Recall that the clique number $\omega(G)$ is the size of the largest complete subgraph of $G$ and the independence number (stability number) $\alpha(G)$ is the size of the largest independent set (i.e., the clique number of the complement of $G$).

Consider $\xi_n(p)$ with $p$ fixed. In fact, to simplify notation, we take $p = \frac{1}{2}$. The clique number $\omega$ (and thus $\alpha$ as well) may be estimated very precisely by the first and second moment methods:

Let $N_k$ be the number of cliques of size $k$. Then

$$P(\omega \geq k) = P(N_k \geq 1) \leq \mathbb{E}N_k = \binom{n}{k} 2^{-k}$$

$$\leq \left( \frac{ne^{-1}}{k} \right) 2^{-k} \rightarrow 0 \text{ as } k \rightarrow \infty$$

The lower bound follows from the second moment method. We need to show that

$$\frac{\text{var}(X_k)}{\mathbb{E}(X_k)^2} = \frac{\mathbb{E}(X_k)^3}{\mathbb{E}(X_k)^2} - 1 \rightarrow 0$$

when the value of $k$ is slightly decreased.

But

$$\frac{\mathbb{E}(X_k)^2}{\mathbb{E}(X_k)^2} - 1 = \left( \frac{\mathbb{E}(X_k)}{\mathbb{E}(X_k)} \right)^2 - 1 \leq \sum_{i=1}^{k} \frac{\binom{k}{i} \binom{k}{i-1} 2^{-i}}{\binom{n}{k} 2^{-k}} \leq \sum_{i=1}^{k} \frac{\binom{k}{i} \binom{k}{n-2} 2^{-i}}{\binom{n}{k} 2^{-k}}$$
After careful bounding (because!) we get the following:

**Theorem** N.H.p.,

\[ 2 \log_2 n - 2 \log_2 \log_2 n \log_2 e - \frac{5}{2} \leq W \leq 2 \log_2 n - 2 \log_2 \log_2 n + 2 \log_2 e + \frac{1}{2} \]

(In \( G(n, p) \) the base of the logs is \( \frac{1}{p} \) for the clique number and \( \frac{1}{1-p} \) for the independence number.)
The chromatic number

Next we study the magnitude of the chromatic number \( \chi \) of \( G(n, p) \) for fixed \( p \). (Again, we may take \( p = \frac{1}{2} \)).

Since in any proper coloring any set of vertices of the same color is an independent set, \( \chi \geq \frac{n}{k} \).

Since \( \alpha \leq 2 \log n - 2 \log \log n \) with \( p \),

\[ \chi \geq \frac{n}{2 \log n (1 - o(1))} \]

It turns out that this bound is tight! (Rödl's '88)

The key ingredient is a sharp estimate for the probability of non-existence for large independent sets:

**Claim.** The probability that \( G(n, \frac{1}{2}) \) has no independent set of size \( 2 \log n - 2 \log \log n \) is at most

\[ \exp \left( - \frac{n^2}{66 \log^2 n} \right) \]

**Proof.** Let \( k = 2 \log n - 2.1 \log \log n \) and use Janson's inequality to bound the probability of non-existence of stable sets of size \( k \). If \( X_k \) denotes the number of stable (independent) sets of size \( k \), then

\[ P(X_k = 0) \leq \exp \left( - \frac{(EX_k)^2}{\sum X_{\alpha} X_{\beta} \mathbb{I}_{\alpha \cap \beta \leq k} \mathbb{I}_{\alpha, \beta \leq \frac{1}{2} n/3} |\alpha \cap \beta| \leq k} \right) \]

indicates that \( B \) is a stable set.
just like before,

\[
\frac{\sum E_{X_A X_B}}{(EX_k)^2} \leq \frac{\sum (i)(k^{-E}) (1)}{(k^2)} \leq \frac{6 \log(n)^5}{\log^2(n)}
\]

Exercise.

We may use the sharp estimate above in combination with the union bound:

\[
P(\exists \text{ a subgraph of } G(n, \varepsilon) \text{ with } \geq \frac{n}{(\log n)^2} \text{ vertices and stable set of size } \geq \frac{2 \log \log(n) - \log \log \log(n)}{2(\log(n) - 2)} \geq \frac{(\log n)^2 - 2}{\log(n) - 2}) 
\leq 2 \exp(-\frac{n^{\frac{3}{2}}}{66 \log \log \log(n)^5}) \leq 2 \exp(-\frac{n^2}{(\log n)^5}) \to 0 
\]

upper bound on the

In particular, we may color the graph as follows: Find a stable set of size \( \geq \frac{2 \log \log(n) - \log \log \log(n)}{2(\log(n) - 2)} \) and give it color #1. Remove those vertices. In the remainder, again there is a stable set of the same size, so color them by color #2.

We can continue until we have \( \frac{n}{(\log n)^2} \) vertices remaining. The rest of the vertices will all get a different color. The number of colors used is at most

\[
\frac{n}{2 \log \log(n) - 7 \log \log(n)} + \frac{n}{(\log n)^2} \leq \frac{n}{2 \log \log(n) - 7 \log \log(n)}
\]

Thus,

\[
X = \frac{n}{2 \log \log(n) (1 + o(1))}
\]
Greedy coloring

Consider again \( \chi(n, \frac{1}{k}) \) and color it greedily as follows: order the vertices \( v_1, v_2, \ldots, v_n \). Color \( v_1 \), then \( v_2 \), then \( v_3 \), etc. such that \( v_i \) gets the first available color not used by any of the vertices among \( \{v_1, \ldots, v_{i-1}\} \) that \( v_i \) is adjacent to.

Then, \( \frac{n}{k} \) h.p., the greedy algorithm colors \( \chi(n, \frac{1}{k}) \) with at most \( \frac{n}{\log n - 3 \log \log n} \) colors.

This is within a factor of 2 of the optimal coloring.

**Proof.** Let \( M \) denote the number of colors used by the algorithm, and let \( k = \left\lceil \frac{n}{\log n - 3 \log \log n} \right\rceil \). Let \( A_i \) be the event that vertex \( i \) is the first to receive color \( k+1 \). Thus,

\[
P(M > k) = \sum_{i=1}^{n} P(A_i).
\]

To bound \( P(A_i) \), we may assume that exactly \( k \) colors have been used by vertices \( 1, \ldots, i-1 \) (otherwise the conditional probability of \( A_i \) is zero). Fix a coloring \( C_1, \ldots, C_k \) of \( 1, \ldots, i-1 \). Then
\[ P(A_i | c_1, \ldots, c_n) = \prod_{i=0}^{k} (1 - \left(\frac{k}{n}\right)^{\frac{3\sqrt{2} \log n}{k}}) \]

Proof that at least one vertex from the color class \( c_i \) is connected to at least one vertex from the color class \( c_j \):

\[
\leq \left(1 - \left(\frac{k}{n}\right)^{\frac{3\sqrt{2} \log n}{k}}\right)^k = \left(1 - \frac{2^{-\frac{3\sqrt{2} \log n}{k}}}{2^{\frac{3\sqrt{2} \log n}{k}}}ight)^k = \left(1 - \frac{1}{2^{\frac{3\sqrt{2} \log n}{k}}}ight)^k = o\left(\frac{1}{n}\right).
\]

Thus, \( \sum_{i=2}^{n} P(A_i) \to 0 \).

See the survey of