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AN EQUILIBRIUM EXISTENCE THEOREM FOR A GENERAL MODEL WITHOUT ORDERED PREFERENCES*

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1. Introduction

In a recent paper the second author has shown that some of the usual hypotheses on consumers' preferences are not needed for the proof of existence of a Walrasian General Equilibrium [Mas-Colell (1974)]. Specifically, it is not necessary that preferences come from a preference *ordering*. The only order property required is irreflexivity (meaning that a given commodity bundle is not preferred to itself). The properties of non-satiation, continuity and convexity of preferred sets turn out to be sufficient to obtain the existence result. The main purpose of the present note is to give a second proof of this fact which seems simpler than that of Mas-Colell (1974), and no more lengthy or complicated than the known equilibrium existence proofs which use ordered preferences.

In two additional respects the model studied here generalizes the usual equilibrium model. The standard Walras, Arrow–Debreu theory treats what might be called the laissez-faire model in which each agent's income is whatever he gets from selling goods plus his share of the profits of any firm in which he may own stock. In the present model the income of a consumer may be any continuous function of the prices, so the laissez-faire income function is included, but so also would any rule for assigning income to consumers (e.g., according to his ability or his need or the color of his eyes). Another way of saying this is that the model includes the possibility of arbitrary lump sum transfers of income among consumers, as might be achieved, for example, by a program of income taxes and subsidies. This substantial economic generalization requires no change whatever in the mathematical argument.

The second generalization concerns production. The only requirement on our production set, besides the usual closure, convexity, and free disposal, is that it intersect the positive orthant in a bounded set. This means that the usual assumption

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that one cannot get a positive output from zero input is replaced by the weaker requirement that one cannot get an infinite output from zero input. This latter condition is clearly necessary in any reasonable theory since an economy which could produce infinity from nothing would in general not even admit Pareto optimal allocations, let alone equilibria.

One interesting question remains unsettled. It is well known that in the standard model if negative prices are allowed the free disposal assumption may be dispensed with. We do not know if this is also true in this more general setup.

2. A fixed point theorem

Our results depend on the following theorem.

Theorem. *Given* $X = \prod_{i=1}^{n} X_i$, *where* X_i *is a non-empty compact convex subset of* R^n, *let* $\varphi_i : X \to 2^{X_i}$ *be n convex (possibly empty) valued mappings whose graphs are open in* $X \times X_i$. *Then there exist x in X such that for each i either* $x_i \in \varphi_i(x)$ *or* $\varphi_i(x) = \emptyset$.

Proof. For every i, let $U_i = \{x \mid \varphi_i(x) \neq \emptyset\}$, then $\varphi_i \mid U_i : U_i \to X_i$ is a convex valued correspondence having an open graph. Let $f_i : U_i \to X_i$ be a continuous function such that $f_i(x) \in \varphi_i(x)$ for $x \in U_i$ [see Michael (1956, Theorem 3.1‴, p. 368)]. Define a correspondence $\Psi_i : X \to X_i$ by $\Psi_i(x) = f_i(x)$ if $x \in U_i$, and $\Psi_i(x) = X_i$ otherwise; for every i, Ψ_i is convex valued and (since U_i is open) upper hemicontinuous. Define $\Psi : X \to X$ by $\Psi(x) = \prod_{i=1}^{n} \Psi_i(x)$. By Kakutani's theorem there is $\bar{x} \in X$ with $\bar{x} \in \Psi(\bar{x})$. By construction this $\bar{x}$ satisfies the conclusion of the theorem. ▮

The special case of the theorem in which $n = 1$ was proved by Fan (1961) and independently by Sonnenschein (1971).

3. The model and equilibrium theorem

There are *m traders*. Corresponding to trader *i* is the *trading set* X_i in R^n and a *preference mapping* P_i from X_i to 2^{X_i}. There is also a subset Y of R^n called the *technology* of the economy.

Definition. An *allocation* x is an m-tuple of vectors $x_1, \ldots, x_m$ where $x_i \in X_i$. Thus $x \in \prod_{i=1}^{m} X_i$. An allocation is feasible if $\sum_{i=1}^{m} x_i \in Y$.

Notation. If $x = (x_1, \ldots, x_m)$ is an allocation, we denote the sum $\sum_{i=1}^{m} x_i$ by x_0. Thus, x is feasible if $x_0 \in Y$.

Let Δ be the unit n-simplex. The elements of Δ will be called price vectors and denoted by p, p', etc. In formulating equilibrium models one must describe some way in which allocations x are associated with price vectors p. In particular, one needs to associate with trader i his *income* $\alpha_i(p)$ at prices p. In the pure exchange model it is assumed that trader i has an *initial endowment* vector ω_i and his income is then given by $\alpha_i(p) = p\omega_i$. The Arrow–Debreu model involves a more complicated set of income functions. It is assumed that the technology Y is the sum of 'subtechnologies' $Y_1, \ldots, Y_r$, to be thought of as firms, and trader i is provided with a 'portfolio vector' $\theta_i = (\theta_{i1}, \ldots, \theta_{ir})$ where θ_{ij} represents trader i's share of firm j. The Arrow–Debreu income functions are then given by

$$\alpha_i(p) = p\omega_i + \sum_{j=1}^{r} \theta_{ij} \sup_{Y_j} pY_j. \tag{1}$$

In the present treatment, as stated in the introduction, we wish to allow for more general income function. For any p in Δ define the *profit function* $\Pi(p)$ by the rule

$$\Pi(p) = \sup pY \tag{2}$$

(in convexity theory Π is called the *support function* of Y). Since Y is unbounded Π may take on the value infinity. We define $\Delta' \subset \Delta$ by

$$\Delta' = \{p \mid \Pi(p) < \infty\}.$$

Our hypotheses on Y will guarantee that Δ' is nonempty. One also verifies easily that Δ' is convex. We now postulate the existence of m real valued functions α_i on Δ' (to be called *income functions*) satisfying

$$\sum_{i=1}^{m} \alpha_i(p) = \Pi(p) \quad \text{for all } p \text{ in } \Delta'. \tag{3}$$

Definition. An *equilibrium* for the model described above consists of a price vector $\bar{p}$ in Δ' and an allocation $\bar{x}$ such that:

$$\bar{p}\bar{x}_i = \alpha_i(\bar{p}), \quad \text{for all } i \quad \text{(budget equation)}; \tag{4}$$

$$\text{if } x_i \in P_i(\bar{x}_i), \text{ then } \bar{p}x_i > \bar{p}\bar{x}_i \quad \text{(preference condition)}; \tag{5}$$

$$\bar{x} \text{ is feasible} \quad \text{(balance of supply and demand)}. \tag{6}$$

These conditions are so familiar as not to require discussion. Observe, however, that the requirement that the vector $\bar{x}_0$ 'maximizes profit' is automatically satisfied since

$$\bar{p}\bar{x}_0 = \sum_{i=1}^{m} \bar{p}\bar{x}_i = \sum_{i=1}^{m} \alpha_i(\bar{p}) = \sup \bar{p}Y.$$

Our main result is the following theorem.

Equilibrium Theorem. The following conditions are sufficient for the existence of equilibrium:

The set Y is closed, convex, contains the negative orthant, and has a bounded intersection with the positive orthant. (7)

The sets X_i are closed, convex, non-empty and bounded below. (8)

The preference mappings P_i are irreflexive [that is, $x_i \notin P(x_i)$], have an open graph in $X_i \times X_i$ and their values are non-empty, convex sets. (9)

The functions $\alpha_i(p)$ are continuous and satisfy $\alpha_i(p) > \inf pX_i$ for all p in Δ'. (10)

The conditions are again quite familiar, but a few points are worth noting:

(A) Our assumptions on Y generalize, for example, those of Debreu (1959) as already mentioned.

(B) The condition that $P_i(x_i)$ is non-empty is a non-satiation condition asserting that there is no 'bliss point' for any trader.

(C) Condition (10) guarantees that no trader will be allowed to starve no matter what the prices are. The need for this sort of condition is familiar. In pure exchange models, for example, it is achieved by the customary, and unpleasant, assumption that all traders have a strictly positive initial endowment. In our present, more general way of looking at equilibrium the assumption becomes more palatable. Not many economies in the present day are so extremely laissez faire as to permit people to starve.

(D) The condition that Y contains the negative orthant is the free disposal hypothesis. It will play a crucial role in our proof.

4. Proof of the equilibrium theorem

Since each X_i is bounded below, there exists a vector e such that for any $S \subset \{1, \ldots, m\}$

$$e < \sum_{i \in S} X_i.$$

Define $\hat{Y} = \{y \mid y \in Y, y \geqq e\}$.

Note that $\hat{Y}$ contains all feasible $x_0 = \sum_{i=1}^{n} x_i$, and from (7) one easily sees that $\hat{Y}$ is bounded above as well as below so there exists a vector f such that $f > \hat{Y}$.

Lemma 1. If x *is feasible, then* $x_i < f-e$ for all i.

▌ By feasibility, $x_0 = \sum_{j=1}^{m} x_j < f$, so

$$x_i < f - \sum_{j \neq i} x_j < f-e$$

by definition of e. ▌

We now define $\hat{X}_i = \{x_i \mid x_i \in X_i \text{ and } x_i \leq f-e\}$. Observe that the sets $\hat{X}_i$ are compact convex. They are called the *truncated trading sets*.

Next define new mappings $\hat{P}_i$ by

$$\hat{P}_i(x_i) = \{x_i' \mid x_i' = \lambda x_i'' + (1-\lambda)x_i \quad \text{for} \quad 0 < \lambda \leq 1, x_i'' \in P_i(x_i)\}. \tag{11}$$

The mappings $\hat{P}_i$ are called the *augmented preference mappings*. Notice that the $\hat{P}_i$ satisfy condition (9) of the Equilibrium Theorem if the P_i do, and also that $P_i(x_i) \subset \hat{P}_i(x_i)$ and $x_i \in \overline{\hat{P}_i(x_i)}$ for all $x_i \in X_i$.

Our proof will give an equilibrium with the X_i replaced by $\hat{X}_i$ and P_i replaced by $\hat{P}_i$. This will also clearly be an equilibrium for the original model.

Define $\Delta'' \subset \Delta'$ by

$$\Delta'' = \{p \mid py = \Pi(p) \text{ for some } y \text{ in } \hat{Y}\}.$$

Geometrically, Δ'' consists of all p in Δ which are normal to Y at some point y of $\hat{Y}$.

Lemma 2. Δ'' is non-empty and closed.

▌ Let y be the point of $\hat{Y}$ for which $qy = \max q\hat{Y}$ for some positive vector q. Then, clearly, there is no point y' of $\hat{Y}$ such that $y' \geq y$ so the set of all $y' \geqq y$ can be separated from Y by a hyperplane whose outward normal p is in Δ (this is the familiar 'efficient point' theorem) so $p \in \Delta''$ and $\Delta'' \neq \phi$.

Further, if $p_n \in \Delta''$ and $p_n \to \bar{p}$, then, by compactness and taking subsequences if necessary, we may suppose there exist y_n in $\hat{Y}$ such that $p_n y_n = \Pi(p_n)$ and $y_n \to \bar{y}$ in $\hat{Y}$. Then, $\bar{p}\bar{y} = \Pi(\bar{p})$ for if there existed y' in $\hat{Y}$ with $py' > \bar{p}\bar{y}$, then $p_n y' > p_n y_n$ for large n contradicting $p_n y_n = \Pi(p_n)$. ▌

Let Δ^* be the convex hull of Δ''. Then $\Delta^* \subset \Delta'$ since as remarked earlier Δ' is convex, and Δ^* is closed since Δ'' is closed by Lemma 2.

For any p in Δ^* define

$$\gamma_i(p) = \{x_i \mid x_i \in \hat{X}_i \quad \text{and} \quad px_i < \alpha_i(p)\}.$$

From (10) $\gamma_i(p)$ is non-empty. Further, since α_i is continuous, one verifies that the correspondence γ_i has an open graph in $\Delta^* \times \hat{X}_i$.

Let $\hat{X} = \prod_{i=1}^m \hat{X}_i$. We define mappings φ_i, $i = 1, \ldots, m$, from $\Delta^* \times \hat{X}$ to 2^{X_i} by the rule

$$\begin{aligned} \varphi_i(p, x) &= \gamma_i(p), & \text{if} \quad px_i > \alpha_i(p), \\ &= \gamma_i(p) \cap \hat{P}_i(x_i), & \text{if} \quad px_i \leq \alpha_i(p). \end{aligned} \tag{12}$$

Define the mapping φ_0 from $\Delta^* \times \hat{X}$ to 2^Δ by the rule

$$\varphi_0(p, x) = \{q \mid q \in \Delta^* \quad \text{and} \quad qx_0 > \Pi(q)\}. \tag{13}$$

Note that all these mappings may be empty valued and that they are clearly convex valued. To show that φ_i has an open graph for $i \geq 1$, let

$$A_i = \{(p, x, z) \mid p \in \Delta^*, x \in \hat{X}, z \in \hat{X}_i \quad \text{and} \quad px_i > \alpha_i(p)\},$$

$$B_i = \{(p, x, z) \mid p \in \Delta^*, x \in \hat{X}, z \in \hat{X}_i \quad \text{and} \quad pz < \alpha_i(p)\},$$

$$C_i = \{(p, x, z) \mid p \in \Delta^*, x \in \hat{X}, z \in \hat{X}_i \cap \hat{P}(x_i)\}.$$

Then A_i, B_i, C_i are open in $\Delta^* \times \hat{X} \times \hat{X}_i$, and therefore the graph of φ_i, which is $(A_i \cap B_i) \cup (C_i \cap B_i)$, is obviously open. The fact that φ_0 has an open graph follows at once from the definition.

We show now

Lemma 3. There exist $(\bar{p}, \bar{x}) \in \Delta^* \times \hat{X}$ *such that* $\varphi_i(p, x) = \varnothing$, $i = 0, 1, \ldots, m$.

Proof. The φ_i satisfy the conditions of the fixed point theorem as it has just been shown. Let then $(\bar{p}, \bar{x})$ satisfy its conclusion. We cannot have $x_i \in \varphi_i(\bar{p}, \bar{x})$ for $i \geq 1$, for if so, then $\bar{x}_i \in \hat{P}_i(\bar{x}_i)$ which contradicts irreflexivity. Thus $\varphi_i(\bar{p}, \bar{x}) = \varnothing$ for $i \geq 1$, which implies $p\bar{x}_i \leq \alpha_i(p)$ for $i \geq 1$, and this

$$\bar{p}\bar{x}_0 \leq \sum_{i=1}^m \alpha_i(\bar{p}) = \Pi(\bar{p});$$

so, $\bar{p} \in \varphi_0(\bar{p}, \bar{x})$, and therefore $\varphi_0(\bar{p}, \bar{x}) = \varnothing$. ∎

To complete the proof we must now show that $(\bar{p}, \bar{x})$ is an equilibrium. We first verify feasibility (6) arguing by contradiction. Suppose x_0 is not in Y. Now $x_0 > e$ so we may choose the smallest λ such that $y_\lambda = \lambda e + (1-\lambda)\bar{x}_0$ is in Y. By the fundamental separation theorem there exists p in Δ such that $py_\lambda = \Pi(p)$

and $px_0 > \Pi(p)$. Noting that $y_\lambda \in \hat{Y}$ it follows that $p \in \Delta''$ so $p \in \varphi_0(\bar{p}, \bar{x})$ contradicting the fact that $\varphi(\bar{p}, \bar{x}) = \emptyset$.

We next verify that the budget equation (4) is satisfied. Since $\varphi_i(\bar{p}, \bar{x}) = \emptyset$, we must have $\bar{p}\bar{x}_i \leq \alpha_i(\bar{p})$ for all i. Suppose $\bar{p}\bar{x}_i < \alpha_i(\bar{p})$ for some i. Since $\bar{x}$ is feasible, we know by Lemma 1 that $\bar{x}_i < f-e$. Since $P_i(\bar{x}_i)$ is non-empty, we can pick an $x_i \in \hat{P}_i(\bar{x}_i)$; then there will exist $0 < \lambda < 1$ such that $x_\lambda = (1-\lambda)\bar{x}_i + \lambda x_i$ satisfies $x_\lambda < f-e$, and $\bar{p}x_\lambda < \alpha_i(\bar{p})$. But then $x_\lambda \in \hat{P}_i(\bar{x}_i)$ by construction of $\hat{P}_i$, and this means $x_\lambda \in \varphi_i(\bar{p}, \bar{x})$ which contradicts $\varphi_i(\bar{p}, \bar{x}) = \emptyset$. Hence $\bar{p}\bar{x}_i = \alpha_i(p)$.

Finally, we prove the preference condition (5), again arguing by contradiction. Suppose $x_i \in \hat{P}_i(\bar{x}_i)$ and $\bar{p}x_i \leqq \alpha_i(\bar{p})$ for some i. From condition (10) there exist $\tilde{x}_i$ in $\hat{X}_i$ such that $\bar{p}\tilde{x}_i < \alpha(\bar{p})$. By condition (9) there exists $0 < \lambda < 1$ such that $x_\lambda = \lambda\tilde{x}_i + (1-\lambda)x_i \in \hat{P}_i(\bar{x}_i)$. But also $\bar{p}x_\lambda < \alpha_i(\bar{p})$ which implies $x_\lambda \in \varphi_i(\bar{p}, \bar{x})$ and contradicts $\phi_i(\bar{p}, \bar{x}) = \emptyset$. This completes the proof.

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