Competitive and Value Allocations of Large Exchange Economies*

ANDREU MAS-COLELL

Departments of Economics and Mathematics, University of California, Berkeley

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INTRODUCTION

Consider an exchange economy whose traders have concave utility functions. It is to be understood, in the present paper, that utility has cardinal significance, so we refer to such economies as being cardinal. As an extension of game theoretic concepts, Shapley [21] proposed the following notion of value allocation for a cardinal economy. Let there be \( n \) traders and let us say that the positive numbers \( \lambda_1, ..., \lambda_n \) are equilibrium weights if there is a reallocation of goods to traders such that the weighted utility that every trader obtains is precisely the value [20] of the game where the worth of a coalition is simply the maximum utility that the coalition can get when the utility of trader \( i \) is given weight \( \lambda_i \). The allocation of goods for which \( \lambda_1, ..., \lambda_n \) is an equilibrium is then called a value allocation. Under standard conditions value allocations exist.

Recent work by, among others, Shapley and Shubik [23], Champsaum [9], and Aumann [3], has established that, as it is the case with the concept of the core (see [19] for an extensive treatment), the value notion provides yet another cooperative game theoretic foundation for the idea of perfect competition. Thus, building on the remarkable theory of values of nonatomic games developed by Aumann and Shapley [4], Aumann [3] has proved under, among others, smoothness hypotheses, that in cardinal economies with a continuum of traders the set of competitive allocations and the set of value allocations coincide.

The purpose of this paper is to give an “asymptotic” version of Aumann’s theorem. As in Hildenbrand [19], by asymptotic it is meant that instead of dealing with economies with a continuum of traders, we deal with sequences of finite, but increasingly large, ones. Thanks to the pioneering investigation by Champsaum [9] of value allocations of replica economies, it is well known that if we consider a sequence of cardinal increasingly large replica economies

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limit, then, up to a few technical assumptions, value allocations satisfying a symmetry requirement are eventually close to competitive allocations. Unlike Aumann, Champsaur did not impose in his model smoothness hypotheses. Without them it may happen that competitive allocations are not eventually close to value allocations, and so it is not possible to get a counterpart to Aumann's theorem even for the replica case. Since this is the aim here we shall place ourselves in the same smoothness framework as Aumann's. This restriction in scope is, we feel, fully justified by the stronger results that can be hoped for. For a general nondifferentiable analysis of cardinal economies with a continuum of traders see Hart [17].

Suppose then that we have a sequence of smooth, increasingly large cardinal economies. We will assume that the sequence converges to a definite limit in a sense already used by Hildenbrand. We wish to find conditions for the set of value and competitive allocations to be eventually close. Since, by Aumann's result, they are identical at the limit, it is patent that, informally speaking, all we need is that the limit be a continuity point of both the competitive equilibrium correspondence and the value allocation correspondence. It is well known that a sufficient condition for continuity at the limit of the equilibrium correspondence is that the "limit economy" be regular in the sense introduced by Debreu [10] and generalized to the continuum of traders context by, among others, Dierker [13]. It turns out that if the limit is regular, then it is also a continuity point of the correspondence which assigns to economies the value allocations satisfying a weak symmetry requirement. Thus, by requiring that the given sequence converge to a regular limit we obtain a very general form of the desired result. The regularity condition is easily describable (i.e., an economy is regular if the Jacobian of the excess demand map has full rank at every equilibrium price vector) and, most important, it is generic, i.e., in an appropriate sense almost all limits are regular (see [10, 13]).

Except for some special arguments at the very end, where regularity is exploited, our line of proof is entirely traditional. As a side product to the main result we investigate, by being very explicit on the approximating bounds, which rate of convergence does the proof yield. After all, to quote Aumann and Shapley [4, p. 210], "the asymptotic results imply a framework within which the manner and rate of convergence can be discussed. The continuous formulation, by its nature, precludes such considerations." We get a convergence of the order of \(1/n^{1/3}\); it is not claimed that it is the best but we have no suggestion on how to improve it (Shapley and Shapiro [22] also got \(1/n^{1/3}\) for the convergence of the value in some weighted majority games). Rate of convergence results for the case of the core have been obtained by Debreu [11] and Grodal [15]. It was Debreu's paper [11] that made very clear the usefulness of the concept of regular limit for the purposes of establishing strong asymptotic theorems.
Section 1 describes the model and assumptions, states the results, discusses open problems, and gives more precise and exhaustive references. Section 2 contains the proofs.

1. THE MODEL

The consumption set is $P = \{x \in \mathbb{R}^k : x \geq 0\}$. In the present cardinal utility context a consumer is a pair $(u, \omega)$, where $\omega \in P$ is the initial endowment vector and $u: P \to \mathbb{R}$ the utility function. We assume:

1. $u: P \to \mathbb{R}$ is a bounded above and below nonnegative $C^2$ function with $Du(x) \geq 0$ and $D^2u(x)$ negative definite for every $x \in P$ (hence $u$ is concave). Moreover, if $x_n \to x \in \partial P$, then $\|Du(x_n)\| \to \infty$.

Two utility functions $u, u'$ are equivalent if $u = au' + b$, $a > 0$. Therefore, for the sake of definiteness and without loss of generality, we will normalize and assume $u(e) = 1$, $u(2e) = 2$, $e = (1, \ldots, 1) \in P$.

Let $\mathcal{U}$ be the space of (normalized) utility functions satisfying (1). We endow $\mathcal{U}$ with the topology of $C^2$ uniform convergence on compacta plus $C^0$ uniform convergence. This is a strong topology; our results are valid (without modification of proofs) for the topology of uniform convergence on compacta, but then our assumptions (yet to be made) do not seem to be independent of the normalization chosen, a situation we want to avoid.

The space of consumer's characteristics is $\mathcal{A} = \mathcal{U} \times P$; we denote $a = (u_a, \omega_a)$, $\mathcal{A}$, with the product topology, is metrizable.

As usual, we define the demand correspondence $\Phi: \mathcal{A} \times P \times \mathbb{R}_+ \to P$ by $\varphi(a, p, w) = \{w \in P: px \leq w$, and $py \leq w$ implies $u_a(x) \geq u_a(y)\}$; (1) guarantees that $p$ is U.H.C. and nonempty valued.

A continuum economy is a measure $\nu$ on $\mathcal{U} \times P$ (endowed with the product Borel $\sigma$ field) with compact support.

Let $S = \{p \in P: \|p\| = 1\}$. Given the continuum economy $\nu$, the aggregate excess demand function $\Phi_v: S \to \mathbb{R}^k$ is given by $\Phi_v(p) = \int_\mathcal{A} (\varphi(a, p, p\omega_a) - \omega_a) \, d\nu(a)$. As in Hildenbrand [19, p. 119] one verifies the boundary condition: $p_n \to p \in \partial S$ implies $\|\Phi_v(p_n)\| \to \infty$. Hence the set $\{p \in S: \Phi_v(p) = 0\}$ is compact.

As in Dierker [13], it is verified that the functions $(p, w) \mapsto \varphi(a, p, w)$ and $p \mapsto \Phi_v(p)$ are $C^1$.

Let $\Pi(\nu) = \{p \in S: \Phi_v(p) = 0\}$. If $p \in \Pi(\nu)$, the linear map $D\Phi_v(p)$ maps $T_p(S)$, the tangent plane to $S$ at $p \in S$, into itself. If $D\Phi_v(p)$ does in fact map onto, i.e., if it has the largest possible rank $(l - 1)$, then it is said that $\Phi_v$ is regular at $p$ (see Dierker [13] and her references).

DEFINITION. $\nu$ is regular if rank $D\Phi_v(p) = l - 1$ whenever $\Phi_v(p) = 0$.

As a corollary of Dierker’s [13] results, it is easy to see that, with a natural topology on the $\nu$’s (which will be specified later on) there is an open, dense set of regular economies.
A finite economy is a mapping \( \mathcal{E} : I \rightarrow \mathcal{A} \), where \( I \) is a finite indexing set. Two finite economies are equivalent if one can be obtained from the other via an automorphism of \( I \). Hence, up to equivalence, a finite economy is completely specified by its cardinality, i.e., \( \#(I) \), and the measure \( \nu_\mathcal{E} \) on \( \mathcal{A} \) defined by

\[
\nu_\mathcal{E}(U) = \#\{i \in I : \mathcal{E}(i) \in U\}/\#(I).
\]

We will always write expressions like \( u_\mathcal{E}(i) \) as \( u_i \).

**DEFINITION.** The sequence of finite economies \( \mathcal{E}_n \) converges to the continuum economy \( \nu \) (written \( \mathcal{E}_n \rightarrow \nu \)) if:

(i) \( \#(I_n) \rightarrow \infty \), where \( \mathcal{E}_n : I_n \rightarrow \mathcal{A} \)

(ii) the measure \( \nu_\mathcal{E} \) converges weakly to the measure \( \nu \); and

(iii) \( \text{supp}(\nu_\mathcal{E}) \) converges to \( \text{supp}(\nu) \) with respect to the Hausdorff distance derived from any metric on \( \mathcal{A} \).

Given a (finite) economy \( \mathcal{E} : I \rightarrow \mathcal{A} \) an allocation is a map \( x : I \rightarrow P \); \( x \) is feasible if \( \sum_{i \in I} x_i \leq \sum_{i \in I} \omega_i \); \( x \) is competitive if it is feasible and, for some \( p \in S \), \( x_i \in \varphi (u_i \cdot p, p \cdot \omega_i) \) for every \( i \in I \), \( p \) is then an equilibrium price system. We let \( W(\mathcal{E}) \), \( \Pi(\mathcal{E}) \) be, respectively, the set of competitive allocations and price systems for \( \mathcal{E} \). The sets \( W(\mathcal{E}) \), \( \Pi(\mathcal{E}) \) are compact.

For a function \( f : I \rightarrow R \), \( \#(I) < \infty \), denote \( |f| = \max_{i \in I} |f(i)| \). By abuse of language, if \( x : I \rightarrow P \), we let \( |x| = \max_{i \in I} \|x_i\| \), where \( \|\cdot\| \) is the euclidean norm. Of course, \( |\cdot| \) is a norm on the set of function from \( I \) to \( P \) (or from \( I \) to \( K \)). With respect to the metric of this norm we let \( d \) be the Hausdorff distance on the compact subsets of \( \{x : I \rightarrow P\} \equiv P^I \). More precisely, if \( A, B \subseteq P \) are compact, let \( \rho(A, B) = \min\{\varepsilon > 0 : \text{ for every } x \in A, \|x - y\| \leq \varepsilon \text{ for some } y \in B\} \). Put then \( d(A, B) = \max\{\rho(A, B), \rho(B, A)\} \). Remember, we always assume \( \#(I) < \infty \).

Following Shapley and Shubik [23] and Aumann [2], we can define, for every finite economy \( \mathcal{E} : I \rightarrow \mathcal{A} \) and \( \lambda : I \rightarrow R_{++} \) (a vector of "weights"), a game in characteristic form by letting the worth of coalition \( \phi \neq C \subseteq I \) be

\[
V(\mathcal{E}, \lambda, C) = \max\{\sum_{i \in I} \lambda_i u_i(x_i) : x_i \in P, \sum_{i \in I} x_i \leq \sum_{i \in I} \omega_i\};
\]

it is easily verified that the maximum exists. Let, also, \( V(\mathcal{E}, \lambda, \emptyset) = 0 \). Note that \( V \) is homogeneous of degree 1 on \( \lambda \), i.e., \( V(\mathcal{E}, \alpha \lambda, C) = \alpha V(\mathcal{E}, \lambda, C) \), \( \alpha \geq 0 \).

The game \( V(\mathcal{E}, \lambda, \cdot) \) has a Shapley value; let it be \( \nu(\mathcal{E}, \lambda) : I \rightarrow R \). The value can be computed as follows. For every \( i \in I \) let \( C_i \) be the set of coalitions from \( I \setminus \{i\} \), player \( i \) has a probability distribution \( \pi_i \) on \( C_i \); \( \pi_i \) is uniquely determined from two principles: (i) if \( \#(C) = \#(C') \), then \( \pi_i(C) = \pi_i(C') \); (ii) \( \pi_i(C \in C_i : \#(C) = m) \) is independent of \( 0 \leq m < \#(I) \). Then \( \nu_i(\mathcal{E}, \lambda) = \sum_{C \in \mathcal{E}} \pi(C)[V(\mathcal{E}, \lambda, C \cup \{i\}) - V(\mathcal{E}, \lambda, C)] \), i.e., the value is an average of marginal contributions. It can be seen that the value is an imputation, i.e., \( \sum_{i \in I} \nu_i(\mathcal{E}, I) = V(\mathcal{E}, \lambda, I) \).

**DEFINITION.** A feasible allocation \( x : I \rightarrow P \) for the finite economy \( \mathcal{E} : I \rightarrow \mathcal{A} \) is a (Aumann-Shapley) value allocation if for some \( \lambda : I \rightarrow R_{++} \),
\[ \lambda_i \mu_i(x_i) = v_i(\mathcal{E}, \lambda) \text{ for all } i \in I. \] The set of value allocations will be denoted \( \mathcal{V}(\mathcal{E}) \).

For the purposes of the theorem we feel the need to consider only some subsets of \( \mathcal{V} \) satisfying a minimal symmetry (i.e., equal treatment) requirement.

**Definition.** Given \( r \geq 0 \), an allocation \( x: I \to P \) for the finite economy \( \mathcal{E}: I \to \mathcal{A} \) is an \( r \)-value allocation if for some \( \lambda: I \to R_{++}, \lambda_i \mu_i(x_i) = v_i(\mathcal{E}, \lambda) \) and \( \lambda_i \geq r \frac{1}{|I|} \) for all \( i \in I \).

The set of \( r \)-value allocations will be denoted \( \mathcal{V}_r(\mathcal{E}) \). So, \( \mathcal{V}_r(\mathcal{E}) \subseteq \mathcal{V}_0(\mathcal{E}) = \mathcal{V}(\mathcal{E}) \). The intended interpretation is that \( r \) is positive but "very small."

We have:

**Theorem.** If \( v \) is regular and \( \mathcal{E}_n: I_n \to \mathcal{A}, \#I_n = n, \) is a sequence of finite economies converging to \( v \), then, for some \( r > 0 \) sufficiently small,

\[ d(\mathcal{V}_r(\mathcal{E}_n), \mathcal{W}(\mathcal{E}_n)) = O(1/n^{1/3}). \]

Presumably, the theorem can be improved in at least three directions.

1. We do not know if (always assuming \( v \) regular) \( d(\mathcal{V}_r(\mathcal{E}_n), \mathcal{W}(\mathcal{E}_n)) \to 0 \). It is true that eventually every competitive allocation is close to a value allocation, but is eventually every value allocation close to a competitive allocation? The key unsettled question is: "Does there exist a compact set \( Q \subseteq P \) such that for every \( x \in \mathcal{V}(\mathcal{E}_n) \) and \( i \in I_n, x_i \in Q ? " The analogous statement for the core holds true (a fact first proved by Bewley [2]) and it plays a fundamental role in the asymptotic theory for the core. For every \( r > 0 \), \( r \)-value allocations have this compactness property: if we could rule out sequences of value allocations "escaping to infinity," then the theorem would hold in all generality.

If we had reduced ourselves to a type-economy universe, i.e., if we had assumed that, for all \( n, \text{supp}(v_{\mathcal{E}_n}) \subseteq K \subseteq \mathcal{A}, \) where \( K \) is finite, then we could have appealed to Champsaur's [9] concept of symmetric value allocations: A value allocation is symmetric if agents with the same characteristics receive the same weight (i.e., \( a_i = a_i \) implies \( \lambda_i = \lambda_i \)). It is easy to check that for symmetric value allocations the compactness property of the last paragraph holds; trivial modifications of the proof yield then the validity of the theorem for that case; more precisely: if \( \text{supp}(v_{\mathcal{E}_n}) \subseteq K \) for all \( n, K \) is finite, and \( \mathcal{V}_r^*(\mathcal{E}_n) \) denote the symmetric value allocations, then \( d(\mathcal{V}_r^*(\mathcal{E}_n), \mathcal{W}(\mathcal{E}_n)) = O(1/n^{1/3}) \).

The \( r \)-value allocations are the ones that satisfy a weak symmetry requirement: Since the agents with characteristics in \( \bigcup_n (\text{supp}(v_{\mathcal{E}_n})) \cup \text{supp}(v) \) (a compact set) are not "too dissimilar" we prevent the possibility of the ratio between maximal and minimal weights becoming unbounded. It is a legitimate and delicate question to determine if the definition of \( \mathcal{V}_r \), or, more
properly, the validity of the theorem is independent of the particular normalization used for the utility functions. The answer is yes (i.e., the theorem, as it stands, remains valid) up to a point: We are not free to choose arbitrary representatives of the (dimension 2) utility functions equivalence classes; the assignment \( a \mapsto u_a \) has to be continuous. Again, in the type-economy universe this is automatically satisfied.

Strictly speaking, we have not shown that the requirement that \( v \) be regular is needed to get \( d(\mathcal{V}(\mathcal{E}_n), \mathcal{W}(\mathcal{E}_n)) \to 0 \), but there can be no doubt that without it a counterexample can be developed. The idea that the notion of regular continuum economies is the key concept for obtaining strong limit theorems (i.e., continuity in the limit theorems) is Debreu's [11].

2. Our method of proof, which is essentially the same as every proof of the value equivalence theorem given so far [2, 3, 9, 24], yields an order of convergence of \( 1/n^{1/3} \). This is to be contrasted with the situation for the core, where a rate of convergence of \( 1/n \) has been proved [11, 15] and with the Shapley and Shubik [23] presumption derived from examples, that a rate of \( 1/n \) may also be the typical case with the value in replica economies. See, however, Shapley and Shapiro [22, 23]. In a nutshell the source of the \( 1/3 \) exponent is the following. The fastest way to bring an expression like \( \epsilon(n) + (1/\epsilon(n)^3) \) to zero is to put \( \epsilon(n) = 1/n^{1/3} \).

3. It is a fact that if \( v \) is regular and \( \mathcal{E}_n \to v \), then, eventually, \( \#\mathcal{W}(\mathcal{E}_n) \) is a constant. It may be conjectured that, eventually, \( \#\mathcal{V}_r(\mathcal{E}_n) = \#\mathcal{W}(\mathcal{E}_n) \) provided, of course, that \( r \) be small enough.

Our whole setup of looking at economies as distributions and concentrating on sequence of finite economies is due to Hildenbrand [19]; the consideration of large economies (in one form or another) for equivalence theorem purposes goes back to Debreu and Scarf [12], Vind [26], and Aumann [2]. The notion of regular, smooth economies is due to Debreu [10]; the appropriate definitions for the present continuum case have been given by Dierker [13]. Debreu and Smale [25] have results on an alternative characterization of regularity which is also implicit, and basic, in our analysis (Proposition 5). The concept of the Shapley value is, of course, Shapley's [20]. For the case of markets with transferable utility a value equivalence theorem is stated and proved in Shapley and Shubik [23]. Shapley [21] and, implicitly, Harsanyi (see [16]) proposed the extension of the value concept to non-side-payment games (and, in particular, markets) through the use of \( \lambda \) weights. So extended, and using the replica-economy device, the value of large, finite economies has been studied by Champasaur [9] without smoothness hypotheses, in which case competitive allocations may not be close to value allocations. In the continuum situation, and with smoothness assumptions, the equality of competitive and value allocations has been proved by Aumann and Shapley [4] (for transferable utility markets) and Aumann [3](general case) as an applica-

For the present paper, Champsaur [9] and Aumann's [3] articles (and, from a different angle, Debreu [11] and Grodal [15]) have been fundamental. Our assumptions on utility functions and on single economies are very close to Aumann's. Champsaur perceived that to get asymptotic theorems for the value a certain well-known family of probability inequalities has to be appealed to. The general outline of the proof (of every proof) seems to go back to Shapley and Shubik [23], which contains also the first published version of the value equivalence theorem.

As a final remark, our boundary assumption on utility functions (1) is certainly impalatable. It makes things easier and isolates the essential points. It should be possible to give a substantially identical version of the theorem with much more acceptable hypotheses; strictly speaking, however, this is yet another open problem.

2. PROOF OF THE THEOREM

2.1. Some definitions and Preliminaries

We observe first that because of our boundary assumption (1) on utility functions there is for every \( a \in \mathcal{A} \), \( \lambda > 0 \), and \( q \in P \) a unique \( x_{a,\lambda}(q) \in P \) such that \( \lambda D u_a(x_{a,\lambda}(q)) = q \). Moreover, \( D^2 u_a \) being nonsingular, the function \( q \mapsto x_{a,\lambda}(q) \) is \( C^1 \). Note that \( x_{a,\lambda}(q) = q(a, q, q x_{a,\lambda}(q)) \).

Let \( \delta_n : I_n \to \mathcal{A} \) and \( v \) satisfy the conditions of the theorem; in particular, \( \delta_n \to v \) and \( v \) is regular. From now on the sequence \( \delta_n \) and \( v \) are kept fixed, but whenever some property is asserted of the sequence \( \delta_n \), it should always be understood that the property holds eventually, i.e., for all but possibly finitely many \( n \). We assume \( \#(I_n) = n \) and let \( I_n = \{1, \ldots, n\} \); \( \delta \) will denote a generic term of the sequence. If \( \delta : I \to \mathcal{A} \), \( x : I \to P \), \( \lambda : I \to R_{++} \), then \( a_i, u_i, \omega_i, x_i, \lambda_i \) stand for \( \delta(i) \), \( u_{\delta(i)} \), \( \omega_{\delta(i)} \), \( x(i) \), \( \lambda(i) \); this notational convention will be used freely (thus \( x_{a_i,\lambda_i} \) becomes \( x_i \), etc.).

As it is seen from examining the corresponding Lagrangean expression, the problem "Max \( \sum_{i \in C} \lambda_i u_i(y_i) \) s.t. \( y_i \in P \) and \( \sum_{i \in C} y_i \leq z' \)" is solved by \( y_i = x_i(q(\delta, \lambda, C, z)) \), where \( q(\delta, \lambda, C, z) \in P \) is uniquely determined by \( \sum_{i \in C} x_i(q(\delta, \lambda, C, z)) = z \). Moreover, \( \sum_{i \in C} D x_i(q) = \sum_{i \in C} [\lambda_i D^2 u_i(x_i)]^{-1} \) is a negative definite, hence nonsingular, matrix, and, therefore, \( z \mapsto q(\delta, \lambda, C, z) \) is \( C^1 \). Let \( q(\delta, \lambda, C, \sum_{i \in C} \omega_i) - q(\delta, \lambda, C), q(\delta, \lambda, I) - q(\delta, \lambda) \). Note that, for \( \alpha > 0 \), \( q(\delta, \alpha \lambda, C) = \alpha q(\delta, \lambda, C) \). Define \( V(\delta, \lambda, C, z) = \text{Max}(\sum_{i \in C} \lambda_i u_i(y_i) : \ldots) \).
\( y_i \in P, \sum_{i \in C} y_i \leq z \). Of course, \( V(\theta, \lambda, C) - V(\theta, \lambda, C, \sum_{i \in C} \omega_i) \). For fixed \( \theta, \lambda, C \) the function \( z \mapsto V(\theta, \lambda, C, z) \) is \( C^2 \), concave, and satisfies \( D^2 V(\theta, \lambda, C, z) = q(\theta, \lambda, C, z) \).

The set \( \mathcal{A}^* = \bigcup_n \text{supp}(v_{i_n}^*) \cup \text{supp}(v_i) \) is compact. Therefore, given our assumption, there is \( M > 0 \) such that for all \( n, i \in I_n \) and \( x \in P \) we have \( u_i(x) \leq M \).

2.2. Lemmata

**Lemma 1.** For any \( s, s' > 0 \) the set \( Q = \{ x \in P : x \leq se, ||Dv_a(x)|| \leq s' \} \) for some \( a \in \mathcal{A}^* \) is compact.

**Proof.** Let \( a \in \mathcal{A}^* \). By (1) if \( x_n \to x \in \partial P \), then \( ||Dv_a(x_n)|| \to \infty \). By the concavity of \( u_a \), this implies that for any \( \varepsilon > 0 \) there is \( \delta > 0 \) such that if \( y \in \{ y \in P : y^j \leq \delta \text{ for some } j \} \), then \( \partial u_a(y) > 1/\varepsilon \) and \( y^j < \varepsilon \) for some \( j \).

Take any \( a \in \mathcal{A}^* \) and \( q \in \mathbb{R}^d \); we claim there is \( z_{a,a} \in P, z_{a,a} < se \) such that whenever \( x_n \to x \), \( x_n \in P, x_n < se \), we have \( u_a(z_{a,a}) - qz_{a,a} > \lim u_a(x_n) - qx \). Indeed, it suffices for this, given the strict concavity of \( u_a \), that \( u_a(z) - qz \) have a maximizer on \( \{ x \in P : x < se \} \), which is clearly true from the observation of the first paragraph.

Suppose the lemma is false. Then \( a_n \to a \in \mathcal{A}^* \), \( x_n \to x \in \partial P \), \( x_n \leq se, ||Dv_a(x_n)|| \leq s' \). We can assume \( q_n = Dv_a(x_n) - q \geq 0 \). For every \( z \in \{ y \in P : y < se \} \) we have \( u_a(x_n) + q_n(z - x_n) > u_a(z) \), which yields \( \lim u_a(x_n) - qx \geq u(z) - qz \). Since \( u_a \) converges to \( u(C^0) \) uniformly, \( \lim u(x_n) - qx \geq u(z) - qz \). Since this should hold for \( z = z_{a,a} \), we get a contradiction. 1

**Lemma 2.** Let \( x_n : I_n \to P \) be a sequence such that \( \lim \inf_n ||x_n|| \to \infty \). Then \( \lim_{n} (1/n) \sum_{i \in I_n} ||x_{ni}|| \to \infty \).

**Proof.** This is standard. If the conclusion is false, then we can assume \( \sum_{i \in I_n} (x_{ni}/n) \to z \). For every \( 1 \leq j \leq l \), let \( I_{n}^j = \{ i \in I_n : x_{ni}^j < 2l^n \} \). Then, for \( n \) large enough, \( \#(I_n^j)/n > (l - 1)/l \). Hence, for every \( n \) large enough, there is \( i \in I \) with \( ||x_{ni}|| \leq 2l ||z|| \), which is a contradiction. 1

**Lemma 3.** Given \( k > 0 \) there is \( \mu > 0 \) such that for all \( \theta, \lambda, C \subset I, z \in P \) with \( z \geq \#(C)ke \) we have \( ||q(\theta, \lambda, C, z)|| \leq ||z||/\mu \).

**Proof.** Pick \( \mu > M^{1/2}/k \). Suppose that for a \( q \in P, q^j \geq ||z||/\mu \). Then \( x_i^j(q) < k \) for all \( i \) (otherwise, \( u_i(x_i(q)) > M \) for some \( i \) because \( \lambda_i Dv_i(x_i(q)) = q \) and \( u_i \) is concave) and so \( \sum_{i \in C} x_i^j(q) < z^j \). Hence \( q \neq q(\theta, \lambda, C) \) and we can conclude \( ||q(\theta, \lambda, C)|| \leq ||z||/\mu \).

**Lemma 4.** For any \( r > 0 \) there is a compact set \( J \subset S \) such that, for all \( \theta, \lambda, C \subset I \) satisfying \( \min \lambda_i \geq r \) \( ||q(\theta, \lambda, C)|| \leq ||q(\theta, \lambda, C)|| \in J \).
Proof. Let \( q(\theta, \lambda, n, C_n) \| q(\theta, \lambda, n, C_n) \| \to p \in S \). By Lemma 2 there is \( i_n \) such that, letting \( q_n = q(\theta, \lambda, n, C_n) \{ x_{i_n}(q_n) \}_{n=1}^\infty \) is a bounded set. So, we can assume \( x_{i_n}(q_n) \to x \in P \). Since \( q_n = \lambda_{i_n} \cdot D_{x_{i_n}}(x_{i_n}(q_n)) \) and \( \lambda_{i_n} \geq r \| \lambda_n \| \) we have, by Lemma 3, \( \| D_{x_{i_n}}(x_{i_n}) \| \leq \mu/r \) for some \( \mu \). Therefore, \( x \in P \) by Lemma 1. Again by the compactness of \( S^* \) we can assume \( a_{i_n} \to a \in S^* \), and then, by continuity, \( p = D_{x_{i_n}}(x) \| D_{x_{i_n}}(x) \| \). Since \( D_{x_{i_n}}(x) \geq 0 \) we conclude \( p \in S \. 

Lemma 5. Let \( Q \subseteq P \) be a compact region. There is \( \epsilon > 0 \) and \( K > 0 \) such that for all \( \theta, \lambda, C, z_1, z_2 \in P \) with \( (1/\| C \|) \sum_{i \in C} \omega_i - z_h \| \leq \epsilon \) and \( x_i(q(\theta, \lambda, C, z_i)) \in Q \) for every \( i \in C \) and \( h = 1, 2 \), we have

\[
| V(\theta, \lambda, C, z_1) - V(\theta, \lambda, C, z_2) - q(\theta, \lambda, C, z_2)(z_1 - z_2) |^2 \\
\leq | \lambda | K \| z_1 - z_2 \| ^2 / \| C \|.
\]

Proof. Denote \( V(\theta, \lambda, C, z_h) = V(z_h) \), \( q(\theta, \lambda, C, z_2) = q, \# C = m \). We have \( q = D^2V(z_2) \) and since \( V \) is concave, \( V(z_1) - V(z_2) - q(z_1 - z_2) \leq 0 \).

Let \( Q \subseteq U \), where \( U \) is open and convex and \( \bar{U} \subseteq P \) is compact. Pick \( \epsilon > 0 \) so that, for all \( i \in C \), \( x_i(q) + ((z_1 - z_2)/m) \in \bar{U} \). Let \( K = \frac{1}{2} \max_{z \in \bar{U}} \max_{q \in S^*} \| D^2u_d(x) \| \).

By definition of \( V \), \( V(z_1) \geq \sum_{i \in C} \lambda_i u_i(x_i(q) + (z_1 - z_2)/m) \). By Taylor's formula (see, for example, [1, p. 124]),

\[
\lambda_i u_i \left( x_i(q) + \frac{z_1 - z_2}{m} \right) \\
= \lambda_i u_i(x_i(q)) + q \frac{z_1 - z_2}{m} + \frac{\lambda_i}{2m^2} (z_1 - z_2) D^2u_i(y_i)(z_1 - z_2),
\]

where \( y_i \in \bar{U} \). Henceforth,

\[
0 \geq V(z_1) - V(z_2) - q(z_1 - z_2) \geq \frac{1}{2m^2} \sum_{i \in C} \lambda_i(z_1 - z_2) D^2u_i(y_i)(z_1 - z_2),
\]

and so,

\[
| V(z_1) - V(z_2) - q(z_1 - z_2) | \leq \frac{| \lambda |}{2m^2} mK \| z_1 - z_2 \| ^2
\]

\[
= \frac{| \lambda |}{m} K \| z_1 - z_2 \| ^2.
\]

Lemma 6. Let \( Q \subseteq P \) be a compact region. Then there is \( \epsilon > 0 \) and \( K > 0 \) such that for all \( \theta, \lambda, C, z \in P \) with \( (1/\| C \|) \sum_{i \in C} \omega_i - z \| \leq \epsilon \) and \( x_i(q(\theta, \lambda, C)) \in Q \) for every \( i \), we have \( \| D^2V(\theta, \lambda, C, z) \| \leq 2 | \lambda | K/\| C \|. \)
Proof. Let $\epsilon$ and $K$ be as in Lemma 4. Denote $D_{z}^{2}V(\theta, \lambda, C, z) = A, \#C = m$. By Lemma 5 and Taylor's formula we have

$$|v'Av| \leq (2 \lambda |K/m||v|^2$$

for all $v \in \mathbb{R}^{l}$.

But this implies $\|A\| \leq 2 \lambda |K/m$ (because, being $A$ symmetric, $\|A\| \leq |\mu_j|$, where $\mu_j$ is the eigenvalue of $A$ with largest absolute value; obviously, $|\mu_j| \leq 2 \lambda |K/m|$).

**Lemma 7.** There is $\xi > 0$ such that, for all $\theta, \lambda, C \subset I$ and $i \in I$,

$$|V(\theta, \lambda, C \cup \{i\}) - V(\theta, \lambda, C)| \leq |\lambda| \xi.$$

**Proof.** Let $k > 0$ be such that $\omega_a > ke$ for all $a \in A^*$. Obviously, we can restrict ourselves to $C$'s with $\#C > m$, where $m$ is chosen so that $\omega_i/m < (k/2)e$. For notational economy we suppress reference to $\theta, \lambda$. By Lemma 2 there is $\mu > 0$ such that, for $C$ with $\#C > m$ and $i \in I$, $\|q(C \cup \{i\}, \sum_{i \in C} \omega_j\| \leq |\lambda| \mu$. Pick any $C \subset I$ with $\#C > m$ and $i \in I$ with $i \notin C$. Denote $\omega = \sum_{j \in C} \omega_j$ and $q = q(C \cup \{i\}, \omega)$. We have

$$V(C \cup \{i\}) - V(C) \leq |V(C \cup \{i\}) - V(C \cup \{i\}, \omega)| + |V(C \cup \{i\}, \omega) - V(C)|.$$

By the concavity of $V$ with respect to $z$, $0 \leq V(C \cup \{i\}) - V(C \cup \{i\}, \omega) \leq q\omega_i$. Since $\sum_{j \in C} x_j(q) \leq \omega$, we have $V(C) \geq \sum_{j \in C} \lambda_j \mu_j(x_j(q))$ and therefore,

$$0 \leq V(C \cup \{i\}, \omega) - V(C) \leq V(C \cup \{i\}, \omega) - \sum_{j \in C} \lambda_j \mu_j(x_j(q)) = \lambda_i u_i(x_i(q)).$$

Summing up,

$$|V(C \cup \{i\}) - V(C)| \leq |q\omega_i| + \lambda_i |u_i(x_i(q))| \leq |\lambda| |\mu| |\omega_i| + |\lambda| M. \quad \blacksquare$$

**Lemma 8.** Let $Q \subset P$ be a compact region. Then there is $K > 0$ and $\epsilon > 0$ such that for $\theta, \lambda, C$ satisfying $x_i(q(\theta, \lambda)) \in Q$ for all $i \in I$ and

$$\left\| \sum_{i \in C} (x_i(q(\theta, \lambda)) - \omega_i) \right\|/\#C \leq \epsilon,$$

we have

$$\|q(\theta, \lambda) - q(\theta, \lambda, C)\| \leq |\lambda| K \left( \left\| \sum_{i \in C} (x_i(q(\theta, \lambda)) - \omega_i) \right\|/\#C \right).$$

**Proof.** This is a trivial consequence of Lemma 5. Let $\epsilon > 0$ and $K > 0$ be given by Lemma 6 and put $V(y) = V(\theta, \lambda, C, y)$, $x = \sum_{i \in C} x_i(q(\theta, \lambda))$, ...
\( \omega = \sum_{i \in C} \omega_i \). Then \( q(\varepsilon, \lambda) = DV(x) \) and \( q(\varepsilon, \lambda, C) = DV(\omega) \). So,

\[
\| q(\varepsilon, \lambda) - q(\varepsilon, \lambda, C) \|
\leq \max_{0 < \varepsilon < 1} \frac{\| D^2 V(\omega) \| \| x - \omega \|}{\# C} \leq \lambda \| K \frac{\| x - \omega \|}{\# C} .
\]

**Lemma 9.** Let \( Q \subset P \) be a compact region. Then there is \( K > 0 \) and \( \varepsilon > 0 \) such that for all \( \varepsilon, \lambda, C \) and \( i \in I \) satisfying \( x_i(q(\varepsilon, \lambda)) \in Q \) for all \( j \) and (denoting \( b_1 = (1/\# C) \| \sum_{i \in C} (x_i(q(\varepsilon, \lambda)) - \omega_j) \|, \ b_2 = (1/\# (C \cup \{i\})) \| \sum_{i \in C \cup \{i\}} (x_i(q(\varepsilon, \lambda)) - \omega_j) \|, \ b = b_1 + b_2 \), \( b_2 \leq \varepsilon \), we have (denoting \( q = q(\varepsilon, \lambda), q(C) = q(\varepsilon, \lambda, C) \), etc.):

\[
\eta \equiv \| \lambda u_i(x_i(q(C \cup \{i\}))) - \lambda u_i(x_i(q))
- q(\omega_i - x_i(q)) + q(C)(\omega_i - x_i(q(C \cup \{i\}))) \| \leq \lambda \| Kb. \]

**Proof.** For some \( r > 0 \) if we have \( x_i(q(\lambda)) \in Q \) for all \( j \), then \( \min_j \lambda_j \geq r |\lambda| \). Since \( \eta \) is linearly homogeneous in \( \lambda \) we can assume \( |\lambda| = 1 \). We have then

\[
\eta \leq |u_i(x_i(q(C \cup \{i\}))) - u_i(x_i(q))|
+ \| q(\omega_i - x_i(q)) - q(C)(\omega_i - x_i(q(C))) \|
+ \| q(C) \| \| x_i(q(C \cup \{i\}) - x_i(q(C)) \|
\leq H_1 \| q(C \cup \{i\}) - q \| + H_2 \| q - q(C) \| + H_3 \| q(C \cup \{i\}) - q(C) \|,
\]

where (use compactness and \( \min_j \lambda_j \geq r \) \( H_1, H_2, \) and \( H_3 \) can be chosen independently of \( \varepsilon, \lambda, \) and \( C \) as long as \( |\lambda| = 1 \)). Since, by Lemma 8, \( \| q - q(C \cup \{i\}) \| \leq Kb_2 \) and \( \| q - q(C) \| \leq Kb_1 \), we get \( \eta \leq K(H_1 + H_2 + H_3)(b_1 + b_2) \).

**Lemma 10.** \( \Phi_{\varepsilon, n} \) converges to \( \Phi_\varepsilon \) in the sense of \( C^1 \) uniform convergence on compacta; i.e., if \( J \subset S \) is compact, then

\[
\max_{p \in J} (\| \Phi_{\varepsilon, n}(p) - \Phi_\varepsilon(p) \| + \| D\Phi_{\varepsilon, n}(p) - D\Phi_\varepsilon(p) \|) \to 0.
\]

**Proof.** This is an immediate corollary of Dierker's results [13, Lemma 3, p. 53].

**Lemma 11.** Let \( Q \) be a compact region. Then there is \( K > 0 \) and \( \varepsilon > 0 \) such that for all \( n, \lambda, \) and \( \lambda' \) satisfying \( x_i(q(\varepsilon_n, \lambda)), x_i(q(\varepsilon_n, \lambda')) \in Q \) for all \( i \in I_n \), we have

\[
\| q(\varepsilon_n, \lambda) - q(\varepsilon_n, \lambda') \| \leq K |\lambda - \lambda'|.
\]

**Proof.** Note first that, by the usual compactness argument, there is \( r > 0 \),
such that for any \( n \), if \( x_i(q(\mathcal{E}_n, \lambda)) \in Q \) for all \( i \), then \( \lambda_i \geq r | \lambda | \) for all \( i \).

Let \( \mu > 0 \) be as in Lemma 3 and pick \( \delta, H \) such that

\[
0 < \delta < \{\|v[D^2u(x)]^{-1}v : a \in \mathcal{A}^*, x \in Q, \|v\| = 1\},
\]

\[
H > (1/|\lambda|)[\|D^2u(x)\|^{-1}q : a \in \mathcal{A}^*, x \in Q, \|q\| \leq |\lambda|/\mu].
\]

From now on we consider a fixed \( n \) and suppress reference to \( \mathcal{E}_n \). It is also convenient to write \( x_i(\lambda_i, q) \) instead of \( x_{\mathcal{E}_n \lambda_i}(q) \). We will show that \( q(\lambda) \) is \( C^1 \) and, for \( \lambda \) satisfying the condition of the lemma, \( \|Dq(\lambda)\| \leq H/\delta \). By the Mean Value Theorem this implies the desired conclusion with \( K = H/\delta \).

Define \( \Psi(q, \lambda) = \sum_{i \in I_n} x_i(\lambda_i, q) \). Since \( q = q(\lambda) \) if and only if \( \Psi(q, \lambda) = \sum_{i \in I_n} \omega_i \) and \( D_q\Psi(q, \lambda) = \sum_{i \in I_n} (1/\lambda_i)[D^2u(x_i(\lambda_i, q))]^{-1} \) is nonsingular, \( q(\lambda) \) is \( C^1 \), and \( Dq(\lambda) = -[D_q\Psi(q(\lambda), \lambda)]^{-1} D_q\Psi(q(\lambda), \lambda) \). Because \( D_q\Psi(q(\lambda), \lambda) = D_qx_i(\lambda_i, q(\lambda)) = -\frac{1}{\lambda_i}[D^2u(x_i(\lambda_i, q(\lambda))]^{-1} q(\lambda) \), we have \( \|D_q\Psi \| \leq n(1/|\lambda| \sup \|v\|) \frac{H}{|\lambda|} = nH/(|\lambda| \sup \|v\|) \). We also have \( vD_q\Psi(q, \lambda)v \geq (n\delta/|\lambda|)\|v\| \) for all \( v \) so that \( |\mu| \geq n\delta/|\lambda| \) for any eigenvalue \( \mu \) of \( D_q\Psi(q, \lambda) \), and therefore \( \|D_q\Psi(q, \lambda)\| \geq n\delta/|\lambda| \). Hence \( \|D_q\Psi(q(\lambda), \lambda)]^{-1} \| \leq |\lambda|/(n\delta) \).

We conclude

\[
\|D_q(\lambda)\| \leq \|D_q\Psi(q(\lambda), \lambda)]^{-1} \| D_q\Psi(q(\lambda), \lambda)\| \leq H/\delta.
\]

2.3. Boundedness of \( r \)-Optimal Allocations

**Proposition 1.** Let \( r > 0 \) be given. Then there is a compact region \( Q \subset P \) such that for all \( \mathcal{E}, C \subset I \) and \( \lambda \) with \( \min_i \lambda_i \geq r | \lambda | \), then \( x_i(q(\mathcal{E}, \lambda, C)) \in Q \) for all \( i \).

**Proof.** Without loss of generality we can confine ourselves to \( C = I \). Let \( J \subset \mathcal{S} \) be the compact set given by Lemma 4. Suppose we have \( \mathcal{E}_n \) and admissible \( \lambda_n \) with, denoting \( x_n = x(q(\mathcal{E}_n, \lambda_n)) \), \( \lim_n \sup_i \| x_n_i \| = \infty \). Suppose \( \| x_n \| \rightarrow \infty \). Without loss of generality we can also assume \( \|q(\mathcal{E}_n, \lambda_n)\| = 1 \). Since the utility functions are concave and uniformly bounded, \( q(\mathcal{E}_n, \lambda_n) \in J \) and \( \| x_n \| \rightarrow \infty \) is possible only if \( \|Du_n(x_n)\| \rightarrow 0 \), and \( \lambda_n \rightarrow \infty \). But then \( \lim_n \inf_{\lambda_n} \lambda_n = \infty \), which yields \( \lim_n \sup_i \| Du_n(x_n)\| = 0 \) and this \( \lim_n \inf_{\lambda_n} \| x_n \| = \infty \). By Lemma 2 this contradicts the feasibility of \( x_n \), and we conclude that \( \{\sup_i \| x_n_i \|\}_{n=1}^{\infty} \) is a bounded set. Henceforth, there is \( s > 0 \) such that \( x_i(q(\mathcal{E}, \lambda)) \leq se \) for all \( \mathcal{E}, \) admissible \( \lambda \) and \( i \).

Observe now that, letting \( \mu \) be as in Lemma 3 and because of Lemma 1,

\[
Q = \{x \in P : x \leq se \text{ and } \|Du(x)\| \leq \mu/r \text{ for some } a \in \mathcal{A}^*\}
\]
is compact. But for any $\varepsilon$ if $\lambda$ is admissible, then, by Lemma 3

$$r \mid \lambda \mid \| Du_i(x_i(q(\varepsilon, \lambda))) \| \leqslant \| q(\varepsilon, \lambda) \| \leqslant \lambda \mid \mu$$

for all $i$,

i.e., $x_i(q(\varepsilon, \lambda)) \in Q$.

### 2.4. Some Probability Estimates

Let $r > 0$ be fixed. Its specific value will be determined later on. Let $Q \subseteq P$ be the compact region given by Proposition 1. Without loss of generality we can assume that $Q$ is of the form $Q = \{x \in P: (1/s) e \leqslant x \leqslant se\}$.

Given $\varepsilon > 0$, $n$, $x \in Q^I$, and $i \in I_n$, let $\beta_n(x, i, \varepsilon)$ be the probability assigned by $i$ to the set of coalitions $C$ with $\| \sum_{j \in C} x_j - w_j \| \geq \#(C) \varepsilon$ and, of course, $i \notin C$. Then we have

**Proposition 2.** For all $\varepsilon > 0$, $n$, $x \in Q^I$, $\sum_{i \in I_n} x_i - \sum_{i \in I_n} \omega_i$,

$$\beta_n(x, i, \varepsilon) \leq (2\ell/n) \left( 1/\left(1 - \exp \left(-\frac{\ell}{2} \left(\frac{\varepsilon}{s} - \frac{1}{n-1}\right)\right)\right) \right).$$

**Proof.** Let $\varepsilon > 0$, $x \in Q^I$, $i \in I_n$ be fixed. Take $1 \leq m \leq n - 1$. We consider first the family of coalitions $C$ with $i \notin C$ and $\#(C) = m$. Denote $z_j = x_j - \omega_j$, $z(C) = \sum_{j \in C} z_j$. We have

$$\text{Prob}[\| z(C)/m \| \geq \varepsilon] \leq \sum_{k=1}^{\ell} \text{Prob}[\| z^h(C)/m \| \geq \ell^{1/2}\varepsilon].$$

But

$$\text{Prob}\left\{ \left\| z^h(C)/m \right\| \geq \ell^{1/2}\varepsilon \right\} \leq \text{Prob}\left\{ \left\| z^h(C)/m \right\| \geq \frac{z^h(C)}{n-1} \geq \ell^{1/2}\varepsilon - \frac{\ell^{1/2}s}{n-1}\right\},$$

and since $-z^h_i/(n-1)$ is the mean of the random variable which gives weight $1/(n-1)$ to $z_j^h \in [-s, s]$, $j \in I_n \{i\}$, we have

$$\text{Prob}[\| z^h(C)/m \| \geq \varepsilon] \leq (2\ell/n) e^{-(\ell/9)((\varepsilon/s)-(1/(n-1)))^2 m}$$

(see Hoeffding [18, Theorem 1 and Section 6]). Further references for exponential inequalities of this type are Bennett [6] and Bahadur [5].

Therefore,

$$\beta_n(x, i, \varepsilon) = \sum_{m=0}^{n-1} \frac{2\ell}{n} e^{-(\ell/2)((\varepsilon/s)-(1/(n-1)))^2 m} \leq \frac{2\ell}{n} \sum_{m=0}^{\infty} e^{-(\ell/2)((\varepsilon/s)-(1/(n-1)))^2 m}$$

$$= (2\ell/n) \left( 1/\left(1 - \exp \left(-\frac{\ell}{2} \left(\frac{\varepsilon}{s} - \frac{1}{n-1}\right)\right)\right) \right).$$
Observe that the expression
\[ 1/(1 - \exp \left\{ -\frac{e}{2} \left( \frac{e}{s} - \frac{1}{n-1} \right)^2 \right\} ) \]
goess to \(+\infty\) as fast as
\[ \left( \frac{e}{s} - \frac{1}{n-1} \right)^2 \]
goess to 0. Hence,

**COROLLARY.** There is \( H > 0, N, \text{ and } \tilde{e} > 0 \) such that for all \( 0 < \epsilon < \tilde{e}, n > N, x \in \mathcal{V}_r(\mathcal{E}_n), \text{ and } i \in I_n, \)
\[ \beta_n(x, i, \epsilon) \leq H/(n\epsilon^3). \]

### 2.5. A Basic Convergence Property

Let \( 0 < r < 1, \) for the moment still undetermined, be as in Sections 2.3 and 2.4. For every \( n, \) put \( A_n = \{ \lambda \in R^n : \lambda_i \geq r \mid \lambda \mid \text{ for all } i \in I_n, \text{ and } \sum_{i \in I_n} \lambda_i = n \}. \) We have then \( 1 \leq \mid \lambda \mid \leq 1/r \text{ for } \lambda \in A_n. \)

Given \( n \) we define two functions \( F_n, G_n : A_n \to R^n \) as follows (denote \( q_n(\lambda) = q(\mathcal{E}_n, \lambda) \)):
\[
F_n(\lambda)_i = q_n(\lambda)(\omega_i - x_i(q_n(\lambda))),
\]
\[
G_n(\lambda)_i = v_i(\mathcal{E}_n, \lambda) - \lambda_i u_i(x_i(q_n(\lambda))).
\]

The functions \( F_n, G_n \) measure, respectively, the departures of optimal "incomes" from the incomes obtained via initial endowments and of value utilities from optimal feasible ones.

**PROPOSITION 3.** \( \max_{\lambda \in A_n} \mid F_n(\lambda) - G_n(\lambda) \mid = O(1/n^{1/3}). \)

**Proof.** Let \( n \) and \( \lambda \in A_n \) be fixed. In the following we will repeatedly appeal to Lemmata 1-11 and Proposition 2. It is understood that this is legitimate for all, but possibly finitely many, \( n \)'s. We suppress the index \( n \) from now on.

Take any \( i \in I; \) it will remain fixed. Let \( \mathcal{C} \) be the set of coalitions from \( I\setminus\{i\}. \) Partition \( \mathcal{C} \) in three subsets:
\[
\mathcal{C}_1 = \{ C \in \mathcal{C} : \#(C) \leq n^{1/3} \};
\]
\[
\mathcal{C}_2 = \{ C \in \mathcal{C} : \#(C) > n^{1/3} \text{ and } (1/\#C) \sum_{j \in C} (x_j(q(\lambda)) - \omega_j) \leq 1/n^{1/3} \};
\]
\[
\mathcal{C}_3 = \mathcal{C} \setminus \mathcal{C}_1 \cup \mathcal{C}_2.
\]
Let $C \in \mathcal{C}$ be fixed. We first estimate (suppressing reference to $\mathcal{C}$): $\Psi(C) = V(\lambda, C \cup \{i\}) - V(\lambda, C) - \lambda_i q_i(\lambda) - q(\lambda)(\omega_i - x_i(q(\lambda)))$. Let $\mu > 0$, $\xi > 0$, $K > 0$ be as in Lemmata 3 to 9. By Lemmas 3 and 7 we have $|\Psi(C)| \leq |\lambda| \xi + |\lambda| M + |\lambda| \mu s \equiv |\lambda| T \leq T/r$. If $C \in \mathcal{C}^2$, then, by Proposition 1 and Lemma 5, we have (suppressing the index $\lambda$):

$$
|V(C \cup \{i\}) - V(C) - \lambda_i q_i(\lambda) - q(\lambda)(\omega_i - x_i(q(\lambda))))|
$$

$$
= \left| V \left(C, \sum_{j \in C} \omega_j + \omega_i - x_i(q(C \cup \{i\})) \right) - V(C) - q(\lambda)(\omega_i - x_i(q(\lambda)))) \right|
$$

$$
\leq |\lambda| K \frac{\|\omega_i - x_i(q(C \cup \{i\}))\|}{\# C} \leq \frac{K}{r} \frac{\ell s}{n^{1/3}},
$$

so that, using Lemma 9,

$$
|\Psi(C)| \leq \frac{K}{r} \left( \frac{2}{n^{1/3}} + \frac{\ell s}{n^{1/3}} \right) = \frac{2K}{r} \frac{\ell s}{n^{1/3}}.
$$

Let $\pi(C)$ be the probability given by $i$ to coalition $C$. Then $G_n(\lambda)_i = F_n(\lambda)_i = \sum_{C \in \mathcal{C}} \pi(C) \Psi(C)$.
Therefore, letting \( H, \gamma \) be as in the corollary to Proposition 2 (and putting \( \epsilon = 1/n^{1/3} \)):

\[
|G_n(\lambda)_i - E_n(\lambda)_i| \\
\leq \sum_{C \in \mathbb{G}} \pi(C) \left| \Psi(C) \right| \leq \frac{T}{r} \left( \sum_{C \in \mathbb{G}_1} \pi(C) + \sum_{C \in \mathbb{G}_3} \pi(C) \right) + 2 \frac{K}{r} \left( \frac{1 + \epsilon}{n^{1/3}} \right)
\]

\[
\leq \frac{T}{r} \left( \frac{1}{n^{1/3}} + \frac{1}{n^{1/3} + \gamma} \right) + 2 \frac{K}{r} \left( \frac{1 + \epsilon}{n^{1/3}} \right) = O\left( \frac{1}{n^{1/3}} \right).
\]

Figure 1 gives an intuitive account of the proof of the theorem to be carried out in Sections 2.6 and 2.7 by using Proposition 3 and the regularity of the limit economy.

2.6. Value Allocations Are Close to Competitive Allocations

Let \( r \) be as in Section 5.

**Proposition 4.** \( \rho(\mathcal{V}(\mathcal{E}_n), \mathcal{W}(\mathcal{E}_n)) = O(1/n^{1/3}) \).

**Proof.** Let \( x_n \in \mathcal{V}(\mathcal{E}_n) \), \( x_n = x_n(q(\mathcal{E}_n, \lambda_n)) \), \( q(\mathcal{E}_n, \lambda_n) = q_n, \| q_n \| = 1 \).

Without loss of generality we can assume that \( q_n \rightarrow p \in J \), where \( J \) is as in Lemma 4. We will show the existence of \( y_n \in \mathcal{W}(\mathcal{E}_n) \) such that \( \max_{i \in I_n} \| x_n - y_n \| = O(1/n^{1/3}) \); it is a trivial matter to verify that every bounding estimate can be chosen uniformly, i.e., independently of the particular \( x_n \) considered; so \( O(1/n^{1/3}) \) will not, in fact, depend on \( x_n \), which yields the proposition.

By Proposition 3, \( F_n(\lambda_n) = O(1/n^{1/3}) \), i.e., \( \max_{i \in I_n} |q_n(\omega_i - x_i(q_n))| = O(1/n^{1/3}) \). So, \( \max_{i \in I_n} \| q(a_i, q_n, q_n \omega_i) - q(a_i, q_n, q_n x_i(q_n)) \| = O(1/n^{1/3}) \) and, therefore, \( (1/n) \| \sum_{i \in I_n} (q(a_i, q_n, q_n \omega_i) - \omega_i) \| = \| \Phi_n(q_n) \| = O(1/n^{1/3}) \).

Since \( \mathcal{E}_n \rightarrow \nu \) and \( q_n \rightarrow p \) we have \( p \in \Pi(\nu) \) (see, for example, [13, Proposition 3, p. 49]). Since \( \nu \) is regular, there are (except possibly for finitely many \( n \)'s) \( p_n \in S \) such that \( p_n \in \Pi(\mathcal{E}_n) \) (of course, \( p_n \in J \)).

Suppose that \( \| \Phi_n(q_n) \| / \| p_n - q_n \| = 0 \). By the Mean Value Theorem \( \| \Phi_n(q_n) \| = \| D\Phi_n(q_n)(q_n - p_n) \| \), where \( q_n \in S \) belongs to the (strictly speaking, curvilinear) segment \( [q_n, p_n] \). Therefore,

\[
\| D\Phi_n(q_n')(q_n - p_n) / \| q_n - p_n \| \rightarrow 0.
\]

We can assume that \( (q_n - p_n) / \| q_n - p_n \| \rightarrow \nu \); of course, \( \nu \neq 0 \) and \( \nu \in T_p(S) \). Since \( \Phi_n \) converges \( C^1 \) continuously to \( \Phi_n \) (Lemma 10), we have \( D\Phi_n(p) \nu = 0 \), but this is impossible because the regularity of \( \nu \) means precisely that \( D\Phi_n(p) \) maps \( T_p(S) \) onto itself. This contradiction proves that, for some \( H > 0 \), \( \| p_n - q_n \| \leq H \| \Phi_n(q_n) \| \) and therefore \( \| p_n - q_n \| = O(1/n^{1/3}) \) which
yields, by the compactness of $\mathcal{A}^*$ and $f$, $\max_{i \in I_n} \| \varphi(a_i, p_n, p_n \omega_i) - \varphi(a_i, q_n, q_n \omega_i) \| = O(1/n^{1/3})$.

The proof is finished, since, remembering $x_t(q_n) = p(a_i, q_n, q_n x_t(q_n))$,

$$
\max_i \| \varphi(a_i, p_n, p_n \omega_i) - x_t(q_n) \|
\leq \max_i \| \varphi(a_i, q_n, q_n \omega_i) - \varphi(a_i, q_n, q_n \omega_i) \|
+ \max_i \| \varphi(a_i, q_n, q_n \omega_i) - \varphi(a_i, q_n, q_n x_t(q_n)) \| = O(1/n^{1/3});
$$

so take $y_n = \varphi(a_i, p_n, p_n \omega_i)$. 

### 2.7. Competitive Allocations Are Close to Value Allocations

Let $r$ be as in the preceding sections. The crucial fact here is:

**Proposition 5.** There is $c > 0$ such that for all (but possibly finitely many) $n$ and $\lambda, \lambda' \in \Lambda_n$ with $F_n(\lambda) = 0$, $| \lambda - \lambda' | \leq c \cdot \| F_n(\lambda') - \lambda' | \leq c \cdot | \lambda' - \lambda |$.

**Proof.** Given any $\lambda \in \Lambda_n$, denote $\hat{\lambda} = \lambda/\| q(\lambda) \|$, then $\| q(\hat{\lambda}) \| = 1$. Clearly, $\{ q(\lambda); \lambda \in \Lambda_n \}$ can be enclosed in a compact set. Let $J \subset S$ be as in Lemma 4.

Suppose that the proposition is false, i.e., $n \to \infty$, $F_n(\lambda_n) = 0$, $\lambda_n, \lambda'_n \in \Lambda_n$, $\lambda_n \neq \lambda'_n$, $| \lambda_n - \lambda'_n | \to 0$, and $| F_n(\lambda'_n) - \lambda_n - \lambda'_n | \to 0$. Clearly, $| F_n(\lambda'_n)/| | \lambda_n - \lambda'_n | \to 0$. We can also assume that $q(\lambda_n) \to p \in S$; since $F_n(\lambda_n) = 0$ implies $\Phi_n(q(\lambda_n)) = 0$, we have $p \in \Pi(\nu)$.

We show now that $| \lambda_n - \lambda'_n | = | \lambda_n - \lambda'_n | \to 0$ is impossible. Suppose it did occur. Denote $d_n = || q(\lambda_n) ||$, $d'_n = || q(\lambda'_n) ||$. Then $d_n \lambda_n - d'_n \lambda'_n = \lambda_n - \lambda'_n \to 0$ and $(1/n) \sum_i (d_n \lambda_{n_i} - d'_n \lambda'_{n_i}) = d_n - d'_n$, which implies $d_n \lambda_n - d'_n \lambda'_n \geq d_n - d'_n$ and this $d_n - d'_n || | \lambda_n - \lambda'_n | \to 0$, or $(d_n - d'_n)/| | \lambda_n - \lambda'_n | \to 0$. However, $d_n \lambda_n - d'_n \lambda'_n = \lambda_n (d_n - d'_n) + d'_n (\lambda_n - \lambda'_n)$, yielding

$$
\lim_n d'_n = 0.
$$

Henceforth, we have $| F_n(\lambda'_n)/| | \lambda_n - \lambda'_n | \to 0$.

By Taylor’s formula, for some $H_1 > 0$,

$$
\| \Phi_n(q(\lambda'_n)) \| = (1/n) \left| \sum_i (\varphi(a_i, q(\lambda'_n), F_n(\lambda'_n) + q(\lambda'_n)(x_t(q(\lambda'_n))) - \omega_i) \right|
\leq H_1 \| F_n(\lambda'_n) \|.
$$

By Lemma 11 $q(\lambda'_n) \to p$; indeed, $\| q(\lambda_n) - q(\lambda'_n) \| \to 0$ (Lemma 11), which
implies \[ \| q(\lambda_n) - q(\lambda_n') \| = (1/\| q(\lambda_n) \|) \| q(\lambda_n) - q(\lambda_n') \| \| q(\lambda_n) \| \| q(\lambda_n') \| \rightarrow 0, \]

because, for any \( \lambda \in A_n \),

\[ 0 < r \min_{a \in e^*, x \in Q} \| D_{a}(x) \| < \| q(\lambda) \|. \]

We have, therefore, \( q(\lambda_n) \rightarrow p \in \Pi(v), q(\lambda_n') \rightarrow p \in \Pi(v), \Phi_n(q(\lambda_n)) = 0, \)

and \( v \) regular. As in the proof of Proposition 4, this implies \( \| q(\lambda_n) - q(\lambda_n') \| \leq H_2 \| \Phi_n(q(\lambda_n')) \| \) for some \( H_2 > 0 \). We conclude \( \| q(\lambda_n) - q(\lambda_n') \| \rightarrow 0. \)

Let \( H_3 > \max_{a \in a^*, p \in \Pi} \| D_{a}(1/\| D_{a}(q(a, p, p\omega_0)) \|) \|. \) Without loss of generality, we can assume \( | \hat{\lambda}_{n1} - \hat{\lambda}_{n1}' | = | \hat{\lambda}_n - \hat{\lambda}_n' |. \)

Define

\[ \xi_n(\eta, w) = 1/\| D_{a}(q(a_n, q(\eta, w))) \|. \]

Then \( \hat{\lambda}_{n1} = \xi_n(\hat{\lambda}_n, q(\lambda_n) \omega_n), \hat{\lambda}_{n1}' = \xi_n(\hat{\lambda}_n', q(\lambda_n') \omega_n) - F_\infty(\hat{\lambda}_{n1}) \); define \( \lambda_{n1} = \xi_n(\lambda_{n1}', q(\lambda_{n1}') \omega_n) \).

Since \( \{ F_n(\lambda_n) \} / \| \hat{\lambda}_{n1} - \hat{\lambda}_{n1}' \| \rightarrow 0 \), an application of Taylor's formula yields \( | \hat{\lambda}_{n1} - \hat{\lambda}_{n1}' | / | \hat{\lambda}_{n1} - \hat{\lambda}_{n1}' | \rightarrow 1. \) On the other hand, the Mean Value Theorem gives

\[ | \hat{\lambda}_n - \hat{\lambda}_{n1}' | \leq H_3 \| q(\lambda_n) - q(\lambda_n') \|, \]

so we can assume \( | \hat{\lambda}_{n1} - \hat{\lambda}_{n1}' | / | \hat{\lambda}_n - \hat{\lambda}_n' | \rightarrow 0, \) contradicting \( | \hat{\lambda}_{n1} - \hat{\lambda}_{n1}' | = | \hat{\lambda}_n - \hat{\lambda}_n' |. \)

Suppose that the \( 0 < r < 1 \) of the previous sections has been chosen small enough to have

\[ \min_{a \in e^*, p \in \Pi(v)} \| D_{a}(q(a, p, p\omega_0)) \| > r \max_{a \in e^*, \lambda \in A_n} \| D_{a}(q(a, p, p\omega_0)) \|. \]

Let \( H > 0 \) be such that, for \( n \) large enough, if \( \lambda \in A_n \), then

\[ | F_n(\lambda) - G_n(\lambda) | \leq H/|\lambda|^3 \] (Proposition 3). Denote \( \delta_n = 2H/(cn^{1/3}) \), where \( c \) is as in Proposition 5.

Let \( x_n \in W(\delta_n) \) and \( p_n \in S \) be the competitive prices associated with \( x_n \). Without loss of generality we can assume \( p_n \rightarrow p \in \Pi(v). \) If, for \( i \in I_n \), we let \( \hat{\lambda}_{ni} = 1/\| D_{a}(q(a_i, p_n, p\omega_0)) \|, \) then \( p_n = q(\delta_n, \hat{\lambda}_n) \) and (for large enough) \( \hat{\lambda}_{ni} > r / | \hat{\lambda}_n | \) for all \( i \in I_n \). If we then define \( \hat{\lambda}_n = (n/(\sum_{i=1}^{\infty} \lambda_{ni})) \lambda_n \),

we get \( \lambda_n \in A_n \). Of course, \( F_n(\lambda_n) = 0 \). Moreover, an easy compactness argument yields the existence of \( \varepsilon > 0 \) (independent of the particular sequence \( x_n \)) such that \( | \lambda_n - \lambda' | < \varepsilon \) and \( \sum_{i=1}^{\infty} \lambda_i = n \) implies \( \lambda' \in A_n \) (again, for \( n \) large enough). So, the set \( \Gamma_n = \{ \lambda \in A_n : | \lambda - \lambda_n | = \delta_n \} \) is (topologically) a sphere.

By Proposition 5, \( \text{Min}_{\lambda \in \Gamma_n} | F_n(\lambda) | \geq 2H/|\lambda|^3 \). Letting, for every \( n \) and \( \lambda \in \Gamma_n \), \( u(\lambda, n) \) be such that \( | F_n(\lambda(\lambda, n)) | > 2H/|\lambda|^3 \), we have, for every \( n \) and \( \lambda \in \Gamma_n : \)

\[ 0 \leq | F_n(\lambda(\lambda, n)) - G_n(\lambda(\lambda, n)) | \leq | F_n(\lambda) - G_n(\lambda) | \leq H/|\lambda|^3 < 2H/|\lambda|^3 \leq | F_n(\lambda(\lambda, n)) |. \]
Therefore, the maps \( F_n \mid \Gamma_n \) are homotopic with respect to \( \mathbb{R}^d \setminus \{0\} \); just take 
\( \alpha F_n + (1 - \alpha) G_n \) for a homotopy.

The situation now is as follows: the map \( F_n \) is \( C^1 \) on \( \Lambda_n \), and since \( \sum_i F_n(\lambda)_i = 0 \) it maps at every \( \lambda \in \Lambda_n \) into the tangent plane of \( \Lambda_n \); if we denote \( \tilde{\Gamma}_n = \{ \lambda \in \Lambda_n : |\lambda - \lambda_n| \leq \delta_n \} \), then \( \Gamma_n \) is the boundary (rel. \( \Lambda_n \)) of \( \tilde{\Gamma}_n \) and, by Proposition 5, \( F_n(\lambda) = 0 \), \( \lambda \in \tilde{\Gamma}_n \), implies \( \lambda = \lambda_n \); also by Proposition 5 the map \( D\tilde{F}_n(\lambda_n) \) is nonsingular. Therefore, degree \( F_n \mid \Gamma_n \neq 0 \) and since \( F_n \mid \Gamma_n, G_n \mid \Gamma_n \) are homotopic relative to \( \mathbb{R}^d \setminus \{0\} \), we conclude that degree \( G_n \mid \Gamma_n = 0 \). This implies the existence of some \( \lambda_n' \in \tilde{\Gamma}_n \) (i.e., \( |\lambda_n - \lambda_n'| < 2H/(cn^{1/2}) \)) with \( G_n(\lambda_n') = 0 \) (for the topological results used in this paragraph, see Guillemin and Pollack [14, Chaps. 2, 14]).

The conclusion of the previous paragraph yields 
\[
\rho(\{\lambda \in \Lambda_n : F_n(\lambda) = 0\}, \{\lambda \in \Lambda_n : G_n(\lambda) = 0\}) = O(1/n^{1/2}).
\]
Since \( r > 0 \) was chosen sufficiently small for \( \{\lambda \in \Lambda_n : F_n(\lambda) = 0\} \) to contain all competitive equilibria, it is immediate from Lemma 11 and the usual compactness argument that this implies 
\[
\rho(\mathscr{H}(\mathscr{E}_n), \mathscr{V}(\mathscr{E}_n)) = O(1/n^{1/2}).
\]

Q.E.D.

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