Problem 1 Let $X$ and $Y$ be metric spaces. A function $f : X \to Y$ is called an open mapping if for every open set $G \subset X$, the set $f(G) = \{y \in Y : \exists x \in G$ such that $f(x) = y\}$ is open. Prove that every homeomorphism $f : X \to Y$ is an open mapping.

Problem 2 Consider the set $X$ of bounded real-valued functions defined on $[0, 1]$ and define the distance of $f, g \in X$ by $d(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)|$. Let $f_n \in X$ be a sequence of functions and suppose that $f_n \to f$ for some $f \in X$ (in the sense of the metric defined above). Prove that for every $x \in [0, 1]$, $\lim_{n \to \infty} f_n(x) = f(x)$.

Problem 3 Let $K \subset \mathbb{R}^k$ be a compact set and for any $r > 0$, define $K_r = \{x \in \mathbb{R}^k : d(x, K) \leq r\}$. Show that for every $r > 0$, $K_r$ is compact. (Recall that $d(x, K) = \inf_{y \in K} ||x - y||$.)

Problem 4 Let $X$ and $d$ be defined as in Problem 2. Show that there exists a sequence $f_n \in X$ and a function $f \in X$ such that for every $x \in [0, 1]$, $\lim_{n \to \infty} f_n(x) = f(x)$ but $f_n$ does not converge to $f$.

Problem 5 Let $f$ be a real-valued function defined on $(0, 1]$ and suppose that $f$ is monotone increasing. Prove that $\lim_{x \to 0} f(x)$ exists.

Problem 6 Let $f : E \to \mathbb{R}$ be a real-valued function defined on an open set $E \subset \mathbb{R}^2$ and let $x_0 \in E$. Assume that $D_1 f(x)$ is continuous at every $x$ and $D_2 f(x_0)$ exists. Prove that $f$ is differentiable at $x_0$.

Problem 7 Again, let $X$ and $d$ be defined as in Problems 2 and 4. Show that $X$ is a complete metric space.

Problem 8 Let $K$ and $K_r$ be defined as in Problem 3. Show that if $G$ is an open set such that $K \subset G$ then there exists $r > 0$ such that $K_r \subset G$. 
