Real Analysis, 1999–Homework set # 1.

Problem 1 Let A be a countable set of real numbers. Prove that $A \cup A_-$ is also countable, where $A_- = \{x : -x \in A\}$.

Problem 2 Prove that the set $\mathcal{Q}$ of all rational numbers is countable.

Problem 3 A real number x is called *algebraic* if it is a root of an algebraic equation, that is, if for some n there exist integers $a_0, a_1, a_2, \ldots, a_n$ such that

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

Clearly, all rational numbers (and much more) are algebraic. Prove that the set of algebraic numbers is countable.

Problem 4 Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Assume that $x_n \neq 0$ and $x \neq 0$. Prove that

(a) $\lim_{n\to\infty}(x_n + y_n) = x + y$; (b) $\lim_{n\to\infty} cx_n = cx$;
(c) $\lim_{n\to\infty} x_n y_n = xy$; (d) $\lim_{n\to\infty} 1/x_n = 1/x$;
(e) $\lim_{n\to\infty} y_n/x_n = y/x$.

Problem 5 Prove that if $\{x_n\}$ is a convergent sequence of real numbers then it is Cauchy.

Problem 6 (*) Prove that if $\langle x_n \rangle$ is a Cauchy sequence of real numbers then it is convergent.

Problem 7 Let $x_1, x_2, \ldots$ be a sequence which contains all rational numbers. (Since the set of rationals is countable, such a sequence exists.) Determine the set of cluster points of $\{x_n\}$.

Problem 8 Construct a sequence of real numbers such that the set of cluster points is the set of positive ingtegers.

Problem 9 Show that if x is a cluster point of the sequence $\{x_n\}$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to x.

Problem 10 Show that $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$.

Problem 11 Show that a sequence $\{x_n\}$ of real numbers is convergent if and only if $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$.

Problem 12 Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers. Show that

$$\begin{aligned} \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n &\leq \liminf_{n\to\infty}(x_n + y_n) \\ &\leq \liminf_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \\ &\leq \limsup_{n\to\infty}(x_n + y_n) \\ &\leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \end{aligned}$$

Real Analysis, 1999–Homework set # 2.

Problem 13 Prove that in any vector space X, for all $x, y, z \in X$ and $\alpha, \beta \in \mathcal{R}$,

(a) $x + y = x + z$ implies $y = z$;
(b) $\alpha x = \alpha y$ implies $x = y$ if $\alpha \neq 0$;
(c) $\alpha x = \beta x$ implies $\alpha = \beta$ if $x \neq \theta$ (θ denotes the null element in X.)

Problem 14 Show that if C and D are convex cones in a vector space then $C \cap D$ and $C + D$ are both convex cones.

Problem 15 Show that if w, x, y, z are elements of a normed linear space, then

$$\Big| \|x - z\| - \|y - u\| \Big| \leq \|x - y\| + \|z - u\|.$$

Problem 16 Let A be an arbitrary subset of a normed linear space. Prove that $\mathring{A}$ is an open set and $\bar{A}$ is a closed set.

Problem 17 Let A, B be open sets in a normed linear space. Show that $A + B$ is open.

Problem 18 Prove that the closure of a set A is the intersection of all closed sets containing A. In other words, $\bar{A}$ is the smallest closed set which contains A.

Problem 19 Let X be a normed linear space, and $A \subset X$. A point $x \in X$ is called a *limit point* of A if for any $\epsilon > 0$, there are infinitely many points of A in the sphere $S(x, \epsilon)$ of radius ϵ, centred at x. (Note that x may or may not be in A.) A point $x \in A$ is an *isolated point* of A if there exists an $\epsilon > 0$ such that $S(x, \epsilon) \cap A = \{x\}$. (Observe that all limit points and all isolated points are closure points.) Prove that every closure point of A is either a limit point or an isolated point.

Problem 20 (*) Given a normed linear space X, let $A \subset X$, and $x \in X$. The number

$$d(x, A) = \inf_{y \in A} \|x - y\|$$

is called the *distance between A and x*. Prove that

1. $x \in A$ implies $d(x, A) = 0$, but not conversely.
2. $d(x, A) = 0$ if and only if x is a closure point of A.
3. For fixed A, $d(x, A)$ is a continuous function of x.
4. $d(x, A) \leq 1$ if and only if x is a closure point of $A + S(0, 1)$.

Real Analysis, 1999–Homework set # 3.

Problem 21 Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in a vector space X are called *equivalent* if there exist positive constants $a, b > 0$ such that for all $x \in X$,

$$a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1.$$

Show that in $\mathcal{R}^n$ the norms

$$\|x\|_1 = \sum_{i=1}^n |\xi_i|, \quad \|x\|_2 = \sqrt{\sum_{i=1}^n \xi_i^2}, \quad \|x\|_\infty = \max_{i \leq n} |\xi_i|$$

are equivalent, where $\xi_1, \ldots, \xi_n$ denote the components of x.

Problem 22 (*) Prove that in a finite-dimensional vector space any two norms are equivalent.

Problem 23 Show that in the euclidean n-space E^n,

$$\lim_{n\to\infty} x_n = x$$

if and only if each component of x_n converges to the corresponding component of x.

Problem 24 Let A, B be compact sets in a normed linear space. Show that $A + B$ is compact.

Problem 25 Let $f : K \to Y$ be a continuous mapping defined on a compact subset K of a normed linear space X. Denote by $f(K)$ the set of elements $f(x)$, where $x \in K$. Show that $f(K)$ is a compact set.

Problem 26 Let $f : X \to X$ be a transformation from a normed linear space X to itself. f is called a *contraction mapping* if there exists a number $a < 1$ such that

$$\|f(x) - f(y)\| \leq a\|x - y\|$$

for all $x, y \in X$. Show that every contraction mapping is uniformly continuous (and therefore continuous).

Problem 27 (*) A real-valued function f defined on the interval $[0, 1]$ is *Lipschitz* if there exists a constant $L > 0$ such that for every $t, s \in [0, 1]$,

$$|f(t) - f(s)| \leq L|t - s|.$$

Let A_L denote the set of Lipschitz functions (with Lipschitz constant L). Show that $A_L \subset C[0, 1]$ and that A_L is a closed subset of $C[0, 1]$.

Problem 28 Let f be a monotone (increasing or decreasing) function defined on an interval $[a, b]$. Show that f is Riemann integrable.

Problem 29 Let f be a bounded function on $[a, b]$, continuous everywhere except for finitely many points. Show that f is Riemann integrable.

Real Analysis, 1999–Homework set # 4.

Problem 30 (*) Show that the space $R_p[0, 1]$ is not complete.

Problem 31 Prove Hölder's inequality for $p = 1$, $q = \infty$.

Problem 32 Prove Minkowski's inequality for $p = 1$ and for $p = \infty$.

Problem 33 Let $x = \{\xi_1\xi_2, \ldots\}$ be a bounded sequence of real numbers such that for some $1 \leq p < \infty$, $\|x\|_p < \infty$. Show that

$$\lim_{p\to\infty} \|x\|_p = \|x\|_\infty.$$

Problem 34 A *closed sphere* of radius $\epsilon > 0$ centered at x in a normed linear space X is defined as

$$S[x, \epsilon] = \{y \in X : \|x - y\| \leq \epsilon\}.$$

Assume that X is a Banach space, and let $S[x_1, \epsilon_1] \supset S[x_2, \epsilon_2] \supset \cdots$ be a nested sequence of closed spheres such that

$$\lim_{n\to\infty} \epsilon_n = 0.$$

Show that the intersection

$$\bigcap_{i=1}^\infty S[x_i, \epsilon_i]$$

is not empty, and contains exactly one point.

Problem 35 Let $T : \mathcal{R}^n \to \mathcal{R}^n$ be a linear transformation given by its matrix $A = (a_{i,j})_{n\times n}$. Consider the norm

$$\|x\| = \max_{i\leq n} |\xi_i| \qquad \text{for } x = (\xi_1, \ldots, \xi_n).$$

Show that if there is a constant $\alpha < 1$ such that for each $i = 1, 2, \ldots, n$

$$\sum_{j=1}^n |a_{i,j}| \leq \alpha,$$

then T is a contraction mapping. Can you derive a similar sufficient condition for contractiveness under the norm $\|x\| = \sum_{i=1}^n |\xi_i|$? (Recall that contractivity under *any* norm implies the existence of a unique fixed point.)

Problem 36 (*) Exhibit a transformation $T : X \to X$, where X is a Banach space such that for all $x, y \in X$,

$$\|T(x) - T(y)\| < \|x - y\| \quad (x \neq y),$$

but T does not have any fixed point. Compare this with the fixed point theorem seen in class.

Real Analysis, 1999–Homework set # 5.

Problem 37 Let f be a continuous function on $[a,b]$, and assume that K is a continuous function defined on the square $[a,b]\times[a,b]$. Prove that

$$g(t)=\int_a^b K(t,s)f(s)ds$$

is continuous.

Problem 38 (*) Let X be a Banach space, and assume that the transformation T mapping X onto X is such that there exists a constant $K>1$ such that

$$\|T(x)-T(y)\|\geq K\|x-y\| \quad \text{for all } x\neq y.$$

Prove that T has a unique fixed point in X.

Problem 39 Let $T:X\to X$ be a contraction mapping defined on a Banach space satisfying

$$\|T(x)-T(y)\|\leq \alpha\|x-y\| \quad \text{for all } x,y\in X,$$

where $\alpha<1$, and let $x_0\in X$. Define $x_n=T^n(x_0)$, and let $x=\lim_{n\to\infty}x_n$. (We saw in class that the limit exists.) Show that for all n,

$$\|x_n-x\|\leq\frac{\|x_0-x_1\|}{1-\alpha}\alpha^n.$$

Thus, the method of successive approximation converges to the fixed point of T at an exponentially fast rate.

Problem 40 Consider the functions

$$x_n(t)=\sin(nt)$$

in the pre-Hilbert space of continuous functions defined on $[-\pi,\pi]$, equipped with the inner product

$$(x|y)=\int_{-\pi}^{\pi}x(t)y(t)dt.$$

Show that the set of these functions is closed and bounded but not compact.

Problem 41 Let M and N be closed subspaces of a Hilbert space X such that $M\perp N$, and let $x\in X$. Show that the direct sum $M\oplus N$ is closed and that the orthogonal projection of x onto $M\oplus N$ is $m+n$, where m is the orthogonal projection of x onto M and n is the orthogonal projection of x onto N.

Real Analysis, 1999–Homework set # 6.

Problem 42 Prove that $(x|y)=\int_a^b x(t)y(t)dt$ is an inner product in the vector space of all continuous functions on $[a,b]$. (The obtained pre-Hilbert space is denoted by $C^2[a,b]$.)

Problem 43 A function $x\in C^2[-1,1]$ is called *even* if for all $t\in[-1,1]$, $f(t)=f(-t)$, and x is *odd* if $f(t)=-f(-t)$ for all $t\in[-1,1]$. Let M and N be the subspaces of all even and odd functions, respectively. Show that $M\perp N$ and that $C^2[-1,1]=M\oplus N$.

Problem 44 Show that if S is a set in a Hilbert space, then

$$S^{\perp\perp}=\overline{[S]},$$

that is, $S^{\perp\perp}$ is the smallest closed subspace containing S.

Problem 45 Find the coefficients a,b of the linear function $x(t)=at+b$ which minimizes $\int_0^1\left(e^t-x(t)\right)^2dt$.

Problem 46 Show that if $\|x_n\|\to\|x\|$ and $(x_n|x)\to(x,x)$ then $x_n\to x$.

Problem 47 (*) Let X be a Hilbert space, and assume that there exists a sequence of vectors $\{x_n\}$ such that every element $x\in X$ may be written as a linear combination of finitely many x_n's. Prove that X is finite-dimensional.

Problem 48 (*) Let X be a normed linear space. Prove that it is possible to define an inner product on X if and only if the norm satisfies the parallelogram law, that is, for all $x,y\in X$,

$$\|x+y\|^2+\|x-y\|^2=2\left(\|x\|^2+\|y\|^2\right).$$

Conclude that l_p is a pre-Hilbert space if and only if $p=2$.

Problem 49 Let X be the pre-Hilbert space of all sequences containing at most finitely many nonzero elements (with the usual inner product $xy=\sum_{i=1}^{\infty}\xi_i\eta_i$). Show that the set K of all $x=(\xi_1,\xi_2,\ldots)\in X$ for which $\sum_{i=1}^{\infty}\xi_i=1$ is a convex set. Prove that there is no element k_0 of K for which $\|k_0\|=\inf_{k\in K}\|k\|$. Which condition of the corresponding Theorem seen in class fails to hold here?

Problem 50 Let $y_1,\ldots,y_n$ be linearly independent elements in a Hilbert space, and let $c_1,\ldots,c_n$ be constants. Show that the constrained optimization problem

$$\text{minimize } \|x\| \quad \text{subject to } (x|y_i)\geq c_i, \quad i=1,\ldots,n$$

has a unique solution.

Real Analysis, 1999–Homework set # 7.

Problem 51 Show that in a finite-dimensional normed linear space every linear functional is continuous (i.e., bounded).

Problem 52 (*) Let X be an arbitrary infinite-dimensional normed linear space. Define a linear functional on X which is not continuous.

Problem 53 Let X^* be the dual of an n-dimensional normed linear space X. Determine the dimension of X^*.

Problem 54 We say that a set H in a normed linear space X is a *hyperplane* if it is given by

$$H = \{x \in X : f(x) = c\},$$

where f is a linear functional on X and c is some constant. We say that H *separates* the set $A \subset X$ from $B \subset X$ if for each element $x \in A$ and $y \in B$,

$$f(x) \geq c \quad \text{and} \quad f(y) < c.$$

Let X be a Hilbert space, let K be a closed convex subset of X, and let $x \notin K$ be an arbitrary vector in X. Show that there exists a hyperplane H which separates K from $\{x\}$.

Problem 55 Let a, b be real numbers. Show that

$$f(x) = ax(0) + bx(1)$$

is a bounded linear functional on $C[0,1]$. Determine $\|f\|$.

Problem 56 Show that

$$f(x) = \int_0^{1/2} x(t)dt - \int_{1/2}^1 x(t)dt$$

is a bounded linear functional on $C[0,1]$. Determine $\|f\|$.

Problem 57 (*) Let c_0 denote the vector space of all sequences $x = \{\xi_i\}$ converging to zero, equipped with the norm

$$\|x\| = \sup_i |\xi_i|.$$

Show that this expression defines a norm. Prove that the dual space c_0^* of c_0 is l_1 in the sense that there exists a "one-to-one" and "onto" transformation $T : c_0^* \to l_1$ which is norm preserving, that is, for all linear functionals $f \in c_0^*$,

$$\|f\| = \|T(f)\|.$$

(We say that c_0^* and l_1 are *isomorphic*.)

Real Analysis, 1999–Homework set # 8.

Problem 58 Let p be a continuous sublinear functional defined on a normed linear space X, and let $f : X \to \mathcal{R}$ be a linear functional satisfying $f(x) \leq p(x)$ for all $x \in X$. Prove that f is continuous.

Problem 59 Let X be a normed linear space. Show that for every $x \in X$ there exists a nonzero bounded linear functional $f \in X^*$ such that

$$f(x) = \|f\| \cdot \|x\|,$$

that is, the supremum in the definition of $\|f\|$ is achieved with equality at this x.

Problem 60 Let X be a normed linear space. Show that for all $x \in X$,

$$\|x\| = \sup_{f \in X^* : \|f\|=1} |f(x)|.$$

Problem 61 (*) Let X be a normed linear space. Prove that if X^* is separable then X is separable.

Problem 62 Show that l_p is separable if $1 \leq p < \infty$.

Problem 63 Show that l_∞ is not separable.

Problem 64 Consider the set $H \subset l_2$ consisting of all sequences $x = (\xi_1, \xi_2, \ldots)$ satisfying

$$|\xi_1| \leq 1, \quad |\xi_2| \leq \frac{1}{2}, \ldots, \quad |\xi_n| \leq \frac{1}{2^{n-1}}, \ldots$$

(H is called the *Hilbert cube*.) Show that S is convex but it doesn't have any interior point.

Real Analysis, 1999–Homework set # 9.

Problem 65 Let X be an arbitrary normed space. Determine the Minkowski functional of the sets $\{x : \|x\| < 1\}$ and $\{x : \|x\| \le 1\}$.

Problem 66 Let $p(x)$ be the Minkowski functional of a convex set K in a normed linear space. Prove that

$$\overset{\circ}{K} = \{x : p(x) < 1\}.$$

Problem 67 (*) Let p be a continuous nonnegative subadditive functional in a normed vector space. Show that for any $c > 0$, the set

$$S = \{x : p(x) \le c\}$$

is convex. Prove that $\overset{\circ}{S} = \{x : p(x) < c\}$. Thus, θ is an interior point of S.

Problem 68 Define the functional

$$f(x) = x(1/2)^2$$

for $x \in C[0, 1]$. Determine the Fréchet differential of f.

Problem 69 Let $g : \mathcal{R} \times [0, 1] \to \mathcal{R}$ be a real-valued function such that its partial derivative $g_x(x, t)$ with respect to the first variable is continuous. Define the functional

$$f(x) = \int_0^1 g(x(t), t)dt$$

on $C[0, 1]$. Show that the Fréchet differential of f is

$$\delta f(x; h) = \int_0^1 g_x(x, t)h(t)dt.$$