Barcelona Graduate School of Mathematics, Fall 2013

Random Structures and the Probabilistic Method

References and Exercises: Markov Chains, November 12-14

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1. References

(1) Levin, Peres and Wilmer, "Markov chains and mixing times", 2009, http://pages.uoregon.edu/dlevin/MARKOV/markovmixing.pdf (Chapters 1, 2, 4, 9, 10.3)

(2) Doyle and Snell, "Random walks and electric networks", 2006, http://www.math.dartmouth.edu/ doyle/docs/walks/walks.pdf (Chapter 1)

(3) Scribes of "Markov chain Monte Carlo: Foundations and Applications" taught by Alistair Sinclair, UC Berkeley, Fall 2009, http://www.cs.berkeley.edu/ sinclair/cs294/f09.html (Lectures 1-3)

(4) Scribes of "Probability and Computation" taught by Costis Daskalakis, MIT, Spring 2011, http://people.csail.mit.edu/costis/6896sp11 (Lectures 1-3)

(5) Scribes of "Randomness and Derandomization"' taught by Dieter van Melkebeek, U of Wisconsin Madison, Fall 2002, http://pages.cs.wisc.edu/ dieter/Courses/2002F-CS830/, (Lecture 12)

(6) Mitzenmacher and Upfal, "Probability and computing", 2005, (Chapter 7)

(7) Motwani and Raghavan, "Randomized Algorithms", 1995, (Chapter 6)

2. Exercises

(1) (LPW 1.5) Let T be a tree. Show that the graph whose vertices are proper 3-colorings of T and whose edges are pairs of colorings which differ at only a single vertex is connected.

(2) Show that if the graph of a Markov chain is acyclic then that chain is reversible.

(3) (MU 7.20) Consider the gambler's ruin problem in the case where the game is not fair; instead the probability of losing a euro each game is 2/3 and the probability of winning a euro each game is 1/3. Suppose that you start with i euros, and finish either when you reach n or lose it all. Let W_t be the amount you gain after t rounds of play. Determine the probability of finishing with n euros. (Hint: consider how 2_t^W depends on t.) Generalize the result to the case where the probability of losing is $p > 1/2$.

(4) (MU 7.26) Let n equidistant points be marked on a circle. Without loss of generality, we think of the points as being labeled clockwise from 0 to $n-1$. Initially, a wolf begins at 0 and there is one sheep at each of the remaining $n-1$ points. The wolf takes a random walk on the circle. For each step, it moves with probability 1/2 to one neighboring point and with probability 1/2 to the other neighboring point. At the first visit to a point, the wolf eats a sheep if there is still one there. Which sheep is most likely to be the last eaten?

(5) (MU 7.10) Let G be a 3-colorable graph. Show that there exists a coloring of the graph with two colors such that no triangle is monochromatic. Then consider the following algorithm for finding such a 2-coloring. The algorithm begins with an arbitrary 2-coloring of G. While there are any monochromatic triangles in G the algorithm chooses one such triangle and changes the color of a randomly chosen vertex of that triangle. Derive an upper bound on the expected number of such recoloring steps before the algorithm finds a 2-coloring with the desired property.