## Real Analysis, 2000–Homework set # 1.

**Problem 1** Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$. Prove that

$$|\mathbf{x}+\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2.$$

Why is this equality called the "parallelogram rule?"

**Problem 2** Let $S$ be a set of real numbers. Show that $\inf S = -\sup S_-$, where $S_- = \{x : -x \in S\}$.

**Problem 3** (*) Assume that the set $B \subset \mathbb{R}$ is bounded below, and let $L$ be the set of all lower bounds of $B$. Show that $L$ is bounded above and $\sup L = \inf B$.

**Problem 4** Let $A$ be a countable set of real numbers. Prove that $A \cup A_-$ is also countable, where $A_- = \{x : -x \in A\}$.

**Problem 5** Prove that the set of all irrational numbers is uncountable.

**Problem 6** Show that the set of all finite sequences from a countable set $A$ is countable.

**Problem 7** Use the previous exercise to show that if $A \subset [0,1]^2$ is the set of points in the unit square with two rational coordinates, then $A$ is countable.

**Problem 8** A real number $x$ is called *algebraic* if it is a root of an algebraic equation, that is, if for some $n$ there exist integers $a_0, a_1, a_2, \dots, a_n$ such that

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

Clearly, all rational numbers (and much more) are algebraic. Prove that the set of algebraic numbers is countable.

**Problem 9** (*) Show that the interval $[0,1]$ is equivalent to the following sets:
- any closed interval $[a,b]$.
- any open interval $(a,b)$.
- the set of real numbers $\mathbb{R}$.
- the set of all points in the plane $\mathbb{R}^2$.
- the set of all lines in $\mathbb{R}^2$.

# Real Analysis, 2000–Homework set # 2.

**Problem 10** Show that a k-cell is convex.

**Problem 11** Let $E$ be an open set. Is every point of $E$ a limit point of $E$? What is the answer if $E$ is closed?

**Problem 12** Let $E^0$ be the set of interior points of a set $E$. $E^0$ is called the *interior* of $E$.
- Prove that $E^0$ is open.
- Prove that $E$ is open if and only if $E^0 = E$.
- Prove that if $G \subset E$ and $G$ is open, then $G \subset E^0$.
- Show by example that the interior of $E$ and the interior of $\bar{E}$ (the closure of $E$) may be different.

**Problem 13** Let $x, y \in \mathbb{R}$, and consider the following functions:

$$\begin{aligned} d_1(x,y) &= (x-y)^2 & d_2(x,y) &= \sqrt{|x-y|} \\ d_3(x,y) &= |x^2 - y^2| & d_4(x,y) &= |x - 2y| \\ d_5(x,y) &= \frac{|x-y|}{1+|x-y|} \end{aligned}$$

Determine for each one whether or not it is a metric.

**Problem 14** Show that every set of the form $\{q : d(p,q) \le r\}$ is closed, where $p$ is a fixed point, and $r > 0$.

**Problem 15** (*) Prove that if $G$ is an open subset of $\mathbb{R}$, then $G$ is the union of countably many disjoint open intervals. *Hint:* For every $x \in G$ take the largest open interval $I_x$ which contains $x$. Show that $G = \cup_{x \in G} I_x$. Show that if $I_x \cap I_y$ is nonempty for $x, y \in G$, then $I_x = I_y$, therefore $G$ is a union of disjoint open intervals. To prove that the union is countable, use the fact that every open interval contains a rational.

**Problem 16** Give an example of a union of infinitely many closed sets which is not closed.

**Problem 17** Consider the following three metrics in $\mathbb{R}^k$:

$$d_1(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{k} |x_i - y_i|, \quad d_2(\mathbf{x},\mathbf{y}) = \sqrt{\sum_{i=1}^{k} (x_i - y_i)^2}, \quad d_\infty(\mathbf{x},\mathbf{y}) = \max_{i=1,\dots,k} |x_i - y_i|,$$

where $\mathbf{x} = (x_1, \dots, x_k)$ and $\mathbf{y} = (y_1, \dots, y_k)$ are two arbitrary points in $\mathbb{R}^k$. Let $E$ be a subset of $\mathbb{R}^k$. Show that the following are equivalent:
- $E$ is open with respect to the metric $d_1$;
- $E$ is open with respect to the metric $d_2$;
- $E$ is open with respect to the metric $d_\infty$.

Show the same thing with the word "open" replaced by "closed" and "bounded". *Remark:* Thus, open, closed, and bounded sets are the same with respect to these three metrics. (In such cases we say that $d_1, d_2$, and $d_\infty$ are *topologically equivalent*.)

# Real Analysis, 2000–Homework set # 3.

**Problem 18** The *diameter* of a set $A$ is defined by

$$\mathrm{diam}(A) = \sup_{x,y \in A} d(x, y).$$

Let $A_1, A_2, \ldots$ be a sequence of closed subsets of the euclidean k-space $\mathbb{R}^k$ such that $A_1 \supset A_2 \supset \cdots$, and $\lim_{n\to\infty} \mathrm{diam}(A_n) = 0$. Prove that the intersection $\cap_{n=1}^{\infty} A_n$ is nonempty.

**Problem 19** Consider the set $\mathbb{Q}$ of rational numbers as a metric space with distance function $d(x, y) = |x - y|$. Let $E$ be a set of all $q \in \mathbb{Q}$ such that $q > 0$, and $2 < q^2 < 3$. Show that $E$ is closed and bounded in $\mathbb{Q}$, but it is not compact. *Remark:* This illustrates that unlike in $\mathbb{R}^k$, it is not true in general that closed and bounded sets are compact.

**Problem 20** (*) Prove that a subset $K$ of a metric space is compact if and only if for every collection $\{F_\alpha\}$ of closed subsets of $K$ such that any finite subcollection has a nonempty intersection, $\cap_\alpha F_\alpha \neq \emptyset$.

**Problem 21** (*) A point $x \in \mathbb{R}$ is called a *condensation point* of a set $A$ if for all $\epsilon > 0$, $A \cap (x - \epsilon, x + \epsilon)$ is uncountable. Denote the set of all condensation points by $A^*$. Prove that for any set $A \subset R$, $A - A^*$ is countable.

**Problem 22** Let $\{s_n\}$ and $\{t_n\}$ be sequences of real numbers such that $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$, and assume that $t, t_n \neq 0$. Prove that $\lim_{n\to\infty} s_n/t_n = s/t$.

**Problem 23** Let $\{x_n\}$ and $\{y_n\}$ be sequences in a metric space such that $x_n \to x$ and $y_n \to y$. Prove that the sequence of real numbers $\{d(x_n, y_n)\}$ converges to $d(x, y)$.

**Problem 24** Create a sequence of real numbers such that the set of its subsequential limits is the set of all positive integers.

**Problem 25** Show a sequence of real numbers such that the set of its subsequential limits is the interval $[0, 1]$. Can you do the same with $(0, 1)$?

## Real Analysis, 2000–Homework set # 4.

**Problem 26** Let $\{s_n\}$ and $\{t_n\}$ be sequences of real numbers. Show that

$$\liminf_{n\to\infty} s_n + \limsup_{n\to\infty} t_n \leq \limsup_{n\to\infty}(s_n + t_n) \leq \limsup_{n\to\infty} s_n + \limsup_{n\to\infty} t_n.$$

**Problem 27** Let $\{s_n\}$ be a sequence of real numbers, and define

$$a_n = \frac{s_1 + s_2 + \cdots + s_n}{n}.$$

Prove that if $\lim_{n\to\infty} s_n = s$, then $\lim_{n\to\infty} a_n = s$. Show that the converse is not true: there exists a divergent sequence $\{s_n\}$ such that $\lim_{n\to\infty} a_n = 0$.

**Problem 28** Let $f$ be a function defined on $\mathbb{R}$ such that for every $x \in \mathbb{R}$, and for every sequence $\langle a_n \rangle$ converging to zero

$$\lim_{n\to\infty} f(x + a_n) - f(x - a_n) = 0.$$

Does this imply that $f$ is continuous?

**Problem 29** Consider the function

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Determine the points at which $f$ is continuous.

**Problem 30** Let $f$ be a continuous function defined on $[0, 1]$ such that for all $x \in [0, 1]$, we have $f(x) \in [0, 1]$. Show that for some $x \in [0, 1]$, $f(x) = x$. (In other words, $f$ has a fixed point.)

**Problem 31** Let $f$ be a continuous function on $\mathbb{R}$. Prove that the set $\{x : f(x) = 0\}$ is closed.

**Problem 32** (*) Let $f$ be a function defined on $\mathbb{R}$ such that for every $a$ and $b$, if $f(a) < c < f(b)$, then $f(x) = c$ for some $x$ between $a$ and $b$. Suppose also that for every rational number $r$, the set $\{x : f(x) = r\}$ is closed. Prove that $f$ is continuous. *Hint:* If $x_n \to x$ but $f(x_n) > r > f(x)$ for some rational $r$, then for some $t_n$ between $x_n$ and $x$, $f(t_n) = r$.

**Problem 33** Let $X, Y, Z$ be metric spaces, and assume that $f : X \to Y$ and $g : Y \to Z$ are uniformly continuous functions. Prove that $h = g \circ f : X \to Z$ is also uniformly continuous.

**Problem 34** (*) Let $A$ be a subset of a metric space $X$, and let $p \in X$. Define the *distance* between $p$ and $A$ by

$$D(A, p) = \inf_{q\in A} d(q, p).$$

Show that
(a) If $p \in A$ then $D(A, p) = 0$.
(b) $D(A, p) = 0$ if and only if $p \in \overline{A}$.
(c) For any fixed $A$, $D(A, p)$ is a continuous function of $p$.
(d) If $X = \mathbb{R}^k$, we define the sum of the sets $A, B \subset \mathbb{R}^k$ by $A + B = \{x + y : x \in A \text{ and } y \in B\}$. Show that $D(A, p) \leq 1$ if and only if $p \in \overline{A + N_1(\mathbf{0})}$.

# Real Analysis, 2000–Homework set # 5.

**Problem 35** Let $\{C_\alpha\}$ be a family of convex sets in $\mathbb{R}^k$. Show that $\bigcap_\alpha C_\alpha$ is convex.

**Problem 36** Show that if $C$ and $D$ are convex sets in $\mathbb{R}^k$, and $a$ is a real number, then $C + D = \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in C \text{ and } \mathbf{y} \in D\}$ and $aC = \{a\mathbf{x} : \mathbf{x} \in C\}$ are convex.

**Problem 37** Show that if $A$ is an open set in $\mathbb{R}^k$, then $\mathrm{co}(A)$ is open.

**Problem 38** Prove that if $C$ is a convex set, then the closure $\bar{C}$ and the interior $C^\circ$ of $C$ are also convex.

**Problem 39** (*) Let $A \subset \mathbb{R}^k$. Show that for any $\mathbf{x} \in \mathrm{co}(A)$ there exist $k+1$ points $\mathbf{x}_1, \ldots, \mathbf{x}_{k+1} \in A$ such that $\mathbf{x}$ equals a convex combination of these $k+1$ points.

**Problem 40** (*) Use the previous exercise to prove that if $A$ is a compact set in $\mathbb{R}^k$, then $\mathrm{co}(A)$ is also compact.

**Problem 41** (*) Let $F \subset \mathbb{R}^k$ be a closed and convex set, and let $\mathbf{x} \in \mathbb{R}^k$. We have seen in class that there exists a unique point $\mathbf{y} \in F$ such that

$$|\mathbf{x} - \mathbf{y}| = D(F, \mathbf{x}).$$

This defines a function $g : \mathbb{R}^k \to F$ by $|\mathbf{x} - g(\mathbf{x})| = D(F, \mathbf{x})$. Show that $g$ is continuous.

**Problem 42** Prove that a closed half space is a closed set, and an open half space is an open set.

**Problem 43** A real-valued function defined on a subset $E$ of a metric space $X$ is called *upper semicontinuous* if for each $a \in \mathbb{R}$, the set $\{x \in E : f(x) < a\}$ is open in $E$. A function $f : E \to \mathbb{R}$ is *lower semicontinuous* if $-f$ is upper semicontinuous. Show that a function $f : E \to \mathbb{R}$ is continuous on $E$ if and only if it is both upper and lower semicontinuous.

**Problem 44** Let $K$ be a compact set, and let $f : K \to \mathbb{R}$ be upper semicontinuous. Show that $f$ achieves its maximum on $K$.

**Problem 45** Show that any contraction is uniformly continuous.

**Problem 46** (*) Exhibit a function $f : X \to X$, where $X$ is a complete metric space such that for all $x, y \in X$,

$$d(f(x), f(y)) < d(x, y) \quad (x \neq y),$$

but $f$ does not have any fixed point. Compare this with the fixed point theorem for contractions seen in class.

## Real Analysis, 2000–Homework set # 6.

**Problem 47** Show that if $x_1, x_2, \dots, x_n$ are affinely independent vectors in $\mathbb{R}^k$, then $n \leq k+1$.

**Problem 48** Show that if $x_1, x_2, \dots, x_n$ are linearly independent vectors, then there exists a vector $x_{n+1}$ such that $x_1, x_2, \dots, x_n, x_{n+1}$ are affinely independent.

**Problem 49** Let $f : A \to \mathbb{R}^k$ be a continuous function defined on a set $E \subset \mathbb{R}^k$. Prove that the set of fixed points of $f$ is closed. If $E$ is compact then the set of fixed points of $f$ is also compact.

**Problem 50** Show that if a metric space $X$ is homeomorphic to the metric space $Y$ then $Y$ is homeomorphic to $X$.

**Problem 51** Let $h : X \to Y$ be a homeomorphism. Show that
(a) if $A \subset X$ is open, then $B = \{y \in Y : \exists x \in X \text{ such that } y = h(x)\}$ is open.
(b) if $A \subset X$ is closed, then $B = \{y \in Y : \exists x \in X \text{ such that } y = h(x)\}$ is closed.
(d) if $A \subset X$ is compact, then $B = \{y \in Y : \exists x \in X \text{ such that } y = h(x)\}$ is compact.
Remark: This exercise shows that two homeomorphic sets have the same metric structure.

**Problem 52** Exhibit a nonconvex compact set $K \subset \mathbb{R}^k$ and a continuous function $f : K \to K$ such that $f$ does not have a fixed point.

**Problem 53** Exhibit a convex closed set $C \subset \mathbb{R}^k$ and a continuous function $f : C \to C$ such that $f$ does not have a fixed point.

**Problem 54** Exhibit a convex bounded set $B \subset \mathbb{R}^k$ and a continuous function $f : B \to B$ such that $f$ does not have a fixed point.

**Problem 55** Let $f : \Delta_n \to \mathbb{R}^{n+1}$ be a continuous function satisfying the strong form of Walras' law $\mathbf{p} \cdot f(\mathbf{p}) = 0$ for all price vectors $\mathbf{p} \in \Delta_n$. Then, by the equilibrium theorem seen in class, there exists a price vector $\mathbf{q} \in \Delta_n$ such that $f(\mathbf{q}) \leq \mathbf{0}$. Show that if for some commodity $i$ the excess demand is negative, that is, if $f_i(\mathbf{q}) < 0$, then its price $q_i$ must be zero.

**Problem 56** Show that if $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a basis of a vector space $X$, then each $\mathbf{x} \in X$ can be uniquely written as a linear combination of the basis vectors.

**Problem 57** Let $A \in L(X, Y)$ be a linear transformation such that $A\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$. Show that $A$ is one-to-one.

**Problem 58** (*) Let $A \in L(\mathbb{R}^n, \mathbb{R})$ be a real-valued linear transformation. Prove that there exists a vector $\mathbf{y} \in \mathbb{R}^n$ such that for all $\mathbf{x} \in \mathbb{R}^n$, $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$. Show that $||A|| = |\mathbf{y}|$.

**Problem 59** Let $c_1, \ldots, c_n$ be fixed real numbers. Show that the set of vectors $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ satisfying $\sum_{i=1}^n c_i x_i = 0$ forms an $n-1$-dimensional vector space.

**Problem 60** Let $X$ and $Y$ be two vector spaces in $\mathbb{R}^n$ such that $X \bigcap Y = \mathbf{0}$. Prove that $\dim(X) + \dim(Y) \leq n$.

**Problem 61** Show that for $A \in L(\mathbb{R}^n, \mathbb{R}^m)$,

$$||A|| = \sup_{\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0}} \frac{|A\mathbf{x}|}{|\mathbf{x}|}.$$

**Problem 62** Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation given by its matrix $A = (a_{i,j})_{n \times n}$ (in the standard basis). Consider the distance

$$d(\mathbf{x}, \mathbf{y}) = \max_{i \leq n} |x_i - y_i| \qquad \text{for } \mathbf{x} = (x_1, \ldots, x_n) \text{ and } \ \mathbf{y} = (y_1, \ldots, y_n).$$

Show that if there is a constant $\alpha < 1$ such that for each $i = 1, 2, \ldots, n$

$$\sum_{j=1}^n |a_{i,j}| \leq \alpha,$$

then $T$ is a contraction. Can you derive a similar sufficient condition for contractiveness under the distance $d(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n |x_i - y_i|$? (Recall that contractivity under *any* distance implies the existence of a unique fixed point.)

**Problem 63** (*) Consider a linear transformation $A \in L(\mathbb{R}^n, \mathbb{R})$, and let $\mathbf{x}_0 \in \mathbb{R}^n$ be any fixed vector such that $A\mathbf{x}_0 \neq 0$. Show that every $\mathbf{x} \in \mathbb{R}^n$ may be represented *uniquely* by a pair $(\alpha, \mathbf{y})$, where $\alpha \in \mathbb{R}$, $\mathbf{y} \in \mathbb{R}^n$ is such that $A\mathbf{y} = 0$, and

$$\mathbf{x} = \alpha\mathbf{x}_0 + \mathbf{y}.$$

# Real Analysis, 2000–Homework set # 8.

**Problem 64** Define $f : \mathbb{R}^2 \to \mathbb{R}$ as

$$f(x,y) = \frac{xy}{x^2+y^2} \quad \text{if} \quad (x,y) \neq (0,0)$$

and $f(0,0) = 0$. Show that the partial derivatives $D_1 f$ and $D_2 f$ exist at every point $(x,y) \in \mathbb{R}^2$ but $f$ is not continuous (and therefore not differentiable) at $(0,0)$.

**Problem 65** (*) Let $f$ be a real-valued function defined on an open set $E \subset \mathbb{R}^n$, and assume that the partial derivatives $D_1 f, \ldots, D_n f$ are bounded in $E$. Prove that $f$ is continuous in $E$.

**Problem 66** Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{x}$, and let $\mathbf{u} \in \mathbb{R}^n$ be any vector with norm 1. Show that the *directional derivative* $(D_{\mathbf{u}} f)(\mathbf{x})$ defined by

$$(D_{\mathbf{u}} f)(\mathbf{x}) = \lim_{t \to 0} \frac{f(\mathbf{x} + t\mathbf{u}) - f(\mathbf{x})}{t}$$

exists. Interpret this definition. Prove also that $(D_{\mathbf{u}} f)(\mathbf{x}) = (\nabla f)(\mathbf{x}) \cdot \mathbf{u}$. (Thus, the partial derivatives determine all directional derivatives.) *Hint:* Use the chain rule.

**Problem 67** (Continuation.) Show that if $(\nabla f)(\mathbf{x}) \neq 0$, then the directional derivative $(D_{\mathbf{u}} f)(\mathbf{x})$ is maximal if

$$\mathbf{u} = \frac{(\nabla f)(\mathbf{x})}{|(\nabla f)(\mathbf{x})|}.$$

**Problem 68** Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be differentiable functions. Show that

$$\nabla(fg) = f\nabla g + g\nabla f.$$

**Problem 69** We say that a real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ has a *local maximum* at a point $\mathbf{x} \in \mathbb{R}^n$ if there exists a neighborhood $N_\epsilon(\mathbf{x})$ such that for all $\mathbf{y} \in N_\epsilon(\mathbf{x})$, $f(\mathbf{x}) \geq f(\mathbf{y})$. Prove that if $f$ is differentiable and has a local maximum at $\mathbf{x}$, then $f'(\mathbf{x}) = 0$.

**Problem 70** Can you construct a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that for all $(x,y) \in \mathbb{R}^2$,

$$(D_1 f)(x,y) = y \qquad \text{and} \qquad (D_2 f)(x,y) = 2x?$$

**Problem 71** (*) Let $\mathbf{f} : \mathbb{R} \to \mathbb{R}^3$ be a differentiable function such that for every $x \in \mathbb{R}$, $|\mathbf{f}(x)| = 1$. Show that for all $x \in \mathbb{R}$,

$$\mathbf{f}'(x) \cdot \mathbf{f}(x) = 0.$$

Give a geometrical interpretation of this equality.