Mathematics for economics and finance–Exam, December 17, 2010.

Please explain every step in detail. With a proper reference you may use anything from class and from the homework exercises.

Problem 1 Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces. A function $f : \mathcal{X} \to \mathcal{Y}$ is called an *open mapping* if for every open set $G \subset \mathcal{X}$, the set $f(G) = \{y \in \mathcal{Y} : \exists x \in G \text{ such that } f(x) = y\}$ is open. Prove that every homeomorphism $f : \mathcal{X} \to \mathcal{Y}$ is an open mapping.

Problem 2 Consider the set $\mathcal{X}$ of bounded real-valued functions defined on $[0, 1]$ and define the distance of $f, g \in \mathcal{X}$ by $d(f, g) = \sup_{x\in[0,1]} |f(x) - g(x)|$. Let $f_n \in \mathcal{X}$ be a sequence of functions and suppose that $f_n \to f$ for some $f \in \mathcal{X}$ (in the sense of the metric defined above). Prove that for every $x \in [0, 1]$, $\lim_{n\to\infty} f_n(x) = f(x)$.

Problem 3 Let $K \subset \mathbb{R}^k$ be a compact set and for any $r > 0$, define $K_r = \{\mathbf{x} \in \mathbb{R}^k : d(\mathbf{x}, K) \leq r\}$. Show that for every $r > 0$, K_r is compact. (Recall that $d(\mathbf{x}, K) = \inf_{\mathbf{y}\in K} \|\mathbf{x} - \mathbf{y}\|$.)

Problem 4 Let $\mathcal{X}$ and d be defined as in Problem 2. Show that there exists a sequence $f_n \in \mathcal{X}$ and a function $f \in \mathcal{X}$ such that for every $x \in [0, 1]$, $\lim_{n\to\infty} f_n(x) = f(x)$ but f_n does not converge to f.

Problem 5 Let f be a real-valued function defined on $(0, 1]$ and suppose that f is monotone increasing. Prove that $\lim_{x\to 0} f(x)$ exists.

Problem 6 Let $f : E \to \mathbb{R}$ be a real-valued function defined on an open set $E \subset \mathbb{R}^2$ and let $\mathbf{x}_0 \in E$. Assume that $D_1 f(\mathbf{x})$ is continuous at every $\mathbf{x}$ and $D_2 f(\mathbf{x}_0)$ exists. Prove that f is differentiable at $\mathbf{x}_0$.

Problem 7 Again, let $\mathcal{X}$ and d be defined as in Problems 2 and 4. Show that $\mathcal{X}$ is a complete metric space.

Problem 8 Let K and K_r be defined as in Problem 3. Show that if G is an open set such that $K \subset G$ then there exists $r > 0$ such that $K_r \subset G$.