Real Analysis and Measure Theory–Exam, December 17, 2009.

Please explain every step in detail. With a proper reference you may use anything from class and from the homework exercises.

Problem 1 Consider a linear transformation $A : \mathbb{R}^n \times \mathbb{R}^n$ whose matrix $(a_{i,j})_{n\times n}$ (in the standard basis) is diagonal, that is, $a_{i,j} = 0$ for all $i \neq j$. Determine the norm of A in terms of the entries of the matrix.

Problem 2 Let $f, g : X \to \mathbb{R}$ be real-valued continuous functions defined on a metric space X. Prove that the function $h(x) = \max(f(x), g(x))$ is continuous.

Problem 3 The *Cartesian product* $A \times B$ of $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ is defined as the subset of $\mathbb{R}^{n+m}$ containing all vectors with components $(x_1, \dots, x_n, y_1, \dots, y_m)$ such that $\mathbf{x} = (x_1, \dots, x_n) \in A$ and $\mathbf{y} = (y_1, \dots, y_m) \in B$. Prove that (a) if A and B are both open then $A \times B$ is open; (b) if A and B are both compact then $A \times B$ is compact. (Here we consider the Euclidean metric.)

Problem 4 A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called *affine* if it can be written as $f(\mathbf{x}) = \mathbf{a} + A\mathbf{x}$ where $\mathbf{a} \in \mathbb{R}^m$ is a fixed vector and $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ is a linear transformation. Prove that if $C \subset \mathbb{R}^n$ is convex then $f(C)$ is a convex subset of $\mathbb{R}^m$. Prove that if $D \subset \mathbb{R}^m$ is convex then $f^{-1}(C)$ is a convex subset of $\mathbb{R}^n$.

Problem 5 Let $f, g : E \to \mathbb{R}^m$ where $E \subset \mathbb{R}^n$ is an open set. Assume that for some $\mathbf{x} \in E$, $f(\mathbf{x}) = 0$, f is differentiable at $\mathbf{x}$, and g is continuous at $\mathbf{x}$. Prove that fg is differentiable at $\mathbf{x}$.

Problem 6 Let X and Y be metric spaces. A function $f : X \to Y$ is called an *open map* if for every open set $G \subset X$, $f(G) = \{f(x) : x \in G\}$ is an open subset of Y. Prove that a function $f : X \to Y$ is an open map if and only if for every $x \in X$ and every $r > 0$, there exists a neighborhood V of $f(x)$ such that $V \subset f(N_r(x))$.

Problem 7 Consider a two-person zero-sum game with payoff matrix $A = (a_{i,j})_{M\times N}$ (for player 1). Let $\mathbf{p} = (p_1, \dots, p_M) \in \mathbb{R}^M$ and $\mathbf{q} = (q_1, \dots, q_N) \in \mathbb{R}^N$ be mixed strategies for the two players such that $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium of the game. Prove that

$$\sum_{i=1}^{M} \sum_{j=1}^{N} p_i q_j a_{i,j}$$

equals the value V of the game. (Recall that V is defined by the common value

$$\sup_{\mathbf{p}'} \inf_{\mathbf{q}'} \sum_{i=1}^{M} \sum_{j=1}^{N} p'_i q'_j a_{i,j} = \inf_{\mathbf{q}'} \sup_{\mathbf{p}'} \sum_{i=1}^{M} \sum_{j=1}^{N} p'_i q'_j a_{i,j}$$

which is guaranteed by Von Neumann's minimax theorem.)

Problem 8 Consider the notation introduced in the previous exercise. Show that the set of Nash equilibria $(\mathbf{p}, \mathbf{q})$ of a two-person zero-sum game is a compact and convex subset of $\mathbb{R}^{M+N}$.