Real Analysis–Exam, December 11, 2002.

Please explain every step in detail. With a proper reference you may use anything from class and from the homework exercises.

Problem 1 Consider a set $A \subset \mathbb{R}^n$. Show that the closure of the set obtained by forming all strictly positive convex combinations of any finite subset of A is the convex hull of A.

Problem 2 Let $\mathbb{X}$ be an arbitrary nonempty set and consider $d(p,q) = I_{p \neq q}$ defined for $p, q \in \mathbb{X}$. Show that d is a metric. Determine which sets are open, closed, bounded, and compact.

Problem 3 Let $C \in \mathbb{R}^n$ be a convex set and let $\delta > 0$. Show that if $x_1, ..., x_k \in \mathbb{R}^n$ are such that $D(C, x_i) < \delta$ for all $i = 1, \ldots, k$ then any convex combination y of $x_1, ..., x_k$ satisfies $D(C, y) < \delta$.

Problem 4 Let $\mathbb{X}, \mathbb{Y}$ be metric spaces and let $f : \mathbb{X} \to \mathbb{Y}$ be uniformly continuous. Show that if $\{x_n\}$ is a Cauchy sequence in $\mathbb{X}$ then $\{f(x_n)\}$ is Cauchy in $\mathbb{Y}$. Does the statement remain true if f is continuous on $\mathbb{X}$ but not necessarrily uniformly?

Problem 5 Show that any closed and convex set C in $\mathbb{R}^n$ equals the intersection of all closed halfspaces containing C.

Problem 6 Let $\mathbb{X}$ be a metric space and let $f : \mathbb{X} \to \mathbb{X}$ be a function such that for all $x, y \in \mathbb{X}$,

$$d(f(x), f(y)) < d(x, y) \ .$$

Give an example when f does not have any fixed point. Show that if $\mathbb{X}$ is compact, f has exactly one fixed point. *Hint:* The function $d(x, f(x))$ achieves its minimum on $\mathbb{X}$.

Problem 7 Let f be a real-valued function defined on an open set $E \subset \mathbb{R}^n$, and assume that the partial derivatives $D_1 f, \ldots, D_n f$ are bounded in E. Prove that f is continuous in E.

Problem 8 Prove that there is no continuous one-to-one and onto function f mapping the interval $[0, 1]$ onto the square $[0, 1] \times [0, 1]$.