Real Analysis and Measure Theory–Exam, December 11, 1998.

Please explain every step in detail. With a proper reference you may use anything from class and from the homework exercises.

Problem 1 Let f and g be uniformly continuous functions defined on $\mathcal{R}$. Define the composition of f and g by $h(x) = f(g(x))$. Show that h is uniformly continuous.

Problem 2 Let $\langle f_n \rangle$ be a sequence of nonnegative measurable functions such that for all $x \in \mathcal{R}$, $f_1(x) \geq f_2(x) \geq \cdots$. Assume that f_1 is integrable. Prove that if $f_n \to f$ a.e. then $\int f_n \to \int f$. Show by example that the statement is not necessarily true if f_1 is not integrable.

Problem 3 Let $f_1, f_2, \ldots$ be convex functions such that for some function f, $f_n(x) \to f(x)$ for all $x \in \mathcal{R}$. Show that f is also convex. (A function g is convex if for any $x, y \in \mathcal{R}$ and $\lambda \in [0,1]$, $\lambda g(x) + (1-\lambda)g(y) \geq g(\lambda x + (1-\lambda)y)$.)

Problem 4 Consider the set A of all sequences $\mathbf{x} = (x_1, x_2, \ldots)$ with $\sum_{n=1}^{\infty} x_n^2 < \infty$. Prove that A is a normed linear space with norm $\|\mathbf{x}\| = \left(\sum_{n=1}^{\infty} x_n^2\right)^{1/2}$. Show that $\{\mathbf{x} : \|\mathbf{x}\| \leq 1\}$ and $\{\mathbf{x} : \|\mathbf{x}\| < 1\}$ are both convex subsets of A. Determine the Minkowksi functional of both sets.

Problem 5 Let $\langle f_n \rangle$ be a sequence of measurable functions defined on $[0,1]$ such that $\operatorname{ess\,sup} f_n \leq 3$. Let f be also defined on $[0,1]$. Are the following assertions true of false?

- If $f_n \to f$ in L^∞, then $\operatorname{ess\,sup} f \leq 3$.
- If $f_n \to f$ a.e. then $\operatorname{ess\,sup} f \leq 3$.
- If $f_n \to f$ uniformly, then $\operatorname{ess\,sup} f \leq 3$.

(If a statement is true, prove it, if it isn't, give a counterexample.)

Problem 6 A point $x \in \mathcal{R}$ is called a *condensation point* of a set A if for all $\epsilon > 0$, $A \cap (x-\epsilon, x+\epsilon)$ is uncountable. Denote the set of all condensation points by A^*. Prove that for any set $A \subset R$, $A - A^*$ is countable.

Problem 7 Let $1 \leq p \leq \infty$, and let $f, f_1, f_2, \ldots \in L^p$ be such that $f_n \to f$ in L^p. Prove that f_n converges to f in measure, that is, for all $\epsilon > 0$,

$$\lim_{n\to\infty} m\left(\{x : |f_n(x) - f(x)| > \epsilon\}\right) = 0.$$

Problem 8 Let $A_1, A_2, \ldots$ be measurable subsets of $\mathcal{R}$. Prove that if $\sum_{n=1}^{\infty} m(A_n) < \infty$ then

$$m\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right) = 0.$$

Use this to show that if $f, f_1, f_2, \ldots$ are measurable functions such that for all $\epsilon > 0$

$$\sum_{n=1}^{\infty} m\left(\{x : |f_n(x) - f(x)| > \epsilon\}\right) < \infty,$$

then $f_n \to f$ a.e.