Real Analysis and Measure Theory–Exam, December 18, 2008.

Please explain every step in detail. With a proper reference you may use anything from class and from the homework exercises.

Problem 1 Let A be an uncountable set and let B be a countable set. Show that $A - B$ is uncountable.

Problem 2 Let $\{p_n\}$ be a Cauchy sequence in a metric space X. Assume that $\{p_n\}$ has a convergent subsequence, converging to some $p \in X$. Prove that $\lim_{n\to\infty} p_n = p$.

Problem 3 Let f be a real-valued function defined on an open interval (a, b). f is *convex* if for all $x, y \in (a, b)$ and $\lambda \in (0, 1)$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \ .$$

Prove that if f is convex then it is continuous on (a, b).

Problem 4 Consider the set C of continuous functions $f : [0, 1] \to [0, 1]$ equipped with the metric

$$d(f, g) = \sup_{x\in[0,1]} |f(x) - g(x)| \ .$$

Show that the obtained metric space is not complete.

Problem 5 Consider the set $L(\mathbb{R}^n)$ of all $n\times n$ square matrices with the distance $d(A, B) = \|A-B\|$. Prove that $L(\mathbb{R}^n)$ is a complete metric space.

Problem 6 Let $\{f_n\}$ be a sequence of functions mapping from a metric space X to a metric space Y. We say that $\{f_n\}$ *converges uniformly* to $f : \mathcal{X} \to Y$ if for every $\epsilon > 0$, there exists an integer N such that for all $n > N$, $\sup_{x\in X} d_Y(f_n(x), f(x)) < \epsilon$. Prove that if f_n is continuous for all n and converges uniformly to f, then f is continuous.

Problem 7 Let $A \in L(\mathbb{R}^n)$ be an $n \times n$ matrix and for each $k = 1, 2, \ldots$ define the matrix

$$M_k = \sum_{i=0}^{k} \frac{1}{i!} A^i$$

Prove that there exists a matrix $M \in L(\mathbb{R}^n)$ such that

$$M = \lim_{k\to\infty} M_k \ .$$

(This is the way of defining the matrix exponential function. M is usually denoted by $M = e^A$.) You may use the fact that $e^x = \lim_{k\to\infty} \sum_{i=0}^{k} \frac{x^i}{i!}$ for all $x \in \mathbb{R}$.

Problem 8 Consider the matrix exponential function $f(A) = e^A$ on $L(\mathbb{R}^n)$ defined in Problem 7. Prove that f is continuous on $L(\mathbb{R}^n)$. ($L(R^n)$ is understood to be equipped with the metric defined by the matrix norm, see Problem 5.) *Hint:* You may use the previous problems of this exam.