INSTANTANEOUS AND NON-INSTANTANEOUS ADJUSTMENT TO EQUILIBRIUM IN TWO-SECTOR GROWTH MODELS
The purpose of this paper is to analyze some local stability properties of steady states in the two-sector growth model introduced by Uzawa [3], [4]. In the model one distinguishes between a set of variables which is always adjusted to fulfill some "momentary equilibrium" conditions and a variable (aggregate capital) in whose dynamic path one is interested. This abstraction rules out short run inefficiency and short run disequilibrium behavior. It is postulated partly to simplify the analysis partly for lack of an adequate theory; it is justified by the fact that, presumably, if the short run rates of accumulation of the variables are sufficiently high relative to the pace of capital accumulation all the qualitative conclusions of the instantaneously adjusted model (I. M. from now on) remain valid (9).

We want to investigate situations in which a variable is not instantaneously adjusted. In Part I we consider the case where labor does not shift instantaneously between sectors, while in Part II capital is the factor that does not move instantaneously. In Part III, as in Uzawa [4], the wage- rental ratio is rigid in the short-run and unemployment of factors is possible. Let us observe that the significance of the questions raised in this paper can be viewed from two standpoints; either as proper of an economy with short run disequilibrium behavior or as corresponding to an economy with possibly inefficient short-run equilibrium (although inefficiency is ruled out for steady states).

Our plan is to formulate conditions which guarantee local stability of a steady state regardless of speeds of adjustment. In particular we are interested in the validity of this statement: Consider a steady-state of an economy subject alternately to the following two dynamic specifications. In the first short run equilibrium is instantaneous, in the second some particular variable is, momentarily, fixed. Then

(9) This article was presented at the European meeting of the Econometric Society, Barcelona, September 1957.
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(10) The term "presumably" is needed because this is not, in general, a true statement. However it can be checked that it holds in the two sector growth model.
the local stability of the steady state under the first specification implies its local stability under the second specification. Of course this is a very strong property which cannot be expected to be generally true. We prove that it holds in the case where capital is not instantaneously substitutable between sectors. In spite of the apparent symmetry it does not hold when labour is the factor which is not instantaneously movable. As was already pointed out by Usawa [4] it does not hold either when the wage-real cost ratio is given in the short-run. However, one can derive, in the last two cases, fairly broad stability conditions.

We observe that the elasticity of substitution conditions (for local stability) seem to generalize more easily to situation with short run disequilibrium than the capital intensity conditions.

We consider only the case with a uniform saving ratio. The calculations have been worked out, too, for the classical saving assumption, and the results are given in footnotes. The analysis and proofs are presented in such a way as to take maximum advantage of the existing results for the L.M.

Finally, one word concerning the restriction of the analysis to local stability. The reason for it is that we want to use fairly general adjustment equations. It seems that in order to obtain global stability results [1] one would need more specific forms of the adjustment equations, at the cost of losing generality in the corresponding local results.

**Introduction**

The momentary equilibrium of the two-sector model with instantaneous adjustment is characterized by:

\[
\begin{align*}
F_1(K_1, L_1) & = 0, \\
L_1 + L_2 & = L, \\
K_1 + K_2 & = K,
\end{align*}
\]

\[
\begin{align*}
\frac{\Delta K_1}{\Delta L_1} & = \frac{\Delta F_1}{\Delta L_1}; \\
\frac{\Delta F_2}{\Delta K_2} & = \frac{\Delta F_2}{\Delta L_2}.
\end{align*}
\]

\[
s(F_1 + \rho F_2) = \rho F_1 \quad (0 < s < 1)
\]

Its dynamic path is given by:

\[
\begin{align*}
\hat{L} & = \bar{m} \\
\hat{K} & = F_1
\end{align*}
\]

[1] Except in Part III where the extension of the analysis is quite straightforward.
We assume the $F_i$'s are linear homogeneous, and
\[
\frac{\partial F_i}{\partial x} > 0 ; \quad \frac{\partial F_i}{\partial y} > 0 ; \quad \frac{\partial^2 F_i}{\partial x^2} < 0 ; \quad \frac{\partial^2 F_i}{\partial y^2} < 0 \quad (i = 1, 2)
\]
When it is convenient we also assume that the Inada Conditions hold. Consider the dynamic system:
\[
\begin{aligned}
\dot{x}_i &= g_i (x_1, x_2) \\
\dot{y}_i &= g_2 (x_1, y_2)
\end{aligned}
\tag{A}
\]
with $g_1, g_2$ continuously differentiable. Let $X = (\hat{x}_1, \hat{x}_2)$ be a stationary point of (A). If
\[
\left| \frac{\partial x_1}{\partial x_1} \right|_{x_1 = \hat{x}_1, y_1 = \hat{y}_1} < 0, \quad \left| \frac{\partial x_2}{\partial x_1} \right|_{x_1 = \hat{x}_1, y_1 = \hat{y}_1} < 0,
\]
\[
\left| \frac{\partial y_1}{\partial x_1} \right|_{x_1 = \hat{x}_1, y_1 = \hat{y}_1} < 0, \quad \left| \frac{\partial y_2}{\partial x_1} \right|_{x_1 = \hat{x}_1, y_1 = \hat{y}_1} < 0,
\]
with $\hat{y}_1 = \frac{\partial y_1}{\partial x_1} \hat{x}_1$. We state as a lemma a well known stability result:

Lemma 1: If
\[
\left| \frac{\partial x_1}{\partial x_1} \right| < 0, \quad \left| \frac{\partial x_1}{\partial y_1} \right| < 0, \quad \left| \frac{\partial y_1}{\partial x_1} \right| < 0, \quad \left| \frac{\partial y_1}{\partial y_1} \right| < 0
\]
the system (A) is locally stable at $\hat{X}$.

This follows since under the stated conditions the Jacobian matrix of the system at $\hat{X}$ is a Hessian matrix which — for the two variables cases — is sufficient for local stability.

**Part I**

We will assume in this part that labor does not move instantaneously and that all markets are cleared. This means that in the short run the wage rate may differ between sectors. We will further assume that the relative labor force in every sector changes over time in the same direction as the wage differential. Formally, let
\[
m = \frac{L_1}{L_2}
\]
\[
k_i = \frac{K_i}{L_i}
\]
\[
f_i (k_i) = F_i (k_i, 1)
\]
\[
x_t = f_t - h f_t'
\]
\[
w_t = \beta (f_t - h f_t')
\]
\[
\omega_{it} = \frac{f_t}{w_t} - k_i \quad (i = 1, 2)
\]
For \( s < m < 1 \), \( k > 0 \) given, the momentary equilibriums of the economy is determined by:

\[
\frac{nhk}{s} + \left( \frac{s - m}{m - k} \right) k = k \quad (6)
\]

\[
f'_{m}(h) = f'_{k}(h) \quad (6)
\]

\[
\frac{1}{[m_{s}(h)]} + \frac{1}{f'(h)} = \frac{1}{(1 - m)(s + h)(u)} \quad (7)
\]

Equations (6) and (7) reduce to

\[
s(s + h) - (1 - s)(1 - m) (s + h) \quad (8)
\]

Under the assumptions in (1), (5) and (8) have a unique solution in \( h \) and \( k \).

The path of \( m(t) \), \( k(s) \) is described by the following dynamic equations:

\[
\dot{m} = cH(m, k), \quad \dot{k} = \text{sign}(m - m_{s}) - e > 0 \quad (9)
\]

\[
\dot{k} = (1 - m)f'_{m} - nh \quad (10)
\]

We assume that \( F \) is continuously differentiable, and the system (5)-(10) has a unique solution for every \( (m_{0}, k_{0}) \) which is continuous with respect to initial conditions. The final condition ensures that the solution stays in the non-specialization region \( k(t) > 0 \), \( 1 > m(t) > 0 \). Expressions for \( \lambda_{1}, \lambda_{2} \) are derived in A.1 of the Appendix. Define

\[
\alpha_{k}(m, k) = \frac{\partial k}{\partial m_{s}} \quad (11)
\]

The elasticities of substitution. It is clear that the steady states of the model (5)-(10) coincide with the steady states of the model (1). Suppose \( \alpha_{k} \), \( \lambda_{k} \) is a steady state. For short let \( \mathbf{t} = (\hat{m}, \hat{k}) \). We will study now the stability properties of this model. If \( x = (\hat{m}, \hat{k}) \), \( \hat{m} = 0 \), then:

\[
s \in (0, \infty), \quad \alpha_{k} = 0, \quad \alpha = 0 \quad (12)
\]

Proof: In \( \hat{x} \),

\[
\dot{m} = \frac{(s - m) [s + h + m_{s}]}{s [s + h + m_{s}]} \quad (13)
\]

(1) In the \( (h_{0}, k_{0}) \) space (8) gives a monotonic increasing relation starting at the origin and (5) is a decreasing one.
From Uzawa [7] we know that, if \( \omega_1 = \omega_0 - \omega_s \), \( \delta \) is a function of \( \omega \)
and
\[
\frac{d\delta}{d\omega} > 0
\]  
\( (12) \)

Combining (11) and (12) and manipulating:

\[
\text{sign} \left[ \frac{dm}{d\omega} \right]_{\omega = \omega_0} = \text{sign} \left[ \frac{\omega + \sigma_\delta \omega_1}{\omega + \delta_1} - \frac{\omega + \sigma_\delta \omega_1}{\omega + \delta_1} \right]
\]
\( (13) \)

where all the terms are functions of \((m, \delta)\).

However

\[
\frac{2m_0}{2\delta} = \frac{\frac{dm}{d\omega}}{\frac{d\delta}{d\omega}} \quad \left[ \frac{dm}{d\omega} \right]_{\omega = \omega_0}
\]
\( (14) \)

and, by A.2 in the Appendix:

\[
\text{sign} \left[ \frac{2m}{2\delta} \right] = \text{sign} \left[ \frac{d\omega_1 \delta_1}{d\delta} \frac{d\omega_1 \delta_1}{d\omega} - \frac{d\omega_1 \delta_1}{d\omega} \frac{d\omega_1 \delta_1}{d\delta} \right]
\]
\( (15) \)

This ends the proof.

b) If \( \sigma_\delta (\omega, \delta) \geq 1 \) or \( \sigma_\delta (\omega, \delta) \cdot \delta \geq \sigma_\delta (\omega, \delta) \cdot \delta \),
then

\[
\frac{\delta \delta}{\delta \delta} < 0
\]
where \( \delta = (\omega, \delta) \) is a steady state.

Proof: From (10)

\[
\frac{\delta \delta}{\delta \delta} = (e - m) f' (\omega) \frac{\delta \delta}{\delta \delta} \left[ \frac{\omega + \delta_1}{\delta + \delta_1} \right]
\]
\( (17) \)

From A-3 in the Appendix,

\[
\frac{\delta \delta}{\delta \delta} = \Omega (m, \delta) m_0 + (1 - m) \delta_0 + (1 - w) \omega_0 \quad \left[ \frac{dm}{d\omega} \right]
\]

(*) When \( \omega = \omega_0 \), \( \frac{dm}{d\omega} = \text{sign} \left[ \frac{\omega_0 - \omega_0}{\delta} \right] \).

(*) For example: \( \sigma_\delta (\omega, \delta) \geq \sigma_\delta (\omega, \delta) \) and \( h_1 (\omega, \delta) \geq h_1 (\omega, \delta) \)
\[ \Omega (m, h) = \frac{\omega}{\sigma_i h_i} \left( 1 + \frac{\omega + h}{h} \right) \]  
(28)

and \( \Omega (m, h) \geq 1 \) if \( \sigma_i \geq 1 \) or if \( \sigma_i h_i \geq \sigma_i h \).

Therefore, under these conditions, and taking (5) into account,

\[ \frac{h_i}{\omega + h_i} \leq \frac{h}{\omega + h} \quad \text{or} \quad \frac{h}{\omega + h} < 0. \]

Combining Lemma 1, a) and b) we can state the following proposition:

**Proposition 1:**

If the steady state \( \bar{k} \) corresponding to the model (1) with uniform saving rate and instantaneous adjustment of labor is locally stable and either 1) \( \sigma_i (\bar{m}, \bar{h}) \geq 1 \) or 2) \( \sigma_i (\bar{m}, \bar{h}) \cdot h_i \geq \sigma_i (\bar{m}, \bar{h}) \cdot h \), the steady state \( (\bar{m}, \bar{h}) \) corresponding to the model (5)-(10) will be locally stable.

In particular, using the results of Uzawa [4] and Drandakis [1], we can assert that the steady state \( (\bar{m}, \bar{h}) \) in the model (5)-(10) will be locally stable if

1) \( \sigma_i (\bar{m}, \bar{h}) \geq 1 \) or
2) \( h_i (\bar{m}, \bar{h}) \geq h (\bar{m}, \bar{h}) \) and \( \sigma_i (\bar{m}, \bar{h}) \geq \sigma_i (\bar{m}, \bar{h}) \).

As the following example shows, the stability of an I.M. steady state does not necessarily yield its stability in the N.I.M. (5)-(10) (\(^*)\).

Let

\[ f_a (k_a) = \frac{1}{1 + 2^{1/3}} \left[ 1 + \frac{1}{1 + 2^{1/3}} \right] \]
\[ f_b (k_b) = \frac{3}{2} \frac{k_b^{2/3}}{1 + 2^{1/3}} \]
\[ s = 3/2, \quad \sigma = 9/14. \]

(\(^*)\) Under the classical savings assumption we have found that the local stability of a steady-state equilibrium in the I.M. implies the local stability of this equilibrium in the N.I.M.
A steady state for this economy is given by:

\[ \hat{h} = 7/5, \quad \hat{w} = 2.5 \]

corresponding to:

\[ \hat{h}_1 = 2, \quad \hat{h}_2 = 1, \quad \hat{f}_1 = 1, \quad \hat{f}_2 = 3/2, \quad \hat{f}_3 = 3/4, \quad \hat{f}_4 = 1/3, \quad \hat{b} = 4/5, \quad \hat{a}_0 = 1, \quad \hat{a}_1 = 1/24, \quad \hat{a}_2 = 1. \]

The capital intensity condition is fulfilled, \( \hat{h}_1 > \hat{h}_2 \); however if we substitute those values in (17) we find:

\[
\left| \frac{2\hat{h}_2}{\hat{h}_3} \right| > \frac{2}{7/5} = \frac{10}{7} \Rightarrow \frac{2\hat{h}_3}{\hat{h}_2} < \frac{7}{2} \Rightarrow \frac{2\hat{h}_2}{\hat{h}_3} > \frac{7}{2} \Rightarrow \frac{2\hat{h}_3}{\hat{h}_2} < \frac{7}{2} \Rightarrow \frac{2\hat{h}_2}{\hat{h}_3} > \frac{7}{2}
\]

By (16)

\[
\left| \frac{\hat{h}_2}{\hat{h}_3} \right| > 0
\]

Therefore the behavior of the system at (2/4, 7/5) is cyclic. There is always a value of \( \epsilon \) for which the trace of the Jacobian at (2/4, 7/5) is positive, i.e., the cycles will be divergent.

**Part II**

In this part we will assume that capital does not move instantly across or that it is not shiftable at all. Competitive conditions prevail in each sector. The short-run rate of return on capital, \( r_s \), may differ between sectors. We will assume that investment functions are such that the direction of movement of the relative amount of capital between sectors according with the current rate-of-return differential (7). Under those conditions the steady state of the model coincides with the steady states of a model with perfect mobility of capital.

In order to make the treatment quite simple we will use a notation which will bring out the formal analogy with the models treated in Part I.

\[
\text{Call } u = \frac{K_1}{K}, \quad \tau_i = \frac{K_i}{K}, \quad f_i(\tau_i) = f_i(x_i, \tau_i),
\]

\[
r_1 = f_1 - \gamma f_2, \quad r_2 = f (\hat{h}_2), \quad w_1 = f_1,
\]

\[
w_2 = f_2, \quad \bar{w}_1 = \frac{r_1}{w_1} = 1/a_1
\]

(1) An interesting analysis of a similar situation has been given by Isoula [2].
Observe that:

$$a_t = \frac{d\theta}{da_t} \frac{d\tau_0}{d\theta_t} = \frac{d\tau_0}{da_t}$$  \tag{19}$$

\(a_t\) and \(\tau_0 = 1/\theta_t\) are state variables.

For given \(0 < u < 1\), \(\tau > 0\) the momentary equilibrium of the economy is determined by:

$$u_t = \frac{(1 - u)}{\tau_0} = \tau$$  \tag{20}$$

$$f_1(\tau_0) = g(f_1(u))$$  \tag{21}$$

$$(1 - s)\left[u_t + (1 - u)\frac{\partial f}{\partial u}\right] = u_t$$  \tag{22}$$

(21) and (22) reduce to:

$$su = \left(\frac{u_t}{\tau_0} + \tau\right) = (1 - s) \frac{(1 - u)}{(\beta u_t + \tau)}$$  \tag{23}$$

The path of \(u(t), \tau(t)\) is described by the following dynamic equations:

$$\dot{u} = cH(u, \tau); \quad \text{sign } \dot{u} = \text{sign } (\tau_0 - \tau)$$  \tag{24}$$

$$\dot{\tau} = \frac{u}{u - (1 - u)f_1(\tau_0)}$$  \tag{25}$$

We assume that \(H\) is continuously differentiable and (24)-(25) have a unique solution for every \((u_0, \tau_0)\) which is continuous with respect to initial conditions. Suppose \((\dot{u}, \dot{\tau}) = 0\) is a steady-state equilibrium of the system (20)-(25). Suppose that \(\dot{u} = 0\) at \(u = \bar{u}\), \(\dot{\tau} = 0\). Then:

a) \[
\begin{align*}
\left| \frac{\dot{u}}{\dot{\tau}} \right| < 0
\end{align*}
\]

Proof: Equations (20), (22) and (24) are formally identical with equations (5), (8) and (9). Taking into account (19) and that

$$\text{sign } \frac{d\tau}{du} = \text{sign } \frac{dx}{du}$$

a simple relabelling of symbols in A-1 and A-2 of the Appendix yields the result.

b) \[
\begin{align*}
\left| \frac{\dot{\tau}}{\dot{u}} \right| < 0, \quad \text{if } \bar{u} \text{ is a steady state.}
\end{align*}
\]
Proof: Since $\delta, \tau = 0$, 
\[
\text{sign} \left( \frac{\Delta r}{\Delta \tau} \right) = \text{sign} \left( \frac{\Delta (\bar{r} / r)}{\Delta \tau} \right) = \text{sign} \left( \frac{(1 - \omega) f_r(v)}{\Delta \tau} \right)
\]
by differentiation of (25). By A.2 of the Appendix, after relabelling, 
\[
\frac{\Delta r}{\Delta \tau} > 0.
\]
Therefore, 
\[
\frac{\Delta r}{\Delta \tau} < 0 \text{ q.e.d.}
\]
Combining a), b) and Lemma 1 we may state (recall that $\tau = 1/b$) the following:

**Proposition 2:**

If the steady state $k$ corresponding to the model (1) with a uniform rate of savings and instantaneous adjustibility of capital is locally stable, then the steady state $[u, \bar{k}, \bar{k}]$ corresponding to the model (20)-(25) is locally stable \([1]\).

Therefore, using the results of Uzawa [4] and Dovolakis [1], this will be the case if

1) \[\sigma_k(u, \theta, \bar{\tau}) > 1 \quad \text{or} \]
2) \[k_\tau(u, \bar{\tau}) \geq k_\tau(u, \bar{\tau}). \]

**PART III**

We will assume in this part that the wage-rental ratio is rigid in the short run and that its rate of adjustment is inversely related to the quantity of labor unemployed and directly related to the quantity of unused capital. The state variables are $k$ and $\tau$. This model has been briefly considered by Uzawa [4] whose general approach we use.

Let $m_0 = \frac{L_0}{k}$; given $0 < \omega < \infty$, \[k > 0 \] the momentary equilibrium is determined by:

\[
s (m_i f_x + m_d u) = \text{pm}_i f_y \quad (26)
\]

\[
f_x = (u + k) f_x \quad (i = 1, 2) \quad (27)
\]

\[
f_y = pf_y \quad (28)
\]

\[
h_m k_{m_k} + h_{k_m} \leq k \quad (29)
\]

\[
m_0 + m_{d, K} \leq 1 \quad (30)
\]

\([1]\) Under the classical savings assumption, again, the local stability of a steady-state equilibrium in the L.M. guarantees the local stability of this equilibrium in the N.E.M.
The model is closed when either (29) or (30) holds with equality. (26)-(28) reduces to:

\[
\frac{x - s}{s} \frac{u + b_{1}(a)}{u + b_{1}(a)} = \frac{m_{1}}{m_{2}}
\]

(31)

Let \(k(a)\) designate the — uniquely determined — capital labor ratio that makes expressions (24) and (30) hold with equality, \(\frac{dk}{da} \geq 0\) (Uzawa (4)). If \(k < k(a)\) there is labor underutilization \((m_{1} + m_{2} < x)\).

If \(k > k(a)\) there is capital underutilization \((m_{1}A + m_{2}B < \delta)\).

The dynamic part of the model is described by:

\[
\begin{align*}
\dot{w} &= cH(\omega, k); \quad \text{sign} \dot{w} = \text{sign} \left[ k - k(a) \right], \quad c > 0 \\
\dot{k} &= m_{f_{2}}(k_{b}) - m_{f_{1}}
\end{align*}
\]

(24) \hspace{1cm} (33)

We assume that \(H\) is continuously differentiable and (32)-(33) have a unique solution for every \((\omega, k)\), which is continuous with respect to initial conditions.

Define \(\omega(a) = -\frac{n}{\delta} \int_{-\infty}^{a} b_{1}(\omega) \left[ k(\omega) + \omega \right] \).

When \(k \geq k(a)\) \(n\omega(a) = m_{f_{1}}(k_{b})\) and (33) may be expressed as (see Uzawa (4)):

\[
\dot{k} = n\omega(a) - m_{b} \cdot k, \quad \text{if} \ k \geq k(a)
\]

(34)

\[
\dot{k} = -n\omega(a) - m_{b} \cdot k, \quad \text{if} \ k < k(a)
\]

(35)

The steady states of this model coincide with the steady states of a model with instantaneous adjustment of \(\omega\). Let \(k\) be a steady state of (24)-(33), then:

(a) \(\left[ \frac{\delta_{0}}{\delta_{0}} \right] < a\), This follows from the fact that \(\frac{dk}{da} > 0\).

Since the system (24)-(33) is not differentiable at \(k\), in order to study local stability at \(k\) we will rely on phase-diagram analysis. The phase diagram can be drawn as in Figure 3. In a steady state \(\omega(a) = k(a)\). We observe at once that if \(\omega(a)\) intersects \(k(a)\) from above the corresponding steady state will be locally stable if

\[
\left| \frac{d \omega(a)}{da} \right| \left| \frac{d \omega(a)}{da} \right| > 0
\]

If \(\delta\) is locally stable in the I, M., (34) [for \(k = k(a)\)] yields the conclusion that \(\omega(a)\) intersects \(k(a)\) from above.
We can state the following proposition:

Proposition: p.

A steady state $\hat{a}, \hat{k}$ of the model (36)- (39) is locally stable if $\sigma_i (\hat{a}, \hat{k}) \geq \sigma_i$.

Proof: $\sigma_i (\hat{a}, \hat{k}) \geq 1$ is a sufficient condition for the stability of the I.M. We only need to prove $\left[ \frac{d \psi (ao)}{da} \right] > 0$ and the result follows.

The logarithmic differentiation of $\psi (ao)$ gives:

$$\frac{d \psi}{\psi \, da} = \frac{1}{h (ao) + \alpha} \left( \frac{dh (ao)}{da} + \frac{1}{\alpha} \right) \quad (36)$$

We consider four cases:

1) If $\sigma_i \geq 1$, $k_o > k_s$, (36) is strictly positive and the result follows.

2) If $\sigma_i \geq 1$, $\sigma_i \geq 1$, then $\frac{d^2 \psi (ao)}{da^2} \geq -\frac{h (ao)}{a}$ \quad (37)

(See Drandakis [1]). Therefore, again, (36) is strictly positive and the result follows.

(9) Under the classical savings assumption the same result holds.
3) If $\sigma > 1$, $k_b \leq h_b$, $h_b < 1$; $\mathcal{P}(\omega)$ intersects $\mathcal{I}(\omega)$ from above so that $m_b$ is a small neighbourhood around $\omega$, if $\omega' < \omega$ then $\mathcal{P}(\omega') > h_b(\omega')$, $\sigma(\omega') > 1$, $\sigma(\omega') < 1$. Therefore $m_b(\omega') = m_b(\omega)$ and $m_b + m_b = 1$. Denote the last two equalities by (38).

Differentiating (34) and taking (35) into account we find:

$$\frac{\partial m_b}{\partial \omega} = \frac{s (1 - m_b) (1 + \frac{dh_b}{d\omega}) - (1 - s) m_b (1 + \frac{dh_b}{d\omega})}{(1 - s) (h_b + \omega) + s (h_b + \omega)}$$

The numerator reduces to [after using (34)]:

$$I + \frac{n dh_b}{\omega'} - \frac{n dh_b}{h_b + \omega'}$$

which is positive since $\sigma > 1$ and $\sigma < 1$.

In this case (39), since:

$$\left[ \frac{n \frac{d\mathcal{P}}{d\omega}}{d\omega} \right] = \left[ \frac{dm_b}{d\omega} \right] = \lim_{\omega' \to \omega} \left[ \frac{\partial m_b}{\partial \omega} \right]$$

(where the minus sign in the middle term means left hand derivative), it follows that:

$$\frac{\partial \mathcal{P}}{d\omega} \bigg|_{\omega'} > 0$$

4) If $\sigma = 1$, $k_b \geq k_b$, $\sigma < 1$, the same result is obtained by a judicious use of $c's$ and $d's$ in the argument above.

(*) By the assumptions in (1), $\mathcal{P}(\omega)$ is continuously differentiable.
APPENDIX

A-1

Differentiating totally the system (5)-(7) and using (8) we find:

\[
\frac{\partial h_x}{\partial \alpha} = \frac{1}{\alpha} [ -pf_x^{(s-1)}(1-i)h_x + (1-m)(1-i)\beta f_x^{(s)}] = \frac{1}{\alpha} \left[ 1 + \frac{\alpha_s}{\alpha + \beta} \right] \frac{\partial h_x}{\partial \alpha} = \frac{1}{\alpha} \left[ 1 + \frac{\alpha_s}{\alpha + \beta} \right] \frac{\partial h_x}{\partial \alpha}.
\]

\[
\frac{\partial h_y}{\partial \alpha} = \frac{1}{\alpha} [ -pf_y^{(s-1)}(1-i)h_y + (1-m)(1-i)\beta f_y^{(s)} + sm f_x^{(s)} f_y^{(s)}] = \frac{1}{\alpha} \left[ 1 + \frac{\alpha_s}{\alpha + \beta} \right] \frac{\partial h_y}{\partial \alpha}.
\]

Where

\[
\begin{align*}
\alpha & = \begin{bmatrix} m & (1-m) & 0 \\
(1-i) & -pf_x^{(s)} & -f_x^{(s)} \\
-pf_y^{(s)} & (1-m)(1-i) f_x^{(s)} & (1-m)(1-i) f_y^{(s)} \\
\end{bmatrix} > 0,
\end{align*}
\]

and

\[
A = \frac{1}{\alpha} \begin{bmatrix} m & \alpha_s \beta h_x & \alpha_s \\
\alpha_s & \alpha_s h_x & \alpha_s + \beta \\
\beta & \beta h_x & \alpha_s + \beta \\
\end{bmatrix} + \frac{1-m}{\alpha} \begin{bmatrix} \alpha_s + \beta h_x & \alpha_s \\
\alpha_s & \alpha_s + \beta \\
\end{bmatrix}.
\]

Taking into account that

\[
\frac{\partial h_x}{\partial \alpha} = \frac{1}{\alpha} \begin{bmatrix} m & \alpha_s \beta h_x & \alpha_s \\
\alpha_s & \alpha_s h_x & \alpha_s + \beta \\
\beta & \beta h_x & \alpha_s + \beta \\
\end{bmatrix} + \frac{1-m}{\alpha} \begin{bmatrix} \alpha_s + \beta h_x & \alpha_s \\
\alpha_s & \alpha_s + \beta \\
\end{bmatrix},
\]

Clearly

\[
\frac{\partial h_x}{\partial \alpha} > 0, \quad \frac{\partial h_y}{\partial \alpha} > 0.
\]

A-2

Substituting the expressions from A-1, we get

\[
\frac{\partial u}{\partial \beta} = \frac{\partial u}{\partial \beta} = 0,
\]

\[
\frac{\partial u}{\partial \beta} = 0,
\]

\[
\frac{\partial u}{\partial \beta} = 0.
\]

Manipulating, this expression reduces to:

\[
\frac{1}{\alpha} \begin{bmatrix} \omega & \omega + \sigma h_x \\
\sigma h_x & \omega + h_x \\
\end{bmatrix} > 0,
\]

which establishes the claim.
Taking the expression for \( \frac{\partial q}{\partial k} \) in A-1 and substituting \( A \) into it we have:

\[
\frac{\partial \lambda}{\partial k} = \frac{m \lambda}{h} \left[ \frac{1}{h_1 \lambda_1 + h} \left( \frac{1}{h_1} \right) \right] (1 - m) \nu + \frac{1}{h_2 \lambda_2 + h} \left( \frac{1}{h_2} \right) \nu + \frac{1}{h_1 \lambda_1 + h} \left( \frac{1}{h_1} \right) \nu + \frac{1}{h_2 \lambda_2} \left( \frac{1}{h_2} \right) \nu + (1 - m) \nu
\]

In \( \lambda, \lambda_1 = \lambda_0 + \nu \), yielding:

\[
\frac{\partial \lambda}{\partial k} = \frac{\nu + \lambda}{\Omega \nu + (1 - m) \nu + (1 - m) \nu}
\]

REFERENCES