Real Analysis and Measure Theory–Exam, December 12, 1996.

Problem 1 Let f be a uniformly continuous function on $(0,1)$. Prove that f is bounded.

Problem 2 Construct an unbounded set whose measure equals one.

Problem 3 Let $\langle x_n \rangle$ be a sequence of real numbers, and let $A \subset \mathcal{R}$ be the set of all cluster points of $\langle x_n \rangle$. Show that A is a closed set.

Problem 4 Let f and g be two real-valued functions defined on the set of real numbers such that $f = g$ a.e. Show that if both f and g are continuous then $f(x) = g(x)$ for every $x \in \mathcal{R}$.

Problem 5 Let $\langle f_n \rangle$ be a sequence of measurable functions defined on $\mathcal{R}$, and let A be the set of points $x \in \mathcal{R}$ at which the sequence of real numbers $\langle f_n(x) \rangle$ is convergent. Show that A is measurable.

Problem 6 Let E be an arbitrary set of real numbers. $x \in \mathcal{R}$ is called an *accumulation point* of E if x is a point of closure of the set $E - \{x\}$. Prove that the set E' of accumulation points of E is a closed set.

Problem 7 A continous function ϕ on $[a,b]$ is called *piecewise linear* if there are finitely many numbers $a = x_0 < x_1 < \cdots < x_n = b$ such that ϕ is linear on each interval $[x_i, x_{i+1}]$, $i = 0, \ldots, n-1$. Let f be an arbitrary continuous function on $[a,b]$, and $\epsilon > 0$. Show that there exists a piecewise linear function ϕ on $[a,b]$ such that $|f(x) - \phi(x)| < \epsilon$ for all $x \in [a,b]$.

Problem 8 Prove the following (stronger) version of Fatou's lemma: if $\langle f_n \rangle$ is a sequence of nonnegative measurable functions then

$$\int \liminf f_n \leq \liminf \int f_n$$

Hint: Use the monotone convergence theorem.

Problem 9 Let $\langle f_n \rangle$ be a sequence of integrable functions, and assume that $f_n \to f$ a.e. for some integrable function f. Show that $\int |f_n - f| \to 0$ if and only if $\int |f_n| \to \int |f|$. Show an example when $\int |f_n - f| \not\to 0$. *Hint:* You may proceed similarly to the proof of Lebesgue's convergence theorem.

Problem 10 Let A be an arbitrary closed set of real numbers. Show that there exists a sequence $\langle x_n \rangle$ of real numbers such that if B denotes the set of all cluster points of $\langle x_n \rangle$, then $B = A$. *Hint:* Assume first that $A \subset [0,1]$.