Problem 1 Consider a linear transformation $A: \mathbb{R}^n \times \mathbb{R}^n$ whose matrix $(a_{ij})_{n \times n}$ (in the standard basis) is diagonal, that is, $a_{ij} = 0$ for all $i \neq j$. Determine the norm of $A$ in terms of the entries of the matrix.

Problem 2 Let $f, g : X \to \mathbb{R}$ be real-valued continuous functions defined on a metric space $X$. Prove that the function $h(x) = \max(f(x), g(x))$ is continuous.

Problem 3 The Cartesian product $A \times B$ of $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ is defined as the subset of $\mathbb{R}^{n+m}$ containing all vectors with components $(x_1, \ldots, x_n, y_1, \ldots, y_m)$ such that $x = (x_1, \ldots, x_n) \in A$ and $y = (y_1, \ldots, y_m) \in B$. Prove that (a) if $A$ and $B$ are both open then $A \times B$ is open; (b) if $A$ and $B$ are both compact then $A \times B$ is compact. (Here we consider the Euclidean metric.)

Problem 4 A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called affine if it can be written as $f(x) = a + Ax$ where $a \in \mathbb{R}^m$ is a fixed vector and $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ is a linear transformation. Prove that if $C \subset \mathbb{R}^n$ is convex then $f(C)$ is a convex subset of $\mathbb{R}^m$. Prove that if $D \subset \mathbb{R}^m$ is convex then $f^{-1}(D)$ is a convex subset of $\mathbb{R}^n$.

Problem 5 Let $f, g : E \to \mathbb{R}^m$ where $E \subset \mathbb{R}^n$ is an open set. Assume that for some $x \in E$, $f(x) = 0$, $f$ is differentiable at $x$, and $g$ is continuous at $x$. Prove that $fg$ is differentiable at $x$.

Problem 6 Let $X$ and $Y$ be metric spaces. A function $f : X \to Y$ is called an open map if for every open set $G \subset X$, $f(G) = \{f(x) : x \in G\}$ is an open subset of $Y$. Prove that a function $f : X \to Y$ is an open map if and only if for every $x \in X$ and every $r > 0$, there exists a neighborhood $V$ of $f(x)$ such that $V \subset f(N_r(x))$.

Problem 7 Consider a two-person zero-sum game with payoff matrix $A = (a_{ij})_{M \times N}$ (for player 1). Let $p = (p_1, \ldots, p_M) \in \mathbb{R}^M$ and $q = (q_1, \ldots, q_N) \in \mathbb{R}^N$ be mixed strategies for the two players such that $(p, q)$ is a Nash equilibrium of the game. Prove that

$$\sum_{i=1}^{M} \sum_{j=1}^{N} p_i q_j a_{ij}$$

equals the value $V$ of the game. (Recall that $V$ is defined by the common value

$$\sup_{p'} \inf_{q'} \sum_{i=1}^{M} \sum_{j=1}^{N} p'_i q'_j a_{ij} = \inf_{q'} \sup_{p'} \sum_{i=1}^{M} \sum_{j=1}^{N} p'_i q'_j a_{ij}$$

which is guaranteed by Von Neumann’s minimax theorem.)

Problem 8 Consider the notation introduced in the previous exercise. Show that the set of Nash equilibria $(p, q)$ of a two-person zero-sum game is a compact and convex subset of $\mathbb{R}^{M+N}$.