Real Analysis, 2001–Homework set # 1.

Problem 1 Define the set X as $X = \{1, 2, 3, 4\}$. What is the smallest algebra of subsets of X containing the sets $\{1\}, \{2, 3\}, \{1, 4\}$?

Problem 2 Let X be the set of the positive integers. What is the smallest algebra of subsets of X which contains the set of even numbers?

Problem 3 Let X be the set of real numbers. Determine the smallest algebra of subsets of X which contains all closed intervals. Show that the obtained algebra is not a σ-algebra.

Problem 4 Let A be a countable set of real numbers. Prove that $A \cup A_{-}$ is also countable, where $A_{-} = \{x : -x \in A\}$.

Problem 5 Prove that the set $\mathcal{Q}$ of all rational numbers is countable.

Problem 6 A real number x is called *algebraic* if it is a root of an algebraic equation, that is, if for some n there exist integers $a_0, a_1, a_2, \dots, a_n$ such that

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

Clearly, all rational numbers (and much more) are algebraic. Prove that the set of algebraic numbers is countable.

Problem 7 (*) Let A and B be two sets. We say that the *cardinality* of A is larger than the cardinality of B if there is no function which maps every element of A to different elements of B. Prove that if S is an arbitrary set containing at least two elements and F denotes the set of all functions mapping from S to S, then the cardinality of F is larger than the cardinality of S.

Problem 8 (*) Let $\mathcal{A}$ be a σ-algebra which contains infinitely many members. Prove that $\mathcal{A}$ is not countable. *Hint:* Show first that there are infinitely many disjoint sets in $\mathcal{A}$.

Real Analysis, 2001–Homework set # 2.

Problem 9 Let $\langle x_n \rangle$ and $\langle y_n \rangle$ be sequences of real numbers such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Assume that $x_n \neq 0$ and $x \neq 0$. Prove that

(a) $\lim_{n\to\infty}(x_n + y_n) = x + y$; (b) $\lim_{n\to\infty} cx_n = cx$;
(c) $\lim_{n\to\infty} x_n y_n = xy$; (d) $\lim_{n\to\infty} 1/x_n = 1/x$;
(e) $\lim_{n\to\infty} y_n/x_n = y/x$.

Problem 10 Prove that if $\langle x_n \rangle$ is a convergent sequence of real numbers then it is Cauchy.

Problem 11 (*) Show that a sequence $\langle x_n \rangle$ converges to a number x if and only if every subsequence of $\langle x_n \rangle$ has a subsequence which converges to x.

Problem 12 Let $x_1, x_2, \ldots$ be a sequence which contains all rational numbers. (Since the set of rationals is countable, such a sequence exists.) Determine the set of cluster points of $\langle x_n \rangle$.

Problem 13 Construct a sequence of real numbers such that the set of cluster points is the set of positive ingtegers.

Problem 14 Show that if x is a cluster point of the sequence $\langle x_n \rangle$, then there exists a subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ which converges to x.

Problem 15 Show that $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$.

Problem 16 Show that a sequence $\langle x_n \rangle$ of real numbers is convergent if and only if $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$.

Problem 17 Let $\langle x_n \rangle$ and $\langle y_n \rangle$ be sequences of real numbers. Show that

$$\begin{aligned}
\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n &\leq \liminf_{n\to\infty}(x_n + y_n) \\
&\leq \liminf_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \\
&\leq \limsup_{n\to\infty}(x_n + y_n) \\
&\leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n
\end{aligned}$$

Problem 18 (*) A point $x \in \mathbb{R}$ is called a *condensation point* of a set A if for all $\epsilon > 0$, $A \cap (x - \epsilon, x + \epsilon)$ is uncountable. Denote the set of all condensation points by A^*. Prove that for any set $A \subset R$, $A - A^*$ is countable.

Real Analysis, 2001–Homework set # 3.

Problem 19 (*) Let $\langle x_n \rangle$ be a sequence of real numbers, and let $A \subset \mathbb{R}$ be the set of all cluster points of $\langle x_n \rangle$. Show that A is a closed set. Show that $\limsup_{n\to\infty} x_n = \sup A$ and that $\limsup_{n\to\infty} x_n \in A$.

Problem 20 Let $\mathcal{A}$ be the smallest σ-algebra containing all open intervals. Show that $\mathcal{A}$ contains every closed set of real numbers. Does the smallest *algebra* containing all open intervals also contain every closed set?

Problem 21 Find two sets such that $A \cap B = \emptyset$ but $\overline{A} \cap \overline{B} \neq \emptyset$.

Problem 22 A number x is called an *accumulation point* of a set E of real numbers if it is a point of closure of the set $E - \{x\}$. Prove that the set of accumulation points of a set is closed.

Problem 23 A point x is called an *interior point* of a set E of real numbers, if there exists a $\delta > 0$ such that $(x - \delta, x + \delta) \subset E$. Let E^0 be the set of interior points of a set E. E^0 is called the *interior* of E.

(a) Prove that E^0 is open.

(b) Prove that E is open if and only if $E^0 = E$.

(c) Prove that if $G \subset E$ and G is open, then $G \subset E^0$.

(d) Show by example that the interior of E and the interior of $\overline{E}$ (the closure of E) may be different.

Problem 24 (*) Show that every compact subset of $\mathbb{R}$ is closed and bounded.

Problem 25 (*) Let A be an arbitrary closed set of real numbers. Show that there exists a sequence $\langle x_n \rangle$ of real numbers such that if B denotes the set of all cluster points of $\langle x_n \rangle$, then $B = A$.

Problem 26 Prove that a function is continuous at x if and only if for every sequence $\langle x_n \rangle$ converging to x, $\lim_{n\to\infty} f(x_n) = f(x)$.

Problem 27 Let $\langle f_n \rangle$ be a sequence of continuous functions defined on a set E. Show that if $\langle f_n \rangle$ converges uniformly to a function f, then f is continuous on E.

Problem 28 Show that if f and g are continuous, then

(a) $f + g$ is continuous;

(b) fg is continuous;

(c) $h(x) = f(g(x))$ is continuous;

(d) $|f|$ is continuous.

Problem 29 Let f be a function defined on $\mathbb{R}$ such that for every $x \in \mathbb{R}$, and for every sequence $\langle a_n \rangle$ converging to zero

$$\lim_{n\to\infty} f(x + a_n) - f(x - a_n) = 0.$$

Does this imply that f is continuous?

Real Analysis, 2001–Homework set # 4.

Problem 30 Consider the function

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Determine the points at which f is continuous.

Problem 31 Let f be a continuous function defined on $[0,1]$ such that for all $x \in [0,1]$, we have $f(x) \in [0,1]$. Show that for some $x \in [0,1]$, $f(x) = x$.

Problem 32 Let f be a continuous function on $\mathbb{R}$. Prove that the set $\{x : f(x) = 0\}$ is closed.

Problem 33 (*) Let A be a measurable set with nonzero measure. Prove that there exist two points in A whose distance is a rational number.

Problem 34 Let f be a uniformly continuous function on $(0,1)$. Prove that f is bounded.

Problem 35 A function f defined on $\mathbb{R}$ is said to be *Lipschitz* if there exists a constant $L > 0$ such that for all $x, y \in \mathbb{R}$

$$|f(x) - f(y)| \leq L|x - y|.$$

Prove that

- f is continuous at every $x \in \mathbb{R}$;
- f is uniformly continuous.

Problem 36 (*) We say that a set $A \subset \mathcal{R}$ is G_δ if it may be written as the intersection of countably many open sets. Prove that

- every closed set is G_δ;
- the set of rational numbers is not G_δ;
- if $f : \mathcal{R} \to \mathcal{R}$ is an arbitrary function, then the set of points at which f is continuous is G_δ.

Problem 37 Let $\mathcal{A}$ be the smallest σ-algebra containing all sets of the form $[a, \infty)$. Show that $\mathcal{A}$ is the class of Borel sets.

Problem 38 Let $\mathcal{M}$ be the smallest σ-algebra containing all sets of the form $(n, n+1)$, where n is an arbitrary integer. What kind of sets are there in $\mathcal{M}$? Can you define a countably additive measure on $\mathcal{M}$?

Problem 39 Prove that the outer measure is translation invariant.

Real Analysis, 2001–Homework set # 5.

Problem 40 Show that if the set E is measurable, then each translate $E+y$ of E is also measurable.

Problem 41 Prove that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$.

Problem 42 Show that for every set $A \subset \mathbb{R}$ and $\epsilon > 0$, there exists an open set O such that $A \subset O$, and $m^*(O) \leq m^*(A) + \epsilon$. Conclude that for every set A there exists a set E such that $A \subset E$, $m^*(A) = m^*(E)$, and E is G_δ.

Problem 43 Show that if A and B are measurable, then $m(A \cap B) + m(A \cup B) = m(A) + m(B)$.

Problem 44 Is the class of nonmeasurable sets an algebra?

Problem 45 Show that the Cantor ternary set (defined in Problem 36 on page 44 of Royden) has zero measure. *Note:* This is an example of an uncountable set with measure zero.

Problem 46 (*) Construct a measurable set $A \subset [0, 1]$ with $m(A) \geq 1/2$ such that for any set B of measure zero, $A \cup B$ does not contain any interval.

Problem 47 Let f be a measurable function. Show that the class of sets E whose inverse image $f^{-1}[E]$ is measurable is a σ-algebra. Prove that if E is a Borel set, then $f^{-1}[E]$ is measurable.

Problem 48 (*) Let f be a function defined on $\mathbb{R}$, and assume that f is continuous everywhere except for countably many points. Show that f is measurable.

Proof of Proposition 22: (four exercises).

Problem 49 Let f be a measurable function on $[a, b]$, which takes values $\pm\infty$ on a set of measure zero. Show that for every $\epsilon > 0$ there exists an M such that $m(\{x \in [a, b] : |f(x)| > M\}) < \epsilon$.

Problem 50 Let f be a measurable function on $[a, b]$. Show that for every $\epsilon > 0$ and M there exists a simple function ϕ such that $|f(x) - \phi(x)| < \epsilon$ for all x such that $|f(x)| \leq M$.

Problem 51 (*) Let ϕ be a simple function on $[a, b]$. Show that for every $\epsilon > 0$ there exists a step function g on $[a, b]$ such that $m(\{x \in [a, b] : g(x) \neq \phi(x)\}) < \epsilon$.

Problem 52 Show that for any step function g on $[a, b]$ there exists a continuous function h such that $m(\{x \in [a, b] : h(x) \neq g(x)\}) < \epsilon$.

Problem 53 Let f be a real-valued function on $[a,b]$. Define the *lower envelope* of f by

$$g(x) = \sup_{\delta>0} \inf_{|x-y|<\delta} f(y),$$

and the *upper envelope* of f by

$$h(x) = \inf_{\delta>0} \sup_{|x-y|<\delta} f(y).$$

Show that $g(x) \le f(x) \le h(x)$. Prove that $g(x) = h(x)$ if and only if f is continuous at x.

Problem 54 (*) Let f be a bounded function on $[a,b]$. Show that

$$R\overline{\int_a^b} f = \int_a^b h,$$

where h is the upper envelope of f (see previous exercise). *Hint:* If $\psi \ge f$ is a step function, then $\psi \ge h$ except at finitely many points. On the other hand, there is a sequence $\langle \psi_n \rangle$ of step functions such that $\psi_n \downarrow h$. Use the bounded convergence theorem to show that $\int_a^b h = \lim_{n\to\infty} \int_a^b \phi_n$.

Problem 55 (*) Use the previous exercise to show that a bounded function f on $[a,b]$ is Riemann integrable if and only if f is continuous almost everywhere on $[a,b]$.

Problem 56 Let f and g be two real-valued functions defined on the set of real numbers such that $f = g$ a.e. Show that if both f and g are continuous then $f(x) = g(x)$ for every $x \in \mathbb{R}$.

Problem 57 Let $\langle f_n \rangle$ be a sequence of measurable functions defined on $\mathbb{R}$, and let A be the set of points $x \in \mathbb{R}$ at which the sequence of real numbers $\langle f_n(x) \rangle$ is convergent. Show that A is measurable.

Problem 58 (*) Let $\langle f_n \rangle$ be a sequence of measurable functions, and let f be also measurable. Prove that

$$\lim_{n\to\infty} f_n = f \quad \text{a.e.} \qquad \text{if and only if} \qquad \text{for all } \epsilon > 0, \quad \lim_{n\to\infty} m\left(\left\{x : \sup_{m\ge n} |f_m(x) - f(x)| > \epsilon\right\}\right) = 0.$$

Conclude that if $f_n \to f$ a.e. then for all $\epsilon > 0$, $m(\{x : |f_n(x) - f(x)| > \epsilon\}) \to 0$.

Remark: if for all $\epsilon > 0$ $m(\{x : |f_n(x) - f(x)| > \epsilon\}) \to 0$, then we say that f_n converges to f *in measure*. Thus, the latter claim states that almost everywhere comvergence implies convergence in measure. This is important in probability theory.

Real Analysis, 2001–Homework set # 7.

Problem 59 Let f be a nonnegative measurable function. Show that if $\int f = 0$ then $f = 0$ a.e.

Problem 60 Let f be a measurable function and assume that $\int_A f = 0$ for every measurable set $A \subset E$. Show that $\int_E f = 0$.

Problem 61 Show that if f is integrable, then $|f|$ is integrable, and

$$\left|\int f\right| \leq \int |f|.$$

Problem 62 Prove the following (stronger) version of Fatou's lemma: if $\langle f_n \rangle$ is a sequence of nonnegative measurable functions then

$$\int \liminf f_n \leq \liminf \int f_n$$

Hint: Use the monotone convergence theorem.

Problem 63 Let $\langle f_n \rangle$ be a sequence of nonnegative measurable functions such that for all $x \in \mathbb{R}$, $f_1(x) \geq f_2(x) \geq \cdots$. Assume that f_1 is integrable. Prove that if $f_n \to f$ a.e. then $\int f_n \to \int f$. Show by example that the statement is not necessarily true if f_1 is not integrable.

Problem 64 (*) Let f be an integrable function, and let $\langle f_n \rangle$ be a sequence of nonnegative measurable functions such that
(i) $\liminf\limits_{n\to\infty} f_n(x) \geq f(x)$ for almost every x;
(ii) $\lim\limits_{n\to\infty} \int f_n = \int f$
Show that $\int |f_n - f|$ converges to zero.

Problem 65 (*) Let $\langle f_n \rangle$ be a sequence of integrable functions, and assume that $f_n \to f$ a.e. for some integrable function f. Show that $\int |f_n - f| \to 0$ if and only if $\int |f_n| \to \int |f|$. Show an example when $\int |f_n - f| \not\to 0$.

Real Analysis, 2001–Homework set # 8.

Problem 66 Let $A_1, A_2, \ldots$ be a sequence of measurable sets in a measure space $(X, \mathcal{B}, \mu)$. Show that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} A_i\right) .$$

Problem 67 The *symmetric difference* of the sets A, B is defined as the set

$$A \triangle B = (A - B) \cup (B - A) .$$

Show that if $(X, \mathcal{B}, \mu)$ is a measurable space and A, B are measurable sets such that $\mu(A \triangle B) = 0$, then $\mu(A) = \mu(B)$. SHow that if μ is a complete measure, $A \in \mathcal{B}$ and $\mu(A \triangle B) = 0$, then $B \in \mathcal{B}$.

Problem 68 Let $X = \{x_1, x_2, \ldots\}$ be a countable set, and let $p_1, p_2, \ldots$ be positive numbers such that $\sum_{i=1}^{\infty} p_i = 1$. On the σ-algebra $\mathcal{B}$ of all subsets of X, define

$$\mu(A) = \sum_{n : x_n \in A} p_n , \quad A \in \mathcal{B} .$$

Show that μ is a measure on $(X, \mathcal{B})$.

Problem 69 (*) Prove Proposition 4 (Chapter 11). *Hint.* First show that the collection $\mathcal{B}_0$, defined as the family of all sets of the form $A \cup B$ with $B \in \mathcal{B}$ and $A \subset C$ for some $C \in \mathcal{B}$ with $\mu(C) = 0$, is a σ-algebra.

Problem 70 Let $(X, \mathcal{B}, \mu)$ be a measurable space and let g be a nonnegative measurable function on X. Define the set function ν by

$$\nu(A) = \int_A g d\mu \quad \text{if } A \in \mathcal{B}.$$

Show that ν is a measure on $(X, \mathcal{B})$. Prove that if f is a nonnegative measurable function, then

$$\int f d\nu = \int f g d\mu .$$

Hint: First prove the result if g is a simple function, and then use the monotone convergence theorem.