Barcelona Graduate School of Mathematics, Fall 2013

Random Structures and the Probabilistic Method

Exercises: Markov Chains, November 19

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(1) Let Ω be the set of k-element subsets of $\{1, 2, \ldots, n\}$. Consider the following Markov chain on Ω: if $X_t = S$, pick an element $a \in S$ and $b \notin S$, independently and uniformly at random, and set $X_{t+1} = S + b - a$. Find the stationary distribution of this Markov chain and give an upper bound on the mixing time. The bound should be at most $O(n \log k)$ (and $O(k \log k)$ is doable).

(2) The problem of generating a uniformly random k-coloring of a tree is much simpler than the case of general graphs. Give an exact algorithm for this problem. Do the same for the problem of generating an independent set - a subset of the vertices of a graph is an *independent* set if no two of the vertices in the set are connected by an edge.

(3) Consider the following algorithm for shuffling a deck of n cards. At each step, two cards are chosen uniformly at random from the deck, and their positions are exchanged. (If both choices give the same card, no change occurs.) Find the stationary distribution of this Markov chain and show that its mixing time is at most $O(n^2)$.