

A REMARK ON A SMOOTHNESS PROPERTY OF CONVEX, COMPLETE PREORDERS*

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Let $\succsim \subset A \times A$ be a complete preorder on A , a closed, convex subset of \mathbf{R}^n . Assume (i) \succsim is closed (continuity), and (ii) if $x \succsim y$, then $\alpha x + (1 - \alpha)y \succsim y$ for all $0 \leq \alpha \leq 1$ (convexity).

Denote

$$B = \{q \in \mathbf{R}^n : \|q\| \leq 1\},$$

and define a ('gradient') mapping $g_{\succsim} : A \rightarrow B$ by

$$g_{\succsim}(x) = \{q \in B : \text{if } y \succsim x, \text{ then } qy \geq qx\};$$

$g_{\succsim}(x)$ is a convex set and it has a well-defined dimension. For every $m \leq n$, let

$$C_m = \{x \in A : \dim g_{\succsim}(x) \geq m\},$$

and denote by μ_m the m -dimensional Hausdorff measure in \mathbf{R}^n [for the definition and some facts about the Hausdorff measure, see Federer (1969, 2.10)]; μ_n coincides with the usual Lebesgue measure. We prove

Theorem. For every $m \leq n$, $\mu_{n-m+2}(C_m) = 0$; in particular, $\mu_n(C_2) = 0$, i.e., g_{\succsim} is a (possibly degenerate) ray a.e. (in the sense of Lebesgue measure).

The theorem gives an analog for convex, complete preorders of the a.e. differentiability properties of concave functions; convex, complete preorders play in 'ordinal' maximization problems, i.e., those where the value of the maximization criterion is inessential and only the maximizers matter, the same

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role that concave functions play in ordinary programming problems. We may remark that, in contrast to concave functions, convex, continuous, complete preorders can be quite pathological; see Mas-Colell (1973) for an example of two distinct \succsim, \succsim' having $g_{\succsim} = g_{\succsim'}$.

The method of proof and the results obtained are related to those of Anderson and Klee (1952) for concave functions.¹

As an application (in economics), let \succsim be a continuous, monotone (not necessarily convex) preference relation (i.e., a complete preorder) in \mathbf{R}^n_+ . The demand correspondence $h: \mathbf{R}^l_{++} \rightarrow \mathbf{R}^l_+$ is given by

$$h(p) = \{x \in \mathbf{R}^l_+ : px \leq 1, \text{ and } py \leq 1 \text{ implies } x \succsim y\}.$$

Then:

Corollary. h is single-valued a.e. (in the sense of Lebesgue measure).

Proof. Define \succsim^* in \mathbf{R}^l_{++} by $p \succsim^* q$ if and only if $h(q) \succ h(p)$. Then \succsim^* is continuous and $\{p: p \succsim^* q\} = \varphi^{-1}([1, \infty))$ where φ is the negative of the support function of $\{-y: y \succ h(q)\}$. Hence \succsim^* is convex and $h(p)$ is single-valued if and only if $g_{\succsim^*}(p)$ is a ray. ■

The corollary has an interesting implication: if budgets are ‘sufficiently’ random (in prices and income) then, even with non-convex preferences, expected maximizers will be unique.

Proof of the theorem. It follows simply from the definition of the Hausdorff measure that if $E \subset \mathbf{R}^n$ is an F_σ set, $H \subset \mathbf{R}^n$ is an m -dimensional subspace and $\#(E \cap H + \{a\}) \leq 1$ for every $a \in \mathbf{R}^n$, then $\mu_{n-m+1}(E) = 0$.

Fix $1 \leq m \leq n$ and let \mathcal{H} be the (countable) collection of $(m-1)$ -dimensional subspaces of \mathbf{R}^n having a base with rational components. For every $x \in A$ define

$$L_x = \{z \in \mathbf{R}^n : pz = 0 \text{ for every } p \in g_{\succsim}(x)\},$$

and let $\text{Int } g_{\succsim}(x)$ denote the interior of $g_{\succsim}(x)$ relative to its linearity space. Note that

$$\dim(\text{Int } g_{\succsim}(x)) + \dim L_x = n.$$

For every $H \in \mathcal{H}$ let

$$C_m(H) = \{x \in C_m : \dim(H + L_x) = \dim H + \dim L_x \text{ and, for some } p \in \text{Int } g_{\succsim}(x), p \neq 0 \text{ is perpendicular to } H\}.$$

¹For those familiar with the concept of upper semicontinuous collections of sets [see Berge (1959, V, 7)], we point out that $\{\{x\} - g_{\succsim}(x) : x \in A\}$ is one such.

It is easily checked that $C_m(H)$ is F_σ . If H' is a translate of H , then $\#(C_m(H) \cap H') \leq 1$ for, otherwise, if $x, y \in C_m(H) \cap H'$ and $x \neq y$, then $px < py$ for some $p \in g_{\succ}(y)$ and $qy < qx$ for some $q \in g_{\succ}(x)$ which contradicts the completeness hypothesis. Therefore $\mu_{n-m+2}(C_m(H)) = 0$ for every $H \in \mathcal{H}$. On the other hand, let $x \in C_m$ and $p \in \text{Int } g_{\succ}(x)$ be perpendicular to $I \subset \mathbf{R}^n$, some $(m-1)$ -dimensional subspace independent of L_x ; if I' is an $(m-1)$ subspace with a rational components base which is sufficiently close to I and we take p' oriented as p and perpendicular to I' and L_x , then $p' \in g_{\succ}(x)$. Hence

$$C_m = \bigcup_{H \in \mathcal{H}} C_m(H)$$

and therefore $\mu_{n-m+2}(C_m) = 0$. For $m \leq 1$ the theorem is obvious. ■

References

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