Real Analysis and Measure Theory–Exam, December 18, 1997.

Please explain every step in detail. With a proper reference you may use anything from class and from the homework exercises.

Problem 1 Let X be a normed linear space. The *unit ball* is the set $B \subset X$ of all elements $f \in X$ satisfying $\|f\| \leq 1$. Show that B is a convex set. Is B a convex body?

Problem 2 A function f defined on $\mathcal{R}$ is said to be *Lipschitz* if there exists a constant $L > 0$ such that for all $x, y \in \mathcal{R}$

$$|f(x) - f(y)| \leq L|x - y|.$$

Prove that

- f is continuous at every $x \in \mathcal{R}$;
- f is uniformly continuous;
- f is absolutely continuous.

Problem 3 Let f be a function such that the set $\{x : f(x) \leq r\}$ is measurable for every rational number r. Show that f is measurable.

Problem 4 Let f be a function defined on $\mathcal{R}$, and assume that f is continuous everywhere except for finitely many points. Show that f is measurable.

Problem 5 Let X denote the set of all Lipschitz functions defined on $[0, 1]$. (For the definition see Problem 2.) Show that X is a linear space. Prove that

$$\|f\| = |f(0)| + \sup_{\substack{x,y\in[0,1] \\ x\neq y}} \frac{|f(x) - f(y)|}{|x - y|}$$

defines a norm.

Problem 6 Let $f \in L^p$, $1 \leq p < \infty$. Show that for every $\epsilon > 0$ there exists a continuous function g such that $\|f - g\|_p < \epsilon$.

Problem 7 Let f be an integrable function, and let $\langle f_n \rangle$ be a sequence of nonnegative measurable functions such that

(i) $\liminf_{n\to\infty} f_n(x) \geq f(x)$ for almost every x;

(ii) $\lim_{n\to\infty} \int f_n = \int f$

Show that $\int |f_n - f|$ converges to zero.

Problem 8 Consider a function $f \in L^\infty$. The *essential range* of f is the set S_f of all real numbers y with the property that for all $\epsilon > 0$

$$m(\{x : |f(x) - y| < \epsilon\}) > 0.$$

Prove that S_f is a compact set.