Real Analysis, 2002–Homework set # 1.

Problem 1 Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$. Prove that

$$|\mathbf{x}+\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2.$$

Why is this equality called the "parallelogram rule?"

Problem 2 Let S be a set of real numbers. Show that $\inf S = -\sup S_-$, where $S_- = \{x : -x \in S\}$.

Problem 3 (*) Assume that the set $B \subset \mathbb{R}$ is bounded below, and let L be the set of all lower bounds of B. Show that L is bounded above and $\sup L = \inf B$.

Problem 4 Let A be a countable set of real numbers. Prove that $A \cup A_-$ is also countable, where $A_- = \{x : -x \in A\}$.

Problem 5 Prove that the set of all irrational numbers is uncountable.

Problem 6 Show that the set of all finite sequences from a countable set A is countable.

Problem 7 Use the previous exercise to show that if $A \subset [0,1]^2$ is the set of points in the unit square with two rational coordinates, then A is countable.

Problem 8 A real number x is called *algebraic* if it is a root of an algebraic equation, that is, if for some n there exist integers $a_0, a_1, a_2, \ldots, a_n$ such that

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

Clearly, all rational numbers (and much more) are algebraic. Prove that the set of algebraic numbers is countable.

Problem 9 (*) Let A and B be two sets. We say that the *cardinality* of A is larger than the cardinality of B if there is no function which maps every element of A to different elements of B. Prove that if S is an arbitrary set containing at least two elements and F denotes the collection of all subsets of S, then the cardinality of F is larger than the cardinality of S.

Real Analysis, 2002–Homework set # 2.

Problem 10 Show that a k-cell is convex.

Problem 11 Let E be an open set. Is every point of E a limit point of E? What if E is closed?

Problem 12 Let E^0 be the set of interior points of a set E. E^0 is called the *interior* of E.
- Prove that E^0 is open.
- Prove that E is open if and only if $E^0 = E$.
- Prove that if $G \subset E$ and G is open, then $G \subset E^0$.
- Show by example that the interior of E and the interior of $\bar{E}$ (the closure of E) may be different.

Problem 13 Let $x, y \in \mathbb{R}$, and consider the following functions:

$$d_1(x,y) = (x-y)^2 \qquad d_2(x,y) = \sqrt{|x-y|} \qquad d_3(x,y) = |x^2 - y^2|$$

$$d_4(x,y) = |x - 2y| \qquad d_5(x,y) = \frac{|x-y|}{1+|x-y|}$$

Determine for each one whether or not it is a metric.

Problem 14 Show that every set of the form $\{q : d(p,q) \leq r\}$ is closed, where p is a fixed point, and $r > 0$.

Problem 15 (*) Prove that if G is an open subset of $\mathbb{R}$, then G is the union of countably many disjoint open intervals. *Hint:* For every $x \in G$ take the largest open interval I_x which contains x. Show that $G = \cup_{x \in G} I_x$. Show that if $I_x \cap I_y$ is nonempty for $x, y \in G$, then $I_x = I_y$, therefore G is a union of disjoint open intervals. To prove that the union is countable, use the fact that every open interval contains a rational.

Problem 16 (*) Let $D_7 \in \mathbb{R}$ be the set of real numbers between 0 and 1 whose decimal expansion $0.a_1a_2\cdots$ contains the digit 7. Some real numbers have two different decimal expansions, a finite and an infinite. D_7 contains only those real numbers such that both decimal expansions contain the digit 7. (For example, for 0.17 we write $0.169999\cdots$, which is not in D_7. Similarly, $0.179999\cdots = 0.18$ is not in D_7.) Prove that D_7 is an open set. Now write D_7 as a union of countably many disjoint intervals. (This may be done by the previous exercise.) Let (a_1, b_1) and (a_2, b_2) be two component intervals with $b_1 < a_2$. Show that there exists another component, (a_3, b_3) such that $b_1 < a_3 < b_3 < a_2$. (In other words, D_7 doesn't have "neighboring" components.)

Problem 17 Give an example of a union of infinitely many closed sets which is not closed.

Problem 18 Consider the following three metrics in $\mathbb{R}^k$: $d_1(\mathbf{x},\mathbf{y}) = \sum_{i=1}^k |x_i - y_i|$, $d_2(\mathbf{x},\mathbf{y}) = \sqrt{\sum_{i=1}^k (x_i - y_i)^2}$, $d_\infty(\mathbf{x},\mathbf{y}) = \max_{i=1,\ldots,k} |x_i - y_i|$, where $\mathbf{x} = (x_1, \ldots, x_k)$ and $\mathbf{y} = (y_1, \ldots, y_k)$ are two arbitrary points in $\mathbb{R}^k$. Let E be a subset of $\mathbb{R}^k$. Show that the following are equivalent:
- E is open with respect to the metric d_1;
- E is open with respect to the metric d_2;
- E is open with respect to the metric d_∞.

Show the same thing with the word "open" replaced by "closed" and "bounded". *Remark:* Thus, open, closed, and bounded sets are the same with respect to these three metrics. (In such cases we say that d_1, d_2, and d_∞ are *topologically equivalent.*)

Real Analysis, 2002–Homework set # 3.

Problem 19 The *diameter* of a set A is defined by

$$\mathrm{diam}(A) = \sup_{x,y \in A} d(x,y).$$

Let $A_1, A_2, \ldots$ be a sequence of closed and nonempty subsets of the euclidean k-space $\mathbb{R}^k$ such that $A_1 \supset A_2 \supset \cdots$, and $\lim_{n\to\infty} \mathrm{diam}(A_n) = 0$. Prove that the intersection $\cap_{n=1}^{\infty} A_n$ is nonempty.

Problem 20 Consider the set $\mathbb{Q}$ of rational numbers as a metric space with distance function $d(x,y) = |x-y|$. Let E be a set of all $q \in \mathbb{Q}$ such that $q > 0$, and $2 < q^2 < 3$. Show that E is closed and bounded in $\mathbb{Q}$, but it is not compact. *Remark:* This illustrates that unlike in $\mathbb{R}^k$, it is not true in general that closed and bounded sets are compact.

Problem 21 (*) Let F_1 and F_2 be disjoint closed sets of real numbers. Show that there exist two disjoint open sets O_1 and O_2 such that O_1 contains F_1 and O_2 contains F_2.

Problem 22 (*) Let A be an arbitrary closed set of real numbers. Show that there exists a sequence $\{x_n\}$ of real numbers such that if B denotes the set of all subsequential limits of $\{x_n\}$, then $B = A$.

Problem 23 Let $\{s_n\}$ and $\{t_n\}$ be sequences of real numbers such that $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$, and assume that $t, t_n \neq 0$. Prove that $\lim_{n\to\infty} s_n/t_n = s/t$.

Problem 24 Let $\{x_n\}$ and $\{y_n\}$ be sequences in a metric space such that $x_n \to x$ and $y_n \to y$. Prove that the sequence of real numbers $\{d(x_n, y_n)\}$ converges to $d(x,y)$.

Problem 25 Create a sequence of real numbers such that the set of its subsequential limits is the set of all positive integers.

Problem 26 Show a sequence of real numbers such that the set of its subsequential limits is the interval $[0,1]$. Can you do the same with $(0,1)$?

Problem 27 Let $\{s_n\}$ be a sequence of real numbers, and define

$$a_n = \frac{s_1 + s_2 + \cdots + s_n}{n}.$$

Prove that if $\lim_{n\to\infty} s_n = s$, then $\lim_{n\to\infty} a_n = s$. Show that the converse is not true: there exists a divergent sequence $\{s_n\}$ such that $\lim_{n\to\infty} a_n = 0$.

Real Analysis, 2002–Homework set # 4.

Problem 28 Let $\{s_n\}$ and $\{t_n\}$ be sequences of real numbers. Show that

$$\liminf_{n\to\infty} s_n + \limsup_{n\to\infty} t_n \le \limsup_{n\to\infty}(s_n + t_n) \le \limsup_{n\to\infty} s_n + \limsup_{n\to\infty} t_n.$$

Problem 29 Let f be a function defined on $\mathbb{R}$ such that for every $x \in \mathbb{R}$, and for every sequence $\langle a_n \rangle$ converging to zero

$$\lim_{n\to\infty} f(x + a_n) - f(x - a_n) = 0.$$

Does this imply that f is continuous?

Problem 30 Consider the function

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Determine the points at which f is continuous.

Problem 31 Let f be a continuous function defined on $[0, 1]$ such that for all $x \in [0, 1]$, we have $f(x) \in [0, 1]$. Show that for some $x \in [0, 1]$, $f(x) = x$. (In other words, f has a fixed point.)

Problem 32 Let f be a continuous function on $\mathbb{R}$. Prove that the set $\{x : f(x) = 0\}$ is closed.

Problem 33 A real-valued function defined on a subset E of a metric space X is called *upper semicontinuous* if for each $a \in \mathbb{R}$, the set $\{x \in E : f(x) < a\}$ is open in E. A function $f : E \to \mathbb{R}$ is *lower semicontinuous* if $-f$ is upper semicontinuous. Show that a function $f : E \to \mathbb{R}$ is continuous on E if and only if it is both upper and lower semicontinuous.

Problem 34 (*) Let X be a metric space, and let $f : X \to \mathbb{R}$ be a real-valued function. For each $x \in X$, define

$$\overline{f}(x) = \lim_{r\to 0} \sup_{y\in N_r(x)} f(y).$$

Show that $\overline{f}$ is well-defined (i.e., the limit exists for each x). Prove that $\overline{f}$ is upper semicontinuous.

Problem 35 Let X, Y, Z be metric spaces, and assume that $f : X \to Y$ and $g : Y \to Z$ are uniformly continuous functions. Prove that $h = g \circ f : X \to Z$ is also uniformly continuous.

Problem 36 (*) Let $C[0, 1]$ denote the class of all continuous real-valued functions defined on the interval $[0, 1]$. Show that $C[0, 1]$ is a metric space with the metric

$$d(f, g) = \sup_{x\in[0,1]} |f(x) - g(x)| .$$

A real-valued function f defined on the interval $[0, 1]$ is *Lipschitz* if there exists a constant $L > 0$ such that for every $t, s \in [0, 1]$,

$$|f(t) - f(s)| \le L|t - s|.$$

Let A_L denote the set of Lipschitz functions (with Lipschitz constant L). Show that $A_L \subset C[0, 1]$ and that A_L is a closed subset of $C[0, 1]$.

Real Analysis, 2002–Homework set # 5.

Problem 37 Let $\{C_\alpha\}$ be a family of convex sets in $\mathbb{R}^k$. Show that $\bigcap_\alpha C_\alpha$ is convex.

Problem 38 Show that if C and D are convex sets in $\mathbb{R}^k$, and a is a real number, then $C + D = \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in C \text{ and } \mathbf{y} \in D\}$ and $aC = \{a\mathbf{x} : \mathbf{x} \in C\}$ are convex.

Problem 39 Show that if A is an open set in $\mathbb{R}^k$, then $\mathrm{co}(A)$ is open.

Problem 40 Prove that if C is a convex set, then the closure $\bar{C}$ and the interior C° of C are also convex.

Problem 41 (*) Let $A \subset \mathbb{R}^k$. Show that for any $\mathbf{x} \in \mathrm{co}(A)$ there exist $k+1$ points $\mathbf{x}_1, \ldots, \mathbf{x}_{k+1} \in A$ such that $\mathbf{x}$ equals a convex combination of these $k+1$ points.

Problem 42 (*) Use the previous exercise to prove that if A is a compact set in $\mathbb{R}^k$, then $\mathrm{co}(A)$ is also compact.

Problem 43 (*) Let $F \subset \mathbb{R}^k$ be a closed and convex set, and let $\mathbf{x} \in \mathbb{R}^k$. We have seen in class that there exists a unique point $\mathbf{y} \in F$ such that

$$|\mathbf{x} - \mathbf{y}| = D(F, \mathbf{x}).$$

This defines a function $g : \mathbb{R}^k \to F$ by $|\mathbf{x} - g(\mathbf{x})| = D(F, \mathbf{x})$. Show that g is continuous.

Problem 44 Prove that a closed half space is a closed set, and an open half space is an open set.

Problem 45 Let K be a compact set, and let $f : K \to \mathbb{R}$ be upper semicontinuous. Show that f achieves its maximum on K.

Real Analysis, 2002–Homework set # 6.

Problem 46 (*) Let C be a convex subset of $\mathbb{R}^k$. Show that the interior C° of C is empty if and only if C is contained in a hyperplane.

Problem 47 Show that any contraction is uniformly continuous.

Problem 48 Show that if $x_1, x_2, \dots, x_n$ are affinely independent vectors in $\mathbb{R}^k$, then $n \le k+1$.

Problem 49 Show that if $x_1, x_2, \dots, x_n$ are linearly independent vectors, then there exists a vector x_{n+1} such that $x_1, x_2, \dots, x_n, x_{n+1}$ are affinely independent.

Problem 50 Let $f : A \to \mathbb{R}^k$ be a continuous function defined on a set $E \subset \mathbb{R}^k$. Prove that the set of fixed points of f is closed. If E is compact then the set of fixed points of f is also compact.

Problem 51 Show that if a metric space X is homeomorphic to the metric space Y then Y is homeomorphic to X.

Problem 52 Let $h : X \to Y$ be a homeomorphism. Show that
(a) if $A \subset X$ is open, then $B = \{y \in Y : \exists x \in X \text{ such that } y = h(x)\}$ is open.
(b) if $A \subset X$ is closed, then $B = \{y \in Y : \exists x \in X \text{ such that } y = h(x)\}$ is closed.
(d) if $A \subset X$ is compact, then $B = \{y \in Y : \exists x \in X \text{ such that } y = h(x)\}$ is compact.
Remark: This exercise shows that two homeomorphic sets have the same metric structure.

Problem 53 Exhibit a nonconvex compact set $K \subset \mathbb{R}^k$ and a continuous function $f : K \to K$ such that f does not have a fixed point.

Problem 54 Exhibit a convex closed set $C \subset \mathbb{R}^k$ and a continuous function $f : C \to C$ such that f does not have a fixed point.

Problem 55 Exhibit a convex bounded set $B \subset \mathbb{R}^k$ and a continuous function $f : B \to B$ such that f does not have a fixed point.

Real Analysis, 2002–Homework set # 7.

Problem 56 (*) Let $A, B \subset \mathbb{R}^k$ be compact and convex sets with nonempty interior (i.e., $A^\circ, B^\circ \neq \emptyset$). Show that A is homeomorphic to B.

Problem 57 Let $f : \Delta_n \to \mathbb{R}^{n+1}$ be a continuous function satisfying the strong form of Walras' law $\mathbf{p} \cdot f(\mathbf{p}) = 0$ for all price vectors $\mathbf{p} \in \Delta_n$. Then, by the equilibrium theorem seen in class, there exists a price vector $\mathbf{q} \in \Delta_n$ such that $f(\mathbf{q}) \leq \mathbf{0}$. Show that if for some commodity i the excess demand is negative, that is, if $f_i(\mathbf{q}) < 0$, then its price q_i must be zero.

Problem 58 Let S_1, S_2 be finite sets, and let p_1 be a real-valued (payoff) function defined on $S_1 \times S_2$. Let Π_1, Π_2 be probability distributions (mixed strategies) over S_1 and S_2, respectively, that is, $\Pi_i \in \Delta_{|S_i|-1}$, $i = 1, 2$. Define the associated expected payoff by

$$p_1(\Pi_1, \Pi_2) = \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \Pi_1(s_1)\Pi_2(s_2)p_1(s_1, s_2) \ .$$

For any $\Pi_2 \in \Delta_{|S_2|-1}$ let $\Phi_1(\Pi_2)$ be the set of those $\Pi_1' \in \Delta_{|S_1|-1}$ that maximize $p_1(\Pi_1', \Pi_2)$. Prove that for any Π_2, the set $\Phi_1(\Pi_2)$ is closed and convex. Show that the set-valued function Φ_1 has the graph continuity property. (These are the propoerties which guarantee that Kakutani's fixed point theorem may be used in the proof of the existence of mixed Nash equilibria.)

Problem 59 Show that if $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a basis of a vector space X, then each $\mathbf{x} \in X$ can be uniquely written as a linear combination of the basis vectors.

Problem 60 Let $A \in L(X, Y)$ be a linear transformation such that $A\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$. Show that A is one-to-one.

Problem 61 (*) Let $A \in L(\mathbb{R}^n, \mathbb{R})$ be a real-valued linear transformation. Prove that there exists a vector $\mathbf{y} \in \mathbb{R}^n$ such that for all $\mathbf{x} \in \mathbb{R}^n$, $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$. Show that $||A|| = |\mathbf{y}|$.

Problem 62 Let $c_1, \ldots, c_n$ be fixed real numbers. Show that the set of vectors $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ satisfying $\sum_{i=1}^n c_i x_i = 0$ forms an $n-1$-dimensional vector space.

Problem 63 Let X and Y be two vector spaces in $\mathbb{R}^n$ such that $X \bigcap Y = \mathbf{0}$. Prove that $\dim(X) + \dim(Y) \leq n$.

Problem 64 Show that for $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, $||A|| = \sup_{\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0}} \frac{|A\mathbf{x}|}{|\mathbf{x}|}$.

Problem 65 Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation given by its matrix $A = (a_{i,j})_{n \times n}$ (in the standard basis). Consider the distance

$$d(\mathbf{x}, \mathbf{y}) = \max_{i \leq n} |x_i - y_i| \qquad \text{for } \mathbf{x} = (x_1, \ldots, x_n) \text{ and } \mathbf{y} = (y_1, \ldots, y_n).$$

Show that if there is a constant $\alpha < 1$ such that for each $i = 1, 2, \ldots, n$

$$\sum_{j=1}^n |a_{i,j}| \leq \alpha,$$

then T is a contraction. Can you derive a similar sufficient condition for contractiveness under the distance $d(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n |x_i - y_i|$? (Recall that contractivity under *any* distance implies the existence of a unique fixed point.)

Problem 66 Define $f : \mathbb{R}^2 \to \mathbb{R}$ as

$$f(x,y) = \frac{xy}{x^2+y^2} \quad \text{if} \quad (x,y) \neq (0,0)$$

and $f(0,0) = 0$. Show that the partial derivatives $D_1 f$ and $D_2 f$ exist at every point $(x,y) \in \mathbb{R}^2$ but f is not continuous (and therefore not differentiable) at $(0,0)$.

Problem 67 (*) Consider the set G of vectors $\mathbf{x} = (x_1, x_2, \ldots, x_9) \in \mathbb{R}^9$ for which the matrix

$$A(\mathbf{x}) = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix}$$

is invertible. Show that G is an open set.

Problem 68 Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{x}$, and let $\mathbf{u} \in \mathbb{R}^n$ be any vector with norm 1. Show that the *directional derivative* $(D_{\mathbf{u}} f)(\mathbf{x})$ defined by

$$(D_{\mathbf{u}} f)(\mathbf{x}) = \lim_{t \to 0} \frac{f(\mathbf{x} + t\mathbf{u}) - f(\mathbf{x})}{t}$$

exists. Interpret this definition. Prove also that $(D_{\mathbf{u}} f)(\mathbf{x}) = (\nabla f)(\mathbf{x}) \cdot \mathbf{u}$. (Thus, the partial derivatives determine all directional derivatives.) *Hint:* Use the chain rule.

Problem 69 (Continuation.) Show that if $(\nabla f)(\mathbf{x}) \neq 0$, then the directional derivative $(D_{\mathbf{u}} f)(\mathbf{x})$ is maximal if

$$\mathbf{u} = \frac{(\nabla f)(\mathbf{x})}{|(\nabla f)(\mathbf{x})|}.$$

Problem 70 Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be differentiable functions. Show that

$$\nabla(fg) = f \nabla g + g \nabla f.$$

Problem 71 We say that a real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ has a *local maximum* at a point $\mathbf{x} \in \mathbb{R}^n$ if there exists a neighborhood $N_\epsilon(\mathbf{x})$ such that for all $\mathbf{y} \in N_\epsilon(\mathbf{x})$, $f(\mathbf{x}) \geq f(\mathbf{y})$. Prove that if f is differentiable and has a local maximum at $\mathbf{x}$, then $f'(\mathbf{x}) = 0$.

Problem 72 Can you construct a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that for all $(x,y) \in \mathbb{R}^2$,

$$(D_1 f)(x,y) = y \qquad \text{and} \qquad (D_2 f)(x,y) = 2x?$$

Problem 73 (*) Let $\mathbf{f} : \mathbb{R} \to \mathbb{R}^3$ be a differentiable function such that for every $x \in \mathbb{R}$, $|\mathbf{f}(x)| = 1$. Show that for all $x \in \mathbb{R}$,

$$\mathbf{f}'(x) \cdot \mathbf{f}(x) = 0.$$

Give a geometrical interpretation of this equality.