Proposition 1 of [7], we have that

$$E(D_s f(X) / \mathcal{F}_s) = E\left(f(X) \delta_s^T \left(\frac{w \int_s^T \sigma S_{u-} \beta_{N_u -} dN_u}{D_w X} \right) \Big/ \mathcal{F}_s \right)$$

for any $w \in C_c^1[s, T]$ *such that* $D_w X \neq 0$ *a.s. on* $\int_s^T \sigma S_{u-} \beta_{N_u -} dN_u \neq 0$. *For example, any positive* $w \in C_c^1[s, T]$ *will satisfy these conditions.* δ_s^T *denotes the adjoint operator of* D *defined on* $[s, T]$. *Therefore* $\xi_s = \delta_s^T \left(\frac{w \int_s^T \sigma S_{u-} \beta_{N_u -} dN_u}{D_w X} \right)$. *If we consider an interval* $[s, T']$ *with* $T' < T$ *then the above theorem can be applied directly. This gives the decomposition of* N *as a* $[0, T)$*-semimartingale for* $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(X)$. *In order to consider the problem in* $[0, T]$ *more refined techniques are needed. At any rate, this example shows that there exists a class of random variables* X *for which the previous theorem can be applied.*

In the next proposition we give a formula for β in the case $G(x, y) = x + y$ is an additive function and $Y_t = Z'_{T-t}$ where Z' is an additive process independent of $\{Z_t\}$. Let Q_t be the law of Z'_t.

Theorem 8 *Suppose that the assumptions of Theorem 4 hold. Let for $t \in [0,T)$, $I_t = X + Y_t$. Then there is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$- measurable function $g_t(x,\omega)$ satisfying*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} 1_B(x+y)g_t(x+y)Q_{T-t}(dy)P_t(dx) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} 1_B(x+y)\alpha_t(x)Q_{T-t}(dy)P_t(dx), \tag{5}$$

for any $B \in \mathcal{B}(\mathbb{R}^d)$ and

$$\beta_t(\omega) = g_t(I_t, \omega).$$

Furthermore,
1. *If Q_{T-t} has a density q_{T-t} then we have for $t \in [0,T)$*

$$\beta_t = \frac{\int_{\mathbb{R}^d} \alpha_t(x)\, q_{T-t}(I_t - x)\, P_t(dx)}{\int_{\mathbb{R}^d} q_{T-t}(I_t - x)\, P_t(dx)} = \frac{\int_{\mathbb{R}^d} q_{T-t}(I_t - x)\, P_t^{(1)}(dx)}{\int_{\mathbb{R}^d} q_{T-t}(I_t - x)\, P_t(dx)} \tag{6}$$

2. *If both Q_{T-t} and $P(\cdot|\mathcal{F}_t)$ are discrete distributions with probability functions $q_{T-t}(y)$ and $p_t(x)$, then $P_t^{(1)}(dx)$ is discrete with probability function $p_t^{(1)}(x) = P_t^{(1)}(\{x\})$ and*

$$\beta_t = \frac{\sum \alpha_t(x)\, q_{T-t}(I_t - x)\, p_t(x)}{\sum q_{T-t}(I_t - x)\, p_t(x)} = \frac{\sum q_{T-t}(I_t - x)\, p_t^{(1)}(x)}{\sum q_{T-t}(I_t - x)\, p_t(x)}.$$

Proof. Define measures by

$$\mu_1(B,t,\omega) = \int_{\mathbb{R}^d} 1_B(x+y)Q_{T-t}(dy)P_t(\omega,dx)$$

and

$$\mu_2(B,t,\omega) = \int_{\mathbb{R}^d} \alpha_t(x)1_B(x+y)Q_{T-t}(dy)P_t(\omega,dx)$$

for $B \in \mathcal{B}(\mathbb{R}^d)$. μ_1 and μ_2 are $(\mathcal{F}_t)_{t\in[0,T)}$ progressively measurable for fixed B and μ_2 is absolutely continuous with respect to μ_1 for almost all (t,ω). We define $g_t(\omega,x)$ as a progressively measurable version of the Radon-Nikodym derivative $\frac{d\mu_2}{d\mu_1}(x,t,\omega)$ which therefore satisfies (5).

For $t \in [0,T)$ we may write, using the independence of $\mathcal{F}_T$ and Y

$$\begin{aligned}
\beta_t &= E(\alpha_t(X)|\mathcal{F}_t \vee \sigma(I_s : s \le t)) \\
&= E(\alpha_t(X)|\mathcal{F}_t \vee \sigma(I_t) \vee \sigma(Y_t - Y_s : s \le t)) \\
&= E(\alpha_t(X)|\mathcal{F}_t \vee \sigma(I_t)).
\end{aligned}$$

We have

$$P(I_t \in B|\mathcal{F}_t) = E(1_B(X+Y_t)|\mathcal{F}_t) = \int_{\mathbb{R}^d} 1_B(x+y)Q_{T-t}(dy)P_t(dx)$$

by the independence of $\mathcal{F}_t$ and Y_t. Then, as g is an $\mathcal{F}$-adapted process, we have

$$\begin{aligned}
E(g_t(I_t)1_B(I_t)|\mathcal{F}_t) &= \int_{\mathbb{R}^d} 1_B(x+y)g_t(x+y)Q_{T-t}(dy)P_t(dx) \\
&= \int_{\mathbb{R}^d} 1_B(x+y)\alpha_t(x)Q_{T-t}(dy)P_t(dx) \\
&= E(\alpha_t(X)1_B(I_t)|\mathcal{F}_t).
\end{aligned}$$

This shows that

$$E(\alpha_t(X)|\mathcal{F}_t \vee \sigma(I_t)) = g_t(I_t),$$

a.e. Hence $\beta_t = g_t(I_t)$ a.e. Now we prove (6). If Q has a density q, then

$$g_t(y) = \frac{\int_{\mathbb{R}^d} \alpha_t(x)\, q_{T-t}(y-x)\, P_t(dx)}{\int_{\mathbb{R}^d} q_{T-t}(y-x)\, P_t(dx)} \tag{7}$$

satisfies (5). The second equality is obvious as $\alpha_t(x) = \frac{dP_t^{(1)}}{dP_t}(x)$, $x \in \mathbb{R}^d, t \in [0,T)$.
The proof of 2 is similar to the proof of 1. ■

The reason for the general formulation introduced so far is because we believe it to be more general than the following easier deduction.

Proposition 9 *Let Z be an adapted process in a subfiltration $\mathcal{B} \subseteq \mathcal{A}$. If Z is a semimartingale in $\mathcal{A}$ with a Doob-Meyer decomposition*

$$Z_t = M_t + \int_0^t \alpha_s dm(s)$$

with M a local martingale and m a finite signed measure then Z is also a $\mathcal{B}$-semimartingale with a Doob-Meyer decomposition

$$Z_t = M_t' + \int_0^t E(\alpha_s|\mathcal{B}_s)dm(s).$$

Note that for this approach to work is necessary to know the semimartingale decomposition of Z in a bigger filtration $\mathcal{A}$ in order to obtain the respective decomposition in the smaller filtration. From this point of view is obvious that our previous results are more general than this proposition. The previous proposition is easy to prove and it applies to a certain class of examples. Neverheless as it has been mentioned after Definition 2, the new approach developed in Theorem 4 is necessary in order to build our extension of Jacod's theorem to random times to be developed in Section 4.

3 Examples

In our first example of application we consider the case where Z is a Lévy processes. The following is an extension of an example of enlargement of filtrations with respect to Lévy processes known as Kurtz theorem (see Jacod and Protter [15] , Chaumont and Yor [6] where reference to the concept of harnesses is stressed). In this example, $Y_t = Z'_{T-t}$, $G(x,y) = x+y$. Let $\{Z_t; t \in [0,T]\}$ and $\{Z'_t; t \in [0,T]\}$ be two independent additive processes (that is, processes with independent increments but they are not necessarily stationary). Let $X = Z_T$ and $Y_t = Z'_{T-t}$. Let $E(e^{i\theta Z_t}) = e^{t\psi(\theta)}$ and $E(e^{i\theta Z'_{T-t}}) = e^{g(T-t)\tilde{\psi}(\theta)}$ where

$$\begin{aligned}
\psi(\theta) &= ib\theta - c^2\theta^2/2 + \int (e^{i\theta x} - 1 - i\theta x 1_{\{|x|\le 1\}}(x))\nu(dx), \\
\tilde{\psi}(\theta) &= i\tilde{b}\theta - \tilde{c}^2\theta^2/2 + \int (e^{i\theta x} - 1 - i\theta x 1_{\{|x|\le 1\}}(x))\tilde{\nu}(dx).
\end{aligned}$$

Here $\int |x| \wedge |x|^2 \nu(dx) < \infty$, $\int 1 \wedge |x|^2 \tilde{\nu}(dx) < \infty$ and $g(t)$ is a continuous strictly increasing function. Note that under this hypotheses (1) is satisfied.

Let $R_t(dx) = P(Z_t \in dx)$. Let h_s be an $\mathcal{F}_s$-measurable bounded random variable h_s with $s \le u < t \le T$. Then

$$\begin{aligned}
& E[(Z_t - Z_u)e^{i\theta Z_T} h_s] \\
= \;& E[\int\int \exp\{i\theta(x+y+z+Z_s)\} y R_{T-t}(dx) R_{t-u}(dy) R_{u-s}(dz) h_s] \\
= \;& E[\int \Big(\int e^{i\theta y} y R_{t-u}(dy) \Big) \exp\{i\theta(x+z+Z_s)\} R_{T-t}(dx) R_{u-s}(dz) h_s] \\
= \;& \frac{1}{i}(t-u)\psi'(\theta)\exp\{(T-s)\psi(\theta)\} E[e^{i\theta Z_s} h_s].
\end{aligned}$$

Hence,

$$E[\frac{Z_t - Z_u}{t-u} e^{i\theta Z_T} h_s] = \frac{1}{i}\psi'(\theta)\exp\{(T-s)\psi(\theta)\} E[e^{i\theta Z_s} h_s].$$

Letting $t = T$, and integrating the both sides with respect to du, we have

$$\int_s^t E[\frac{Z_T - Z_u}{T-u} e^{i\theta Z_T} h_s] du = \frac{1}{i}(t-s)\psi'(\theta)\exp\{(T-s)\psi(\theta)\}E[e^{i\theta Z_s} h_s].$$

Hence

$$E[(Z_t - Z_s)e^{i\theta Z_T} h_s] = E[\int_s^t \frac{Z_T - Z_u}{T-u} du e^{i\theta Z_T} h_s].$$

Hence $Z_T \in \mathcal{L}_1$ with $m(du) = du$, $P_u^{(1)}(dx) = \frac{x - Z_u}{T-u} P_u(dx)$, where $P_t(dx)$ is the regular conditional law of Z_T given $\mathcal{F}_t$ and

$$\alpha_t(x) = \frac{x - Z_t}{T-t}.$$

Next, note that $E\int_0^T \left| P_u^{(1)} \right| du = \int_0^T E\left| \frac{Z_T - Z_u}{T-u} \right| du < \infty$. Therefore by 2 of Remark 3 with $X = Z_T$, we have that $Z_t - \int_0^t \alpha_u(Z_T) du$ is a $\mathcal{F} \vee \sigma(Z_T)$-martingale in $[0,T]$. Note that $E\int_0^T \left(\frac{Z_T - Z_u}{T-u}\right)^2 = \infty$. This quantity is of importance when considering applications in insider problems. In particular, if the previous integral is finite it implies that the bounded variation part of the semimartingale decomposition of Z in the enlarged filtration is square integrable. One way to solve this problem is to consider the enlarged filtration $\mathcal{G}$.

Therefore our goal now is to compute as explicitely as possible β using Theorem 8. For this, we have $I_t = Z_T + Z'_{T-t}$, $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(I_s, s \le t)$ and let $L_t = Z_T - Z_t + Z'_{T-t}$. U_{T-t} will denote the law of L_t. As before Q_t denotes the law of Z'_t. We compute first the numerator of β appearing in (6)

$$\begin{aligned} & \int \alpha_t(x) Q_{T-t}(B - x) P_t(dx) \\ = & \int \alpha_t(z + Z_t) Q_{T-t}(B - z - Z_t) R_{T-t}(dz) \\ = & \frac{1}{T-t} \int Q_{T-t}(B - z - Z_t) z R_{T-t}(dz). \end{aligned} \tag{8}$$

On the other hand the denominator of β is:

$$\begin{aligned} \int Q_{T-t}(B - x) P_t(dx) &= \int Q_{T-t}(B - z - Z_t) R_{T-t}(dz) \\ &= U_{T-t}(B - Z_t). \end{aligned}$$

The next two results are classical results on existence of densities for Lévy processes. See Sato, Theorem 27.7 in [23].

Lemma 10 *If either*
1. $c > 0$, *or*
2. $c = 0$ *and* $\nu(dx)$ *has a density* $n(x)$ *and satisfies* $\int_{|x|\le 1} \nu(dx) = \infty$,
then R_{T-t} *has a density* r_{T-t}.

Similarly, one also has the following result.

Lemma 11 *If either*
1. $\tilde{c} > 0$ *or*
2. $\tilde{c} = 0$ *and* $\tilde{\nu}(dx)$ *has a density* $\tilde{n}(x)$ *and satisfies* $\int_{|x|\le 1} \tilde{\nu}(dx) = \infty$,
then Q_{T-t} *has a density* q_{T-t}.

With these results one can find formulas for β in various cases. Let $t \in [0,T)$ and $\tilde{R}_{T-t}(dx) = x R_{T-t}(dx)$.

Theorem 12 *The signed measure $\tilde{R}_{T-t}$ is a finite measure, $Q_{T-t} * \tilde{R}_{T-t}$ is absolutely continuous with respect to U_{T-t} and*

$$\beta_t = \frac{1}{T-t} \frac{d(Q_{T-t} * \tilde{R}_{T-t})}{dU_{T-t}}(L_t).$$

The following are valid : If Q_{T-t} has a density q_{T-t}, then U_{T-t} has a density u_{T-t} and

$$\beta_t = \frac{\int y q_{T-t}(L_t - y) R_{T-t}(dy)}{(T-t) u_{T-t}(L_t)};$$

If R_{T-t} has a density r_{T-t}, then U_{T-t} has a density u_{T-t} and

$$\beta_t = \frac{L_t}{T-t} - \frac{\int y r_{T-t}(L_t - y) Q_{T-t}(dy)}{(T-t) u_{T-t}(L_t)};$$

If both Q_{T-t} and R_{T-t} are discrete with probability functions q_{T-t} and r_{T-t}, then U_{T-t} is discrete with probability function u_{T-t} and

$$\beta_t = \frac{\sum_y y q_{T-t}(L_t - y) r_{T-t}(y)}{(T-t) u_{T-t}(L_t)} = \frac{L_t}{T-t} - \frac{\sum_y y r_{T-t}(L_t - y) q_{T-t}(y)}{(T-t) u_{T-t}(L_t)}.$$

Theorem 13 *If $\tilde{\psi}(\theta) = \psi(\theta)$, then*

$$\beta_t = \frac{L_t}{T - t + g(T-t)}.$$

Proof. We calculate the Fourier transform of the measure in (8).

$$\begin{aligned} & \int e^{i\theta u} \frac{1}{T-t} \int Q_{T-t}(du - z - Z_t) z R_{T-t}(dz) \\ = \;& \frac{1}{i} \psi'(\theta) e^{i\theta Z_t} E(e^{i\theta L_t}) \qquad (9) \\ = \;& \frac{1}{T-t+g(T-t)} \int e^{i\theta u} (u - Z_t) U_{T-t}(du - Z_t) \\ = \;& \int e^{i\theta u} \frac{u - Z_t}{T-t+g(T-t)} \int Q_{T-t}(du - x) P_t(dx). \end{aligned}$$

Hence $g_t(\omega, x) = \frac{x - Z_t(\omega)}{T-t+g(T-t)}$. Therefore by Theorem 8 , we have

$$\beta_t = \frac{L_t}{T - t + g(T-t)}.$$

■

Note that in Theorem 13, densities do not need to exist and the process L_t is not necessarily additive in time as the function g is not necessarily linear. Also, note that in general, L is not a Lévy process. As explained before, note that $E \int_0^T \beta_s^2 ds < \infty$ if $g(T-t) = O((T-t)^a)$ for $a < 1$. Therefore adding Lévy process Z' is justified if we want to obtain that the bounded variation part of the semimartingale decomposition of Z is square integrable.

We now study when we can define the compensator up to T . That is, we specify under which situation β is locally integrable. For this we consider two cases.

Proposition 14 *$Z_t - \int_0^t \beta_s ds$ is a $\mathcal{G}$-martingale in $[0, T]$.*

Proof.

$$\begin{aligned}
& E(|\beta_t|) \\
= \quad & \frac{1}{T-t}\int \Big| \frac{d(Q_{T-t} * \tilde{R}_{T-t})}{dU_{T-t}}(u)\Big| dU_{T-t}(du) \\
\leq \quad & \frac{1}{T-t}\int \{ \int Q_{T-t}(du-x)\} |x| R_{T-t}(dx) \\
= \quad & \frac{1}{T-t}\int |x| R_{T-t}(dx) \\
\leq \quad & \frac{|b|(T-t) + c\sqrt{T-t} + \sqrt{T-t}\int_{\{|z|\leq 1\}} z^2\nu(dz) + (T-t)\int_{\{|z|>1\}} |z|\nu(dz)}{T-t} \\
= \quad & |b| + \int_{\{|z|>1\}} |z|\nu(dz) + \frac{1}{\sqrt{T-t}}\Big(c + \int_{\{|z|\leq 1\}} z^2\nu(dz)\Big)
\end{aligned}$$

Hence

$$\int_0^T E(|\beta_t|)dt < \infty. \tag{10}$$

Therefore we have proved that $Z_t - \int_0^t \beta_s ds$ is a $\mathcal{G}$- martingale in $[0,T]$.

■

Now we reconsider Remark 3.4 and suppose that Z is not necessarily an integrable process.

Proposition 15 *Assume that $\tilde{\psi}(\theta) = \psi(\theta)$ without the assumption $\int_{\{|x|>1\}} |x|\nu(dx) < \infty$. Then Z is a $\mathcal{G}$-semimartingale in $[0,T]$.*

Proof. Define $Z_t^2 = \sum_{s\leq t} \Delta Z_s 1(|\Delta Z_s| > 1)$ and $Z_t^1 = Z_t - Z_t^2$. Note that Z^1 and Z^2 are also independent Lévy processes with $E|Z^1| < \infty$. Regard Z_t^1 as Z_t and $Z_T^2 + Z'_{T-t}$ as Z'_{T-t} in the previous arguments of Theorem 12 and Proposition 14. This gives that Z_t^1 is a semimartingale in the enlarged filtration and a formula for its compensator β. That is, $Z_t^1 - \int_0^t \beta_s ds$ is a $\mathcal{G}$-martingale. As Z_t^2 is adapted to the filtration $\mathcal{G}$ and it is a process of bounded variation the conclusion follows.

■

A similar result can be obtained in the discrete case. In the next proposition we give an alternative expression of β_t.

Proposition 16 *Let $t \in [0,T)$. Suppose any of the following cases:*
1. If either
1a. $c^2 + \tilde{c}^2 > 0$,
1b. $c^2 + \tilde{c}^2 = 0$ and $\liminf_{r\downarrow 0} r^{\alpha-2}\int_{[-r,r]} z^2(\nu+\tilde{\nu})(dz) > 0$ for some $0 < \alpha < 2$ or
1c. $c^2 + \tilde{c}^2 = 0$ and $\nu(dx)$ and $\tilde{\nu}(dx)$ have respective densities $n(x)$ and $\tilde{n}(x)$ such that

$$\lim_{x\downarrow 0} |x| \{n(x) + n(-x) + \tilde{n}(x) + \tilde{n}(-x)\} = \infty,$$

then U_{T-t} has a bounded density u_{T-t} with bounded derivative u'_{T-t} and

$$\beta_t = \frac{\int \{u_{T-t}(L_t - z) - u_{T-t}(L_t)1_{\{|z|\leq 1\}}\} z\nu(dz) - c^2 u'_{T-t}(L_t)}{u_{T-t}(L_t)} + b. \tag{11}$$

2. If $c^2 + \tilde{c}^2 = 0$, $\nu(dx) + \tilde{\nu}(dx)$ is absolutely continuous, $\int_{\{|x|\leq 1\}}(\nu(dx) + \tilde{\nu}(dx)) = \infty$ and $\int_{\{|x|\leq 1\}} |x|(\nu(dx) + \tilde{\nu}(dx)) < \infty$, then U_{T-t} has a density and β satisfies (11) with $c = 0$.
3. If $c^2 + \tilde{c}^2 = 0$, $\int(\nu(dx) + \nu'(dx)) < \infty$ and both ν and $\tilde{\nu}$ are discrete, then U_{T-t} is discrete. Let u_{T-t} be its probability function. Then β satisfies (11) with $c = 0$.
If $b = \int_{|x|\leq 1} x\nu(dx)$ holds in cases (2) and (3), then

$$\beta_t = \frac{\int u_{T-t}(L_t - z) z\nu(dz)}{u_{T-t}(L_t)}.$$

Proof. First we start proving that in any of the cases considered in 1, U_{T-t} has a smooth density with bounded derivatives. In Case 1a, $|E(e^{i\theta L_t})| \leq e^{-(c^2+\tilde{c}^2)\theta^2/2}$. Hence U_{T-t} has a C^∞ density whose all derivatives are bounded.
Case 1b. If $\nu + \tilde{\nu}$ satisfies the assumption of Case 1b, then $(T-t)\nu + g(T-t)\tilde{\nu}$ also satisfies the assumption. Hence

$$\begin{aligned} |E(e^{i\theta L_t})| &\leq \exp\Big[-\int_{1/|\theta|}^{1/|\theta|} (\cos(\theta z) - 1)\{(T-t)\nu + g(T-t)\tilde{\nu}\}(dz)\Big] \\ &\leq \exp\Big[-\frac{1}{2}\int_{-1/|\theta|}^{1/|\theta|} (\theta^2 z^2)\{(T-t)\nu + g(T-t)\tilde{\nu}\}(dz)\Big] \\ &\leq Ce^{-|\theta|^\alpha} \quad \textit{for large } |\theta|, \end{aligned}$$

where C is a positive constant. Then $U_{T-t}(dx)$ has a C^∞ density u_{T-t} whose all derivatives are bounded. See Orey [20]. Case 1c. Let $k \geq 0$. For each $t \in [0,T)$ and $M > k+1$, there is $\delta > 0$ such that $(T-t)\{n(z) + n(-z)\} + g(T-t)\{\tilde{n}(z) + \tilde{n}(-z)\} > M/z$ for $0 < z < \delta$. Hence

$$\begin{aligned} |E(\theta^k e^{i\theta L_t})| &\leq |\theta|^k \exp\{\int_{1/|\theta|}^{\delta} (\cos|\theta|z - 1)\frac{M}{z}dz\} \\ &\leq |\theta|^{k-M} \exp\{M(-\log\delta + \int_1^{|\theta|\delta} \frac{\cos z}{z}dz)\} \\ &\leq C|\theta|^{k-M} \quad \textit{for large } \theta, \end{aligned}$$

where C is a positive constant independent of θ. Hence $U_{T-t}(dx)$ has a C^∞ density u_{T-t} whose all derivatives are bounded.

Since u_{T-t} is bounded and has a bounded derivative in all these cases, we have that

$$\int_{\{|z|\leq 1\}} \{u_{T-t}(x-z) - u_{T-t}(z)\}|z|n(z)dz + \int_{\{|z|>1\}} u_{T-t}(x-z)|z|n(z)dz < \infty$$

for each $x \in \mathbb{R}$. (9) gives

$$\begin{aligned} &\int \alpha_t(x) q_{T-t}(I_t - x)P_t(dx) \\ = &\int \Big\{u_{T-t}(L_t - z) - u_{T-t}(L_t)1_{\{|z|\leq 1\}}\Big\} z\nu(dz) - c^2 u'_{T-t}(L_t) + bu_{T-t}(L_t). \end{aligned}$$

This is easily verified computing the characteristic function on both sides of the previous equality. This shows (11).

In Case 2, U_{T-t} has a density u_{T-t}. Since u_{T-t} and $zn(z)$ are integrable, we have (11) with $c = 0$.
Case 3 is obvious. ■

Remark 17 *Under the assumption of Proposition 16 , we have*

$$\beta_t = \frac{1}{T-t+g(T-t)}\Big[L_t - (\widetilde{b} - b)g(T-t) + (\widetilde{c}^2 - c^2)g(T-t)\frac{u'(L_t)}{u(L_t)}\Big]$$

if $\nu = \widetilde{\nu}$. This shows that harness property does not hold in general in progressive enlargement case.

In the following example we consider the case where the information is given on the form of the number of jumps up of a certain type in the interval $[0,T]$. For this, we introduce notation that will be also used in the next section. We use the representation of a Lévy process using Poisson random measures. In fact, if Z is a Lévy process, then

$$Z_t = bt + cW_t + \int_0^t \int_{|x|\leq 1} x\widetilde{N}(dx, ds) + \int_0^t \int_{|x|>1} xN(dx, ds)$$

where $N(dx, ds)$ denotes the Poisson random measure $\overline{N}(dx, ds) = F(dx)ds$ denotes its compensator and $\widetilde{N}(dx, ds) = N(dx, ds) - \overline{N}(dx, ds)$ denotes the compensated martingale measure. Sometimes we also use the notation N_t to denote certain Poisson processes. The difference should be clear from the context and the subscript.

Example 18 (n *jumps of absolute size bigger than* a *in the interval* $[0, T]$) *Suppose that* $a > 1$. *Let* $N_t = \int_0^t \int_{|x|>a} N(dx, ds)$, $\lambda = E(N_1)$, $X_t = \int_0^t \int_{|x|>a} xN(dx, ds)$ *and* $Y_t = Z_t - X_t$. *In this example,* $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(1\{N_T = n\})$. *Then*

$$Z_t - E(X_1)\Big[\int_0^t \frac{n - N_u}{\lambda(T-u)}du\mathbf{1}\{N_T = n\} + \int_0^t E[\frac{N_T - N_u}{\lambda(T-u)}|\mathcal{G}_u]du\mathbf{1}\{N_T \neq n\}\Big] - E(Y_t)$$

is a $\mathcal{G}$*-local martingale in* $[0, T]$.

Proof. To prove this, consider $0 \leq s \leq u < t \leq T$, and bounded $\mathcal{F}_s$-measurable function h_s,

$$\begin{aligned}
& E[(N_t - N_u)\mathbf{1}\{N_T = n\}h_s] \\
= & \sum_{\ell+m=0}^{n} E[(N_t - N_u)\mathbf{1}\{N_t - N_u = m\}\mathbf{1}\{N_u = \ell\}\mathbf{1}\{N_T - N_t = n - \ell - m\}h_s] \\
= & \sum_{\ell+m=0}^{n} mP(N_t - N_u = m)E[\mathbf{1}\{N_u = \ell\}\mathbf{1}\{N_T - N_t = n - \ell - m\}h_s] \\
= & \sum_{\ell+m=0;m\geq 1}^{n} \lambda(t-u)P(N_t - N_u = m-1)E[\mathbf{1}\{N_u = \ell\}\mathbf{1}\{N_T - N_t = n - \ell - m\}h_s] \\
= & \lambda(t-u)E[\mathbf{1}\{N_T = n-1\}h_s].
\end{aligned}$$

We also have

$$E[(N_t - N_u)\mathbf{1}\{N_T \neq n\}h_s] = \lambda(t-u)E[\mathbf{1}\{N_T \neq n-1\}h_s].$$

Hence

$$\int_s^t E(\frac{N_T - N_u}{T-u}\mathbf{1}\{N_T = n\}h_s)du = \lambda(t-s)E(\mathbf{1}\{N_T = n-1\}h_s)$$

and

$$\int_s^t E(\frac{N_T - N_u}{T-u}\mathbf{1}\{N_T \neq n\}h_s)du = \lambda(t-s)E(\mathbf{1}\{N_T \neq n-1\}h_s).$$

Now, to obtain $E(Z_t|\mathcal{G}_s)$, we compute

$$\begin{aligned}
& E[(Z_t - Z_s)\mathbf{1}\{N_T = n\}h_s] \\
= & E[(X_t - X_s)\mathbf{1}\{N_T = n\}h_s) + E((Y_t - Y_s)\mathbf{1}\{N_T = n\}h_s) \\
= & \sum_{l+m=0}^{n} \int_{|x|>a} xF(dx)mP(N_t - N_s = m)E(\mathbf{1}\{N_s = \ell\}\mathbf{1}\{N_T - N_t = n - \ell - m\}h_s) \\
& +E(Y_t - Y_s)E(\mathbf{1}\{N_T = n\}h_s) \\
= & \sum_{l+m=0;m\geq 1}^{n} \int_{|x|>a} xF(dx)\lambda(t-s)P(N_t - N_s = m-1)E[\mathbf{1}\{N_s = \ell\}\mathbf{1}\{N_T - N_t = n - \ell - m\}h_s] \\
& +E(Y_t - Y_s)E(\mathbf{1}\{N_T = n\}h_s) \\
= & E(X_1)(t-s)E\left(\mathbf{1}\{N_T = n-1\}h_s\right) + E(Y_t - Y_s)E(\mathbf{1}\{N_T = n\}h_s) \\
= & E(X_1)\int_s^t E(\frac{N_T - N_u}{\lambda(T-u)}\mathbf{1}\{N_T = n\}h_s)du + E(Y_t - Y_s)E(\mathbf{1}\{N_T = n\}h_s).
\end{aligned}$$

Similarly, we have

$$\begin{aligned} & E[(Z_t - Z_s)\mathbf{1}\{N_T \neq n\}h_s] \\ = \;& E(X_1)\int_s^t E(\frac{N_T - N_u}{\lambda(T-u)}\mathbf{1}\{N_T \neq n\}h_s)du + E(Y_t - Y_s)E(\mathbf{1}\{N_T \neq n\}h_s). \end{aligned}$$

Finally we obtain that for any function f

$$\begin{aligned} & E[(Z_t - Z_s)f(\mathbf{1}\{N_T = n\})h_s] \\ = \;& E[X_1]\int_s^t E[\frac{N_T - N_u}{\lambda(T-u)}f(\mathbf{1}\{N_T = n\})h_s]du + E[Y_t - Y_s]E[f(\mathbf{1}\{N_T = n\})h_s]. \end{aligned}$$

From here the conclusion follows. Furthermore one also has

$$E(N_T - N_u|\,\mathcal{G}_u) = \begin{cases} n - N_u & if \quad N_T = n \\ \lambda(T-u) & if \quad N_T \neq n \text{ and } N_u > n \\ \lambda(T-u)\,(n - N_u)\,\frac{(n-N_u-1)!-e^{-\lambda(T-u)}(\lambda(T-u))^{n-N_u-1}}{(n-N_u)!-e^{-\lambda(T-u)}(\lambda(T-u))^{n-N_u}} & if \quad N_T \neq n \text{ and } N_u \leq n \end{cases}$$

■

4 Jacod's Theorem for random times

In this section we consider an extension of the previous result to random times. First, we consider a setup for initial enlargement of filtrations. That is, $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\tau)$. In most situations when a random time is considered, one does not have that the measure $P_t^{(1)} << P_t$. Nevertheless this is mostly due to the possible point measure at time τ. Therefore we consider a version of Jacod's theorem which excludes this point. Let $P_t(dx)$ be the regular conditional probability of τ given $\mathcal{F}_t$.

Definition 19 *We say that a random time τ belongs to the class $\mathcal{L}^*$, denoted by $\tau \in \mathcal{L}^*$, if there exists random kernels $P_t^{(i)}(\omega, dx)$, $i = 1, 2$ and a finite deterministic signed measure m such that*

1. *For every Borel set B in the positive real line, $\{P_t^{(i)}(B), t \in [0,T)\}$ is an $(\mathcal{F}_t)_{t\in[0,T)}$-progressively measurable process.*
2. *For every $(t,\omega) \in [0,T) \times \Omega$, $P_t^{(i)}(\omega, dx)$ is a signed measure on the real line.*
3. *For every $t \in [0,T)$, $E\int_0^t \left|P_u^{(i)}\right| |m|(du) < \infty$.*
4. *For any bounded and $(\mathcal{F}_t)_{t\in[0,T]}$-adapted process $h : \Omega \times [0,T] \to \mathbb{R}$, for any bounded and measurable function $f : [0,\infty] \to \mathbb{R}$, and for every $t \in [0,T)$, we have*

$$\begin{aligned} E\left(f(\tau)1(\tau < s)\left(Z_t - Z_s\right)h_s\right) &= E\left(\int_s^t \int_0^s f(x)P_u^{(1)}(dx)m(du)h_s\right) \\ E\left(f(\tau)1(t < \tau)\left(Z_t - Z_s\right)h_s\right) &= E\left(\int_s^t \int_t^T f(x)P_u^{(2)}(dx)m(du)h_s\right). \end{aligned}$$

Theorem 20 *Suppose that τ is a random time in the class $\mathcal{L}^*$ and Z is a semimartingale satisfying (1) such that $E\,|\Delta Z(\tau)| < \infty$. Assume that for almost all (t,ω), the signed measures $P_t^{(i)}(dx)$, $i = 1, 2$ are absolutely continuous with respect to $P_t(dx)$, and set*

$$\alpha_t^{(i)}(x) = \frac{dP_t^{(i)}}{dP_t}(x).$$

We can choose a version of $\alpha_t^{(i)}(x)$ which is $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})$-measurable, where $\mathcal{P}$ denotes the $\mathcal{F}_t$-progressive σ-field. Set the progressively measurable version of the compensator

$$\beta(u) = \alpha_u^{(1)}(\tau)1(u \geq \tau) + \alpha_u^{(2)}(\tau)1(u < \tau).$$

Then $Z_t - \int_0^t \beta(u)m(du) - \Delta Z(\tau)1(t \geq \tau)$ is a martingale with respect to the filtration $(\mathcal{G}_t)_{t\in[0,T)}$.

Proof. We choose versions of $\alpha_t^{(i)}(x)$ which is $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})$-measurable for $i = 1, 2$. Let h be a measurable adapted bounded process and f a bounded measurable function on $\mathbb{R}$. Set $F = f(\tau)$. Then we have

$$\begin{aligned}
& E\left((Z_t - Z_s)F1(t<\tau)h_s\right) = E\left((Z_t - Z_s)f(\tau)1(t<\tau)h_s\right) \\
= \; & E\left(\int_s^t \int_t^T f(x)P_u^{(2)}(dx)m(du)h_s\right) \\
= \; & E\left(\int_s^t \int_t^T f(x)\alpha_u^{(2)}(x)P_u(dx)m(du)h_s\right) \\
= \; & E\left(\int_s^t f(\tau)\alpha_u^{(2)}(\tau)m(du)1(t<\tau)h_s\right) \\
= \; & E\left(\int_s^t \alpha_u^{(2)}(\tau)m(du)F1(t<\tau)h_s\right).
\end{aligned}$$

Similarly, one obtains that

$$E\left((Z_t - Z_s)F1(\tau<s)h_s\right) = E\left(\int_s^t \alpha_u^{(1)}(\tau)m(du)F1(\tau<s)h_s\right).$$

To finish the proof we consider the general case. Let $\pi = \{t_0 < s = t_1 < ... < t_{n-1} = t < t_n\}$ be a partition with $|\pi| = \max\{t_k - t_{k-1}; 1 \leq k \leq n\}$.

$$\begin{aligned}
E\left((Z_t - Z_s)Fh_s\right) \; = \; & E\left(\left(1(\tau \leq t_0)\int_s^t \alpha_u^{(1)}(\tau)m(du) + 1(t_n < \tau)\int_s^t \alpha_u^{(2)}(\tau)m(du)\right)Fh_s\right) \\
& + \sum_{j=1}^{n-2}\sum_{k=0}^{n-1} E\left[(Z_{t_{j+1}} - Z_{t_j})F1(t_k < \tau \leq t_{k+1})h_s\right].
\end{aligned}$$

Let's consider the last term

$$\begin{aligned}
\sum_{j=1}^{n-2}\sum_{k=0}^{n-1} E\left[(Z_{t_{j+1}} - Z_{t_j})F1(t_k < \tau \leq t_{k+1})h_s\right] \; = \; & E\sum_{j<k}\left[(Z_{t_{j+1}} - Z_{t_j})F1(t_k < \tau \leq t_{k+1})h_s\right] \\
& + E\sum_{k=1}^{n-2}\left[(Z_{t_{k+1}} - Z_{t_k})F1(t_k < \tau \leq t_{k+1})h_s\right] \\
& + \sum_{j>k} E\left[(Z_{t_{j+1}} - Z_{t_j})F1(t_k < \tau \leq t_{k+1})h_s\right].
\end{aligned}$$

Now each term can be rewritten as follows:

$$\begin{aligned}
E\left[\sum_{j<k}(Z_{t_{j+1}} - Z_{t_j})F1(t_k < \tau \leq t_{k+1})h_s\right] \; = \; & E\left(\sum_{j<k}\int_{t_j}^{t_{j+1}} \alpha_u^{(2)}(\tau)m(du)F1(t_k < \tau \leq t_{k+1})h_s\right) \\
= \; & E\left(\sum_{j=1}^{n-2}\int_{t_j}^{t_{j+1}} \alpha_u^{(2)}(\tau)m(du)F1(t_{j+1} < \tau \leq t_n)h_s\right) \\
\to \; & E\left(\int_s^t 1(u < \tau \leq t)\alpha_u^{(2)}(\tau)m(du)Fh_s\right),
\end{aligned}$$

$$\sum_{k=1}^{n-2} E\left[(Z_{t_{k+1}} - Z_{t_k})F1(t_k < \tau \le t_{k+1})h_s\right] \to E\left[\Delta Z(\tau)F1(s < \tau \le t)h_s\right]$$

and

$$\begin{aligned}
E\left[\sum_{j>k}(Z_{t_{j+1}} - Z_{t_j})F1(t_k < \tau \le t_{k+1})h_s\right] &= E\left(\sum_{j>k} F1(t_k < \tau \le t_{k+1}) \int_{t_j}^{t_{j+1}} \alpha_u^{(1)}(\tau)m(du)h_s\right) \\
&= E\left(\sum_{j=1}^{n-1} F1(t_0 < \tau \le t_j) \int_{t_j}^{t_{j+1}} \alpha_u^{(1)}(\tau)m(du)h_s\right) \\
&\to E\left(\int_s^t F1(s < \tau \le u)\alpha_u^{(1)}(\tau)m(du)h_s\right)
\end{aligned}$$

as $|\pi| \downarrow 0$. Therefore $Z_t - B(t)$ is a martingale in the filtration $(\mathcal{G}_t)_{t\in[0,T)}$. ■

Now we consider some simple examples of the above situation. One corresponds to a stopping time and the other to a honest time. The first example treats the situation where the filtration is enlarged by the time of the n-th jump of size bigger than $a > 0$ in absolute value. Obviously, the generalization to any set is straightforward. In these examples we use the representation of a Lévy process using Poisson random measures as explained before in Example 18.

Example 21 *(n-th jump of absolute size bigger than a) Let $N_t = \int_0^t \int_{|x|>a} N(dx, ds)$ and let T_n be the n-th jump time of N_t. In this example we have that $\mathcal{F}_t = \sigma(Z_u; u \le t)$ and $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(T_n)$. Further define $X_t = \int_0^t \int_{|x|>a} xN(dx, ds)$ and $Y_t = Z_t - X_t$. Let $\mathcal{F}_t^1 = \sigma(N_u; u \le t)$, $\mathcal{G}_t^1 = \mathcal{F}_t^1 \vee \sigma(T_n)$, $\mathcal{G}_t^2 = \sigma(X_u; u \le t) \vee \sigma(T_n)$ and $\mathcal{G}_t^3 = \sigma(Y_u; u \le t)$. To avoid studying many different casess we assume that $n \ge 2$ and $a \ge 1$. The general case can be obtained doing a careful calculation. Also, as the time of the $n-$th jump of size bigger than a has a range in $(0, \infty)$ we use the extension of the previous theory to this time interval without any further comment.*
By the independence of X and Y, we have

$$\begin{aligned}
E[N_t|\mathcal{G}_s] &= E[N_t|\mathcal{G}_s^1], \\
E[X_t|\mathcal{G}_s] &= E[X_t|\mathcal{G}_s^2], \\
E[Y_t|\mathcal{G}_s] &= E[Y_t|\mathcal{G}_s^3].
\end{aligned}$$

Let $\lambda = E(N_1)$. For $s < t$,

$$E[1(T_n \le t)|\mathcal{F}_s] = P[N_t \ge n|\mathcal{F}_s] = \sum_{k=(n-N_s)\vee 0}^{\infty} \frac{\lambda^k (t-s)^k}{k!} e^{-\lambda(t-s)}.$$

Hence for bounded measurable function f, we have

$$E[f(T_n)|\mathcal{F}_s] = \begin{cases} \int_s^{+\infty} \lambda^{n-N_s} \frac{(u-s)^{n-1-N_s}}{(n-1-N_s)!} e^{-\lambda(u-s)} f(u)du & if\ T_n > s, \\ f(T_n) & if\ T_n \le s. \end{cases}$$

Therefore

$$P_s(dx) = 1(x \le s)\delta_{T_n}(dx) + 1(x > s)\lambda^{n-N_s} \frac{(x-s)^{n-1-N_s}}{(n-1-N_s)!} e^{-\lambda(x-s)} dx$$

First we will compute the measure $P^{(2)}$. For this, consider h an adapted bounded process

$$\begin{aligned}
&E\left((Z_t - Z_s) f(T_n)1(t < T_n)h_s\right) \\
= \;& E\left((X_t - X_s) f(T_n)1(t < T_n)h_s\right) + E\left((Y_t - Y_s) f(T_n)1(t < T_n)h_s\right)
\end{aligned}$$

Here, one easily obtains that

$$E\left((Y_t - Y_s) f(T_n)1(t < T_n)h_s\right) = E(Y_1)(t-s)E\left(f(T_n)1(t < T_n)h_s\right).$$

The first term above is

$$
\begin{aligned}
& E\left(\left(X_t-X_s\right) f(T_n) 1(t<T_n) h_s\right) \\
= \; & E((X_t-X_s)\, E\left(\left. f(T_n)\right| \mathcal{F}_t\right) 1(t<T_n) h_s) \\
= \; & E\left((X_t-X_s) \int_t^{\infty} \lambda^{n-N_t} \frac{(u-t)^{n-1-N_t}}{(n-1-N_t)!} e^{-\lambda(u-t)} f(u) du 1(t<T_n) h_s\right) \\
= \; & E(X_1) E\left((N_t-N_s) \int_t^{\infty} \lambda^{n-N_t} \frac{(u-t)^{n-1-N_t}}{(n-1-N_t)!} e^{-\lambda(u-t)} f(u) du 1(t<T_n) h_s\right) \\
= \; & E(X_1) E\left(\sum_{l=1}^{n-N_s-1} \int_t^{\infty} \lambda^{n-N_s} \frac{(u-t)^{n-1-l-N_s}}{(n-1-l-N_s)!} \frac{(t-s)^l}{(l-1)!} e^{-\lambda(u-s)} f(u) du h_s\right) \\
= \; & E(X_1)(t-s) E\left(\sum_{l=0}^{n-N_s-2} \int_t^{\infty} \lambda^{n-N_s} \frac{(u-t)^{n-2-l-N_s}}{(n-2-l-N_s)!} \frac{(t-s)^l}{l!} e^{-\lambda(u-s)} f(u) du h_s\right).
\end{aligned}
$$

If one uses the same ideas to compute $E\left(\frac{n-1-N_t}{T_n-t} f(T_n) 1(t<T_n) h_s\right)$ *one obtains that*

$$
E(X_1)(t-s) E\left(\frac{n-1-N_t}{T_n-t} f(T_n) 1(t<T_n) h_s\right)=E\left((X_t-X_s) f(T_n) 1(t<T_n) h_s\right)
$$

In particular, supposing that X is a Poisson process, we have that

$$
E\left(\left.(N_t-N_s)\right| \mathcal{G}_s\right) 1(t<T_n)=(t-s) E\left(\left.\frac{n-1-N_t}{T_n-t}\right| \mathcal{G}_s\right) 1(t<T_n).
$$

From here one easily obtains that on the set $\{t<T_n\}$

$$
E\left(\left.\frac{N_t}{T_n-t}-\frac{N_s}{T_n-s}\right| \mathcal{G}_s\right)=\frac{(n-1)(t-s)}{(T_n-s)(T_n-t)}
$$

By this equality, we have that $\frac{n-1-N_u}{T_n-u}$ *is a $\mathcal{G}$-martingale on the set $\{u<T_n\}$. Through a similar calculation one also obtains that*

$$
E\left(f(T_n) 1(t<T_n) \int_s^t E(X_1) \frac{n-1-N_u}{T_n-u} du h_s\right)=E(X_1)(t-s) E\left(f(T_n) 1(t<T_n) \frac{n-1-N_t}{T_n-t} h_s\right).
$$

Therefore we have that $m(du)=du$ and

$$
P_u^{(2)}(dx)=1(u<x)\left(E(X_1) \frac{n-1-N_u}{x-u}+E(Y_1)\right) P_u(dx)
$$

with $E \int_0^{\infty}\left|P_u^{(2)}\right| du<\infty$. *Computing $P^{(1)}$ is easier since*

$$
E\left(f(T_n) 1(T_n<s)\left(Z_t-Z_s\right) h_s\right)=E\left(f(T_n) 1(T_n<s) h_s\right) E\left(Z_t-Z_s\right)
$$

therefore $P_u^{(1)}(dx) \equiv E(Z_1) 1(x \leq u) \delta_{T_n}(dx)$. *From here we easily see that the conditions of Theorem 20 are satisfied and*

$$
Z(t)-\int_0^t\left(\left(E\left(X_1\right) \frac{n-1-N_u}{T_n-u}+E(Y_1)\right) 1(u<T_n)+E(Z_1) 1(u \geq T_n)\right) du-\Delta Z(T_n) 1(t \geq T_n)
$$

is a $\mathcal{G}$ martingale in $[0, T]$ due to the integrability of the compensator. With some further calculation one can also obtain that $E \int_0^T\left|\beta_s\right| ds<\infty$ *and* $E \int_0^T\left|\beta_s\right|^2 ds=\infty$.

The following is a classical example of a random time which is not a stopping time.

Example 22 *(the last jump of absolute size bigger than a before T) Let X_t, Y_t, N_t and T_n be the same as Example 21. Let τ be the last jump time of N_t before T. Let $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\tau)$ and $\mathcal{G}_t^1 = \sigma(N_u; u \le t) \vee \sigma(\tau)$. Let*

$$\tau_t = \inf\{s > 0 : N_t - N_s = 0\}.$$

Then $\tau_T = \tau$ and

$$P(\tau_t \le s) = P(N_t - N_s = 0) = e^{-\lambda(t-s)}.$$

We have for $T \ge v \ge t$, by Markov property,

$$\begin{aligned} E[f(\tau_T)1(\tau_T > v)|\mathcal{F}_t] &= E_{N_t}[f(\tau_{T-t}+t)1(\tau_{T-t} > v-t)] \\ &= E[f(\tau_{T-t}+t)1(\tau_{T-t} > v-t)] \\ &= \lambda \int_v^T e^{-\lambda(T-y)} f(y)dy =: g(v). \end{aligned}$$

Hence, we have

$$P_t(dx) = 1(x \le t)e^{-\lambda(T-t)}\delta_{\tau_t}(dx) + 1(x > t)\lambda e^{-\lambda(T-x)}dx$$

To compute $P^{(2)}$, we consider for $0 < s < t < u$ and for h an adapted bounded process,

$$\begin{aligned} E[(Z_t - Z_s)f(\tau)1(\tau > u)h_s] &= E[(Z_t - Z_s)h_s E[f(\tau)1(\tau > u)|\mathcal{F}_u]] \\ &= E[(Z_t - Z_s)h_s g(u)] \\ &= E[Z_t - Z_s]E[h_s]g(u) \\ &= E[E[Z_t - Z_s]f(\tau)1(\tau > u)h_s] \\ &= E[(t-s)E(Z_1)f(\tau)1(\tau > u)h_s]. \end{aligned}$$

Hence $P_u^{(2)}(dx) = E(Z_1)P_u(dx)$. Next, in order to compute $P^{(1)}$, consider $0 < u < s < t$,

$$E[(X_t - X_s)f(\tau)1(\tau < u)h_s] = 0$$

and

$$\begin{aligned} E[(Y_t - Y_s)f(\tau)1(\tau \le u)h_s] &= E[f(\tau)1(\tau \le u)h_s E[(Y_t - Y_s)]] \\ &= E[(t-s)E(Y_1)f(\tau)1(\tau \le u)h_s]. \end{aligned}$$

Therefore $P_u^{(1)}(dx) = E(Y_1)P_u(dx)$. Finally we have that

$$Z_t - E(Z_1)\,(t \wedge \tau) - E(Y_1)((t \vee \tau) - \tau) - \Delta Z(\tau)1(t \ge \tau)$$

is a $\mathcal{G}$ martingale in $[0,T]$.

The next theorem is an extension of Theorem 20 for progressive enlargement in the form $I(t) = G(\tau, Z'(\tau - t))$. Note this extension is different from the one taken in Proposition 8. From now on, $\mathcal{G}_t$ denotes the smallest filtration that makes Z and I adapted. $\{Z'(t); t \in \mathbb{R}\}$ is a process of independent increments which is independent of Z. In such a case we have the following result.

Theorem 23 *Assume the same conditions as in Theorem 20. Then we have that Z is a semimartingale in the enlarged filtration $\mathcal{G}$ and*

$$Z(t) - \int_0^t E\left(\beta(u)|\,\mathcal{G}_u\right)dm(u) - E\left(\Delta Z(\tau)1(t \ge \tau)|\,\mathcal{G}_t\right)$$

is a $\mathcal{G}$ martingale in $[0,T)$.

Proof. Let $0 \leq s_1 < ... < s_n \leq s < t$ and consider as before

$$\begin{aligned}
& E\left((Z_t - Z_s)F1(t<\tau)h_s\right) \\
= \ & E\left((Z_t - Z_s)f(G(\tau, Z'(\tau - s_1)), \ldots, G(\tau, Z'(\tau - s_n)))1(t<\tau)h_s\right) \\
= \ & E\left(\int_s^t \int_t^T f(G(x, Z'(x - s_1)), \ldots, G(x, Z'(x - s_n)))P_u^{(2)}(dx)m(du)h_s\right) \\
= \ & E\left(\int_s^t \int_t^T f(G(x, Z'(x - s_1)), \ldots, G(x, Z'(x - s_n)))\alpha_u^{(2)}(x)P_u(dx)m(du)h_s\right) \\
= \ & E\left(\int_s^t f(G(\tau, Z'(\tau - s_1)), \ldots, G(\tau, Z'(\tau - s_n)))\alpha_u^{(2)}(\tau)m(du)1(t<\tau)h_s\right) \\
= \ & E\left(\int_s^t E\left(1(t<\tau)\alpha_u^{(2)}(\tau)\Big|\, \mathcal{G}_u\right)m(du)Fh_s\right).
\end{aligned}$$

Therefore repeating the same arguments as in Theorem 20 for the other case $\tau < s < t$, we have that

$$Z_t - \int_0^t E\left(\beta(u)|\, \mathcal{G}_u\right)m(du) - E\left(\Delta Z(\tau)1(t \geq \tau)|\, \mathcal{G}_t\right)$$

is a $\mathcal{G}$ martingale. ■

Now we can also write a result parallel to Theorem 8. Let $P_u(dt)$ be the regular conditional probability of τ given $\mathcal{F}_u$.

Proposition 24 *Assume the same conditions as in Theorem 20. Let $I_t = \tau + Z'(\tau - t)$ and $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(I_u; u \leq t)$. Then there is a $\mathcal{B}(\mathbb{R}) \otimes \mathcal{P}$- measurable functions $g_t^{(i)}(y, \omega)$, $i = 1, 2$ such that*

$$\int_{A^i} \alpha_u^{(i)}(t)Q_{t-u}(B - t)P_u(dt) = \int_{A^i}\Big(\int_B g_u^{(i)}(y)Q_{t-u}(dy - t)\Big)P_u(dt), \tag{12}$$

with $A^1 = (0, u)$ and $A^2 = (u, T)$. Then,

$$E[\beta_u|\mathcal{G}_u] = g_u^{(1)}(I_u)P(u \geq \tau|\mathcal{G}_u) + g_u^{(2)}(I_u)P(u < \tau|\mathcal{G}_u) \ a.s.$$

Moreover, if Q_{t-u} has a density q_{t-u}, then

$$E[\beta_u|\mathcal{G}_u] = \sum_{i=1}^{2} \frac{\int_{A^i} \alpha_u^{(i)}(t)q_{t-u}(I_u - t)P_u(dt)}{\int_{A^i} q_{t-u}(I_u - t)P_u(dt)} P(\tau \in A^i|\mathcal{G}_u) \ a.s. \tag{13}$$

Proof. We do part of the proof for the case $\tau > u$ ($i = 2$). To simplify the notation we delete all reference to the index $i = 2$. The other case follows similarly. Define two measures

$$\mu_1(B, u, \omega) = \int_{(u,T)} Q_{t-u}(B - t)P_u(dt)$$

and

$$\mu_2(B, u, \omega) = \int_{(u,T)} \alpha_u(t)Q_{t-u}(B - t)P_u(dt)$$

for $B \in \mathcal{B}(\mathbb{R})$. Then μ_1 and μ_2 are $(\mathcal{F}_t)$-progressively measurable w.r.t. (u, ω) for fixed B and μ_2 is absolutely continuous with respect to μ_1 for a.a. (u, ω). Then, we can choose a $\mathcal{B}(\mathbb{R}) \otimes \mathcal{P}$-measurable version $g_u(y, \omega)$ of $\frac{d\mu_2}{d\mu_1}(y, u, \omega)$ and g_u satisfies (12). If (12) holds, then for any bounded $\mathcal{B}(\mathbb{R})$ measurable function f,

$$\begin{aligned}
\int_u^T \alpha_u(t)E[f(t + Z'(t - u))]P_u(dt) &= \int_u^T \int \alpha_u(t)f(t + y)Q_{t-u}(dy)P_u(dt) \\
&= \int_u^T \int g_u(t + y)f(t + y)Q_{t-u}(dy)P_u(dt) \\
&= \int_u^T E[g_u(t + Z'(t - u))f(t + Z'(t - u))]P_u(dt).
\end{aligned}$$

For $0 \leq s_1 < s_2 < \cdots < s_n = u$, h an adapted bounded process and bounded measurable functions F_1, F_2, define $\psi(u - s_{n-1}, ..., s_2 - s_1) = EF_2(Z'(t - s_{n-1}) - Z'(t - u), \ldots, Z'(t - s_1) - Z'(t - s_2))$. We have, by the independence of $\mathcal{F}_T$ and Z',

$$\begin{aligned}
& E[1(u < \tau)\alpha_u(\tau)F_1(\tau + Z'(\tau - u))F_2(Z'(\tau - s_{n-1}) - Z'(\tau - u), \ldots, Z'(\tau - s_1) - Z'(\tau - s_2))h_u] \\
= \; & E[\int_u^T \alpha_u(t)E[F_1(t + Z'(t - u))F_2(Z'(t - s_{n-1}) - Z'(t - u), \ldots, Z'(t - s_1) - Z'(t - s_2))]P_u(dt)h_u] \\
= \; & E[\int_u^T \alpha_u(t)E[F_1(t + Z'(t - u))]P_u(dt)h_u]\psi(u - s_{n-1}, ..., s_2 - s_1) \\
= \; & E[\int_u^T E[g_u(t + Z'(t - u))F_1(t + Z'(t - u))]P_u(dt)h_u]\psi(u - s_{n-1}, ..., s_2 - s_1) \\
= \; & E[\int_u^T E[g_u(t + Z'(t - u))F_1(t + Z'(t - u))F_2(Z'(t - s_{n-1}) - Z'(t - u)), \ldots, Z'(t - s_1) - Z'(t - s_2))] \\
& \quad P_u(dt)h_u] \\
= \; & E[g_u(I_u)F_1(I_u)F_2(Z'(\tau - s_{n-1}) - Z'(\tau - u)), \ldots, Z'(\tau - s_1) - Z'(\tau - s_2))h_u 1(u < \tau)]
\end{aligned}$$

Hence, $g_u(I_u)P(u < \tau|\mathcal{G}_u) = E(\alpha_u(\tau)1(u < \tau)|\mathcal{G}_u)$. Now, assume that Q_{t-u} has a density q_{t-u}. Then $g_u(y) = \frac{\int_u^T \alpha_u(t)q_{t-u}(y-t)P_u(dt)}{\int_u^T q_{t-u}(y-t)P_u(dt)}$ satisfies (12). Hence we have (13) after applying Theorem 23. ■

Remark 25 *Note that if $\alpha_u^{(i)}(t) \equiv c_i$ on $t \in A^i$, then $g_u^{(i)}(y) \equiv c_i$ for $i = 1, 2$.*

Let us consider the previous two examples when the filtrations are enlarged through additional perturbing noises. From now on, note that $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(I_u; u \leq t)$ and not like in the previous section.

Example 26 *(n-th jump of absolute size bigger than a. Continuation). If we suppose that the time is perturbed as $I_t = T_n + Z'(T_n - t)$ where Z' is a Lévy process. Then using Theorem 23, the compensator is*

$$\begin{aligned}
& Z(t) - \int_0^t \left(E(X_1)\, E\left(1(u < T_n)\frac{n - 1 - N_u}{T_n - u}\Big|\mathcal{G}_u\right) + E(Y_1)E\left(1(u < T_n)|\,\mathcal{G}_u\right) \right. \\
& \left. + E(Z_1)E\left(1(u \geq T_n)\Big|\mathcal{G}_u\right)\right)du - E\left(\Delta Z(T_n)1(t \geq T_n)\Big|\mathcal{G}_u\right).
\end{aligned}$$

Now we give an explicit expression for the integrand if Z'_t has a density $q_t(\cdot)$. Then using Proposition 24 :

$$E\left(1(u < T_n)\frac{n - 1 - N_u}{T_n - u}\Bigg|\,\mathcal{G}_u\right) = \frac{\lambda P(u, I_u, n - 1)}{P(u, I_u, n)}$$

where

$$P(u, z, n) = \int_u^\infty \frac{(y - u)^{n-1-N_u}}{(n - 1 - N_u)!}e^{-\lambda(y-u)}q_{y-u}(z - y)dy.$$

The result is similar if instead Z' has a discrete distribution.

Similarly, we have

Example 27 (*last jump of absolute size bigger than a before T. Continuation*) *In this case we apply Theorem 23 and Proposition 24 to compute the compensator which in this case becomes*

$$\int_0^t \left\{E(Z_1)P\left(u < \tau|\,\mathcal{G}_u\right) + E(Y_1)P\left(u \geq \tau|\,\mathcal{G}_u\right)\right\}du + E\left(\Delta Z(\tau)1(t \geq \tau)|\,\mathcal{G}_t\right).$$

Nevertheless, in this particular case, from a modelling point of view it maybe difficult to interpret such a model. It may be more sensible to consider the model $I_t = \tau + Z'(T - t)$. This seems to be *the case with all honest times. In this situation we can also combine techniques shown in previous calculation of this example to obtain that*

$$\begin{aligned}&P\left(u<\tau\right|\mathcal{G}_u)\\=\;&\frac{\int_u^T P^\tau(dy)q_{T-u}(I_u-y)}{\int_0^T P^\tau(dy)q_{T-u}(I_u-y)}\\=\;&\frac{\int_u^T e^{-\lambda(T-y)}q_{T-u}(I_u-y)dy}{e^{-\lambda T}q_{T-u}(I_u)+\int_0^T e^{-\lambda(T-y)}q_{T-u}(I_u-y)dy}\end{aligned}$$

and

$$E\left(\Delta Z(\tau)1(t\geq\tau)\right|\mathcal{G}_t)=\Delta Z(\tau_t)P\left(t\geq\tau\right|\mathcal{G}_t).$$

References

[1] Amendinger, J., Imkeller, P., Schweizer, M., 1998. Additional logarithmic utility of an insider. Stochastic Proc. Appl. 75, 263-286.

[2] Baudoin, F., 2003. Modelling anticipations in financial markets, In Paris-Princeton Lectures on Mathematical Finance 2002, Lecture Notes in Mathematics 1814, Springer-Verlag, Berlin.

[3] Bichteler K., Gravereaux J.B., Jacod, J., 1987 : Malliavin Calculus for processes with jumps, Stochastics Monographs 12, Gordon and Breach.

[4] Carlen, E., Pardoux E., 1990 : Differential calculus and integration by parts on Poisson space. In S. Albeverio, Ph. Blanchard and D. Testard, eds. Stochastics, Algebra and Analysis in Classical and Quantum Dynamics (Marseille, 1988), Math. Appl. 59, 63-73 Kluwer Acad. Publ. Dordrecht.

[5] Corcuera J. M., Imkeller P., Kohatsu-Higa A., Nualart D., 2004 : Additional utility of insiders with imperfect dynamical information. Finance and Stochastics, 8, 437-450.

[6] Chaumont L., Yor M., 2004 : Exercises in Probability, Cambridge University Press.

[7] El-Khatib Y., Privault, N., 2004 : Computations of greeks in a market with jumps via the Malliavin Calculus, Finance and Stochastics 8, 161-179.

[8] Grorud, A., 2000 : Asymmetric information in a financial market with jumps. International Journal of Theoretical and Applied Finance, 3, 641-659.

[9] Grorud, A., Pontier, M., 1998 : Insider Trading in a continuous Time Market Model. International Journal of Theoretical and Applied Finance 1, 331-347.

[10] Imkeller, P., 1996 : Enlargement of the Wiener filtration by an absolutely continuous random variable via Malliavin's calculus. Probab. Th. Rel . Fields 106, 105-135.

[11] Imkeller, P., 1997 : Enlargement of the Wiener filtration by a manifold valued random element via Malliavin's calculus. In Statistics and Control of Stochastic Processes, The Lipster Festschrift, Y.M. Kabanov, B.L. Rosovskii, A.N. Shiryaev (eds.), World Scientific, Singapore.

[12] Imkeller, P., 2002 : Random times at which insiders can have free lunches, Preprint.

[13] Imkeller, P., Pontier, M., Weisz, F., 2001 : Free lunch and arbitrage possibilities in a financial market with an insider, Stochastic Proc. Appl. 92, 103-130.

[14] Jacod, J., 1985 : Grossissement initial, hypothèse (H'), et théorème de Girsanov. In : Grossissements de filtrations: exemples et applications, T Jeulin, M. Yor (eds.) Lect. Notes in Maths. 1118, Springer-Verlag. Berlin.

[15] Jacod J., Protter P., 1988 : Time reversal of Lévy processes, Ann. Probab. 16, 620-641.

[16] Karatzas, I., Pikovsky, I., 1996 : Anticipative portfolio optimization, Adv. Appl. Prob. 28, 1095–1122.

[17] Kohatsu-Higa., A., 2004: Enlargement of Filtrations and Models for Insider Trading . In Stochastic Processes and Applications to Mathematical Finance. Proceedings of the Ritsumeikan International Symposium. World Scientific 2004.

[18] Liptser, R.S., Shiryaev, A.N., 1997 : Statistics of Random Processes I. General Theory, Springer-Verlag, New York.

[19] Meyer, P. A., 1996 : Probability and Potentials, Blaisdell Publishing Company, Waltham, Mass.

[20] Orey S., 1968 : On continuity properties of infinitely divisble ditribution functions, Ann. Math. Statist. 39, 936-937.

[21] Protter, P., 1990 : Stochastic Integration and Differential Equations, A New Approach, Springer-Verlag, New York.

[22] Rogers L.C.G., Williams D., 1994: Diffusions, Markov Processes, and Martingales. John Wiley & Sons.

[23] Sato K., 1999 : Lévy processes and infinitely divisible distributions, Cambridge University Press.

[24] Yor, M., 1985 : Grossissement de filtrations et absolue continuite de noyaux. In : Grossissements de filtrations: exemples et applications. T Jeulin, M. Yor (eds.) Lect. Notes in Maths. 1118, Springer-Verlag, Berlin.