Real Analysis and Measure Theory–Homework set # 1.

Problem 1 Define the set X as $X = \{1, 2, 3, 4\}$. What is the smallest algebra of subsets of X containing the sets $\{1\}, \{2, 3\}, \{1, 4\}$?

Problem 2 Let X be the set of the positive integers. What is the smallest algebra of subsets of X which contains the set of even numbers?

Problem 3 Let A be a countable set of real numbers. Prove that $A \cup A_-$ is also countable, where $A_- = \{x : -x \in A\}$.

Problem 4 Prove that the set $\mathcal{Q}$ of all rational numbers is countable.

Problem 5 A real number x is called *algebraic* if it is a root of an algebraic equation, that is, if for some n there exist integers $a_0, a_1, a_2, \ldots, a_n$ such that

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

Clearly, all rational numbers (and much more) are algebraic. Prove that the set of algebraic numbers is countable.

Problem 6 (*) Let A and B be two sets. We say that the *cardinality* of A is larger than the cardinality of B if there is no function which maps every element of A to different elements of B. Prove that if S is an arbitrary set containing at least two elements and F denotes the set of all functions mapping from S to S, then the cardinality of F is larger than the cardinality of S.

Problem 7 Let $\mathcal{F}$ be a *field* (i.e., any set which satisfies the field axioms). Show the following properties (where x, y, z are arbitrary elements of $\mathcal{F}$:
(a) If $x + y = x + z$ then $y = z$ (cancellation law).
(b) If $x + y = x$ then $y = 0$ (uniqueness of the zero element).
(c) If $x + y = 0$ then $y = -x$ (uniqueness of the additive inverse).
(d) $-(-x) = x$.
(e) If $x \neq 0$ and $xy = 1$ then $y = 1/x$.
(f) $0x = 0$.
(g) If $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.
(h) $(-x)y = -(xy)$.

Problem 8 Let F be an ordered field as defined in class. Show that
(a) if $x > 0$ then $-x < 0$;
(b) if $x > 0$ and $y < z$ then $xy < xz$;
(c) if $x < 0$ and $y < z$ then $xy > xz$;
(d) if $x \neq 0$ then $x^2 > 0$. In particular, $1 > 0$;
(e) if $0 < x < y$ then $0 < 1/y < 1/x$.

Problem 9 (*) Let a and b be fixed positive rational numbers, and consider the set S of all numbers of the form

$$p + q\sqrt{a} + r\sqrt{b} + s\sqrt{ab},$$

where p, q, r, s take all rational values. Show that S is a field.

Problem 10 Let S be a set of real numbers whose greatest lower bound exists. Show that $\inf S = -\sup S_-$, where $S_- = \{x : -x \in S\}$.

Real Analysis and Measure Theory–Homework set # 2.

Problem 11 *The field of complex numbers.* Define the set of complex numbers $\mathcal{C}$ as the set of ordered pairs (a, b) of real numbers. ("Ordered" means that (a, b) and (b, a) are different if $a \neq b$.) Introduce addition and multiplication of complex numbers as follows: let $x = (a, b)$ and $y = (c, d)$ be two complex numbers. Then

$$x + y = (a + c, b + d) \qquad \text{and} \qquad xy = (ac - bd, ad + bc).$$

(a) Prove that with these operations $\mathcal{C}$ is a field (i.e., it satisfies the field axioms A1–A9). What are the zero and unity elements?

(b) Show that if a and b are real numbers then

$$(a, 0) + (b, 0) = (a + b, 0) \qquad \text{and} \qquad (a, 0)(b, 0) = (ab, 0),$$

so we can identify the complex number $(a, 0)$ with the real number a.

(c) Define $i = (0, 1)$. Show that $i^2 = -1$.

(d) Show that if a and b are real numbers then $(a, b) = a + bi$.

(e) Prove that it is impossible to define an order on $\mathcal{C}$ such that it becomes an ordered field (i.e., it satisfies the axioms of order B1–B4). *Hint:* -1 is a square.

Problem 12 Let F be an ordered set, and assume that every subset of F which is bounded above has a least upper bound. Let L and U be subsets of F such that $L \cup U = F$, and for every $l \in L$ and $u \in U$, we have $l < u$. Show that either L has a greatest element (i.e., an element $x \in L$ such that $x \geq l$ for all $l \in L$), or R has a smallest element.

Problem 13 Let $\langle x_n \rangle$ and $\langle y_n \rangle$ be sequences of real numbers such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Assume that $x_n \neq 0$ and $x \neq 0$. Prove that

(a) $\lim_{n\to\infty}(x_n + y_n) = x + y$; (b) $\lim_{n\to\infty} cx_n = cx$;
(c) $\lim_{n\to\infty} x_n y_n = xy$; (d) $\lim_{n\to\infty} 1/x_n = 1/x$;
(e) $\lim_{n\to\infty} y_n/x_n = y/x$.

Problem 14 Prove that if $\langle x_n \rangle$ is a convergent sequence of real numbers then it is Cauchy.

Problem 15 (*) Show that a sequence $\langle x_n \rangle$ converges to a number x if and only if every subsequence of $\langle x_n \rangle$ has a subsequence which converges to x.

Problem 16 Let $x_1, x_2, \ldots$ be a sequence which contains all rational numbers. (Since the set of rationals is countable, such a sequence exists.) Determine the set of cluster points of $\langle x_n \rangle$.

Problem 17 Construct a sequence of real numbers such that the set of cluster points is the set of positive ingtegers.

Problem 18 Show that if x is a cluster point of the sequence $\langle x_n \rangle$, then there exists a subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ which converges to x.

Problem 19 Show that $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$.

Problem 20 Show that a sequence $\langle x_n \rangle$ of real numbers is convergent if and only if $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$.

Problem 21 (*) Let $\langle a_n \rangle$ be a sequence of real numbers. The *series* $\sum_n a_n$ is defined as the sequence $s_n = \sum_{i=1}^n a_i$. The series $\sum_n a_n$ is called *convergent* if the sequence $\langle s_n \rangle$ is convergent. If the limit of $\langle s_n \rangle$ is s, then we write $\sum_{n=1}^\infty a_n = s$. A series $\sum_n a_n$ is *absolutely convergent* if $\sum_n |a_n|$ is convergent. Prove that if a series is absolutely convergent then it is convergent.

Given two series $\sum_n a_n$ and $\sum_n b_n$, we define the *Cauchy product* of them by the series $\sum_n c_n$, where for each n, $c_n = \sum_{i=0}^n a_i b_{n-i}$. Prove that the Cauchy product of two absolutely convergent series is convergent.

Real Analysis and Measure Theory–Homework set # 3.

Problem 22 Let $\langle x_n \rangle$ and $\langle y_n \rangle$ be sequences of real numbers. Show that

$$\begin{aligned} \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n &\leq \liminf_{n\to\infty}(x_n + y_n) \\ &\leq \liminf_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \\ &\leq \limsup_{n\to\infty}(x_n + y_n) \\ &\leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \end{aligned}$$

Problem 23 (*) Let $\langle x_n \rangle$ be a sequence of real numbers, and let $A \subset \mathcal{R}$ be the set of all cluster points of $\langle x_n \rangle$. Show that A is a closed set. Show that $\limsup_{n\to\infty} x_n = \sup A$ and that $\limsup_{n\to\infty} x_n \in A$.

Problem 24 Let $\mathcal{A}$ be the smallest σ-algebra containing all open intervals. Show that $\mathcal{A}$ contains every closed set of real numbers. Does the smallest *algebra* containing all open intervals also contain every closed set?

Problem 25 Find two sets such that $A \cap B = \emptyset$ but $\overline{A} \cap \overline{B} \neq \emptyset$.

Problem 26 A number x is called an *accumulation point* of a set E of real numbers if it is a point of closure of the set $E - \{x\}$. Prove that the set of accumulation points of a set is closed.

Problem 27 A point x is called an *interior point* of a set E of real numbers, if there exists a $\delta > 0$ such that $(x - \delta, x + \delta) \subset E$. Let E^0 be the set of interior points of a set E. E^0 is called the *interior* of E.

(a) Prove that E^0 is open.

(b) Prove that E is open if and only if $E^0 = E$.

(c) Prove that if $G \subset E$ and G is open, then $G \subset E^0$.

(d) Show by example that the interior of E and the interior of $\overline{E}$ (the closure of E) may be different.

Problem 28 (*) Show that every compact subset of $\mathcal{R}$ is closed and bounded.

Problem 29 (*) Let A be an arbitrary closed set of real numbers. Show that there exists a sequence $\langle x_n \rangle$ of real numbers such that if B denotes the set of all cluster points of $\langle x_n \rangle$, then $B = A$.

Problem 30 Prove that a function is continuous at x if and only if for every sequence $\langle x_n \rangle$ converging to x, $\lim_{n\to\infty} f(x_n) = f(x)$.

Problem 31 Let $\langle f_n \rangle$ be a sequence of continuous functions defined on a set E. Show that if $\langle f_n \rangle$ converges uniformly to a function f, then f is continuous on E.

Problem 32 Show that if f and g are continuous, then

(a) $f + g$ is continuous;

(b) fg is continuous;

(c) $h(x) = f(g(x))$ is continuous;

(d) $|f|$ is continuous.

Problem 33 Let f be a function defined on $\mathcal{R}$ such that for every $x \in \mathcal{R}$, and for every sequence $\langle a_n \rangle$ converging to zero

$$\lim_{n\to\infty} f(x + a_n) - f(x - a_n) = 0.$$

Does this imply that f is continuous?

Real Analysis and Measure Theory–Homework set # 4.

Problem 34 (*) Let $A \subset \mathcal{R}$ be a set which is both open and closed. Prove that either $A = \mathcal{R}$ or $A = \emptyset$.

Problem 35 Consider the function

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Determine the points at which f is continuous.

Problem 36 Let f be a continuous function defined on $[0,1]$ such that for all $x \in [0,1]$, we have $f(x) \in [0,1]$. Show that for some $x \in [0,1]$, $f(x) = x$.

Problem 37 Let f be a continuous function on $\mathcal{R}$. Prove that the set $\{x : f(x) = 0\}$ is closed.

Problem 38 (*) Let f be a function defined on $\mathcal{R}$ such that for every a and b, if $f(a) < c < f(b)$, then $f(x) = c$ for some x between a and b. Suppose also that for every rational number r, the set $\{x : f(x) = r\}$ is closed. Prove that f is continuous. *Hint:* If $x_n \to x$ but $f(x_n) > r > f(x)$ for some rational r, then for some t_n between x_n and x, $f(t_n) = r$.

Problem 39 Let f be a uniformly continuous function on $(0,1)$. Prove that f is bounded.

Problem 40 A function f defined on $\mathcal{R}$ is said to be *Lipschitz* if there exists a constant $L > 0$ such that for all $x, y \in \mathcal{R}$

$$|f(x) - f(y)| \leq L|x - y|.$$

Prove that
- f is continuous at every $x \in \mathcal{R}$;
- f is uniformly continuous.

Real Analysis and Measure Theory–Homework set # 5.

Problem 41 (*) We say that a set $A \subset \mathcal{R}$ is G_δ if it may be written as the intersection of countably many open sets. Prove that
• every closed set is G_δ;
• the set of rational numbers is not G_δ;
• if $f : \mathcal{R} \to \mathcal{R}$ is an arbitrary function, then the set of points at which f is continuous is G_δ.

Problem 42 Let $\mathcal{A}$ be the smallest σ-algebra containing all sets of the form $[a, \infty)$. Show that $\mathcal{A}$ is the class of Borel sets.

Problem 43 Let $\mathcal{M}$ be the smallest σ-algebra containing all sets of the form $(n, n+1)$, where n is an arbitrary integer. What kind of sets are there in $\mathcal{M}$? Can you define a countably additive measure on $\mathcal{M}$?

Problem 44 Prove that the outer measure is translation invariant.

Problem 45 Show that if the set E is measurable, then each translate $E + y$ of E is also measurable.

Problem 46 Prove that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$.

Problem 47 Show that for every set $A \subset \mathcal{R}$ and $\epsilon > 0$, there exists an open set O such that $A \subset O$, and $m^*(O) \leq m^*(A) + \epsilon$. Conclude that for every set A there exists a set E such that $A \subset E$, $m^*(A) = m^*(E)$, and E is the intersection of countably many open sets.

Problem 48 Show that if A and B are measurable, then $m(A \cap B) + m(A \cup B) = m(A) + m(B)$.

Problem 49 Is the class of nonmeasurable sets an algebra?

Problem 50 Show that the Cantor ternary set (defined in Problem 36 on page 44 of Royden) has zero measure. *Note:* This is an example of an uncountable set with measure zero.

Problem 51 (*) Construct a measurable set $A \subset [0, 1]$ with $m(A) \geq 1/2$ such that for any set B of measure zero, $A \cup B$ does not contain any interval.

Real Analysis and Measure Theory–Homework set # 6.

Problem 52 Let f be a measurable function. Show that the class of sets E whose inverse image $f^{-1}[E]$ is measurable is a σ-algebra. Prove that if E is a Borel set, then $f^{-1}[E]$ is measurable.

Problem 53 (*) Let f be a function defined on $\mathcal{R}$, and assume that f is continuous everywhere except for countably many points. Show that f is measurable.

Proof of Proposition 22: (four exercises).

Problem 54 Let f be a measurable function on $[a, b]$, which takes values $\pm\infty$ on a set of measure zero. Show that for every $\epsilon > 0$ there exists an M such that $m(\{x \in [a, b] : |f(x)| > M\}) < \epsilon$.

Problem 55 Let f be a measurable function on $[a, b]$. Show that for every $\epsilon > 0$ and M there exists a simple function ϕ such that $|f(x) - \phi(x)| < \epsilon$ for all x such that $|f(x)| \leq M$.

Problem 56 (*) Let ϕ be a simple function on $[a, b]$. Show that for every $\epsilon > 0$ there exists a step function g on $[a, b]$ such that $m(\{x \in [a, b] : g(x) \neq \phi(x)\}) < \epsilon$.

Problem 57 Show that for any step function g on $[a, b]$ there exists a continuous function h such that $m(\{x \in [a, b] : h(x) \neq g(x)\}) < \epsilon$.

Problem 58 Let f be a real-valued function on $[a, b]$. Define the *lower envelope* of f by

$$g(x) = \sup_{\delta > 0} \inf_{|x-y|<\delta} f(y),$$

and the *upper envelope* of f by

$$h(x) = \inf_{\delta > 0} \sup_{|x-y|<\delta} f(y).$$

Show that $g(x) \leq f(x) \leq h(x)$. Prove that $g(x) = h(x)$ if and only if f is continuous at x.

Problem 59 (*) Let f be a bounded function on $[a, b]$. Show that

$$R\overline{\int_a^b} f = \int_a^b h,$$

where h is the upper envelope of f (see previous exercise). *Hint:* If $\psi \geq f$ is a step function, then $\psi \geq h$ except at finitely many points. On the other hand, there is a sequence $\langle \psi_n \rangle$ of step functions such that $\psi_n \downarrow h$. Use the bounded convergence theorem to show that $\int_a^b h = \lim_{n\to\infty} \int_a^b \phi_n$.

Problem 60 (*) Use the previous exercise to show that a bounded function f on $[a, b]$ is Riemann integrable if and only if f is continuous almost everywhere on $[a, b]$.

Real Analysis and Measure Theory–Homework set # 7.

Problem 61 Let f and g be two real-valued functions defined on the set of real numbers such that $f = g$ a.e. Show that if both f and g are continuous then $f(x) = g(x)$ for every $x \in \mathcal{R}$.

Problem 62 Let $\langle f_n \rangle$ be a sequence of measurable functions defined on $\mathcal{R}$, and let A be the set of points $x \in \mathcal{R}$ at which the sequence of real numbers $\langle f_n(x) \rangle$ is convergent. Show that A is measurable.

Problem 63 (*) Let $\langle f_n \rangle$ be a sequence of measurable functions, and let f be also measurable. Prove that

$$\lim_{n\to\infty} f_n = f \quad \text{a.e.} \qquad \text{if and only if} \qquad \text{for all } \epsilon > 0, \quad \lim_{n\to\infty} m\left(\left\{x : \sup_{m\geq n} |f_m(x) - f(x)| > \epsilon\right\}\right) = 0.$$

Conclude that if $f_n \to f$ a.e. then for all $\epsilon > 0$, $m(\{x : |f_n(x) - f(x)| > \epsilon\}) \to 0$.

Remark: if for all $\epsilon > 0$ $m(\{x : |f_n(x) - f(x)| > \epsilon\}) \to 0$, then we say that f_n converges to f *in measure.* Thus, the latter claim states that almost everywhere comvergence implies convergence in measure. This is important in probability theory.

Problem 64 Let f be a nonnegative measurable function. Show that if $\int f = 0$ then $f = 0$ a.e.

Problem 65 Let f be a measurable function and assume that $\int_A f = 0$ for every measurable set $A \subset E$. Show that $\int_E f = 0$.

Problem 66 (*) Let f be a nonnegative integrable function. Show that the function

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

is continuous.

Problem 67 Show that if f is integrable, then $|f|$ is integrable, and

$$\left|\int f\right| \leq \int |f|.$$

Problem 68 (*) Let f be an integrable function, and let $\langle f_n \rangle$ be a sequence of nonnegative measurable functions such that
(i) $\liminf_{n\to\infty} f_n(x) \geq f(x)$ for almost every x;
(ii) $\lim_{n\to\infty} \int f_n = \int f$
Show that $\int |f_n - f|$ converges to zero.

Real Analysis and Measure Theory–Homework set # 8.

Problem 69 Show that $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$.

Problem 70 Let f be a bounded measurable function on $[a,b]$. Show that

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

Problem 71 Show that if $f \in L^1$ and $g \in L^\infty$ then

$$\int |fg| \le \|f\|_1 \|g\|_\infty.$$

Problem 72 Jensen's inequality. Let f be an integrable function defined on $[0,1]$, and let ϕ be convex. Prove that

$$\phi\left(\int_0^1 f(x)dx\right) \le \int_0^1 \phi(f(x))dx$$

Problem 73 Let $p \ge 1$ be a real number. We say that a sequence $\langle x_n \rangle$ of real numbers is in the l^p space of sequences if

$$\sum_{n=1}^{\infty} |x_n|^p < \infty.$$

For such a sequence, define

$$\|\langle x_n\rangle\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}.$$

Show that this quantity is a *norm*.

Problem 74 Hölder's inequality for sequences. Assume that p and q are such that $(1/p)+(1/q)=1$. Prove that if $\langle x_n\rangle \in l^p$ and $\langle y_n\rangle \in l^q$ then

$$\sum_{n=1}^{\infty} |x_n y_n| \le \|\langle x_n\rangle\|_p \cdot \|\langle y_n\rangle\|_q.$$

Problem 75 Minkowski's inequality for sequences. Prove that

$$\|\langle x_n + y_n\rangle\|_p \le \|\langle x_n\rangle\|_p + \|\langle y_n\rangle\|_p.$$

Problem 76 (*) Consider a function $f \in L^\infty$. The *essential range* of f is the set S_f of all real numbers y with the property that for all $\epsilon > 0$

$$m(\{x : |f(x)-y| < \epsilon\}) > 0.$$

Prove that S_f is a compact set.