On Revealed Preference Analysis

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1. INTRODUCTION

In this paper we shall be concerned with the theory of revealed preference understood in its original sense (Samuelson [19]), namely, as a theory of rational consumer's behaviour in competitive market situations; (see Richter [18] for the formulation and development of a general, abstract theory). As with practically all previous work in the field we will focus on the relationship between the theory of demand derived from revealed preference analysis and the one based on preference maximization; we denote the latter theory as the preference hypothesis.

In 1938 Samuelson proposed as a new foundation for consumer theory the weak axiom of revealed preference (WA); the WA asserts that if a commodity bundle y is affordable, given a budget at which x is chosen (for short, if x is revealed preferred to y), then y cannot be revealed preferred to x. Although it has turned out that the demand theory implied by the WA is broader than the one generated by the preference hypothesis (Gale [6]; see also, Kihlstrom, Mas-Colell, Sonnenschein [14]) it is clear that they are conceptually close and their relationships were in need of clarification. The decisive step was given in 1951 by Houthakker [10]. He introduced the strong axiom of revealed preference (SA) and convincingly argued its equivalences with the preference hypothesis. Roughly speaking, the SA postulates the cyclical consistence of the WA.

Since Houthakker's contribution the main line of research (see, for example, Uzawa [22], Stigum [21], etc.) has concentrated in finding sufficient conditions on demand functions guaranteeing that the consumer acts as a preference maximizer *and* his preferences are completely " determined " by his choice behaviour. This programme calls for both the *existence* of preferences underlying choice and their *uniqueness* in some class (usually the class defined by the property of continuity) and it is clear that, to successfully complete it, conditions on demand functions (besides, of course, the SA) will have to be imposed. So far, positive results have been obtained by combining a substantive requirement, the well definiteness of indirect demand functions, with a quite weak regularity one, the income Lipschitz property (this term will be defined later on).

A second approach, initiated in different contexts by Richter [18] and Afriat [1] (see also, Hurwicz and Richter [12]), recognizes that if one sets aside the uniqueness question, then the equivalence between the SA and the preference hypothesis can be established with the utmost generality. Thus, appealing to the set-theory based methods of Richter [18], one directly gets that for any demand function satisfying SA there exists a preference relation rationalizing it. In the specific competitive markets framework in which we are interested one can go much further: for example, Afriat [1] has offered a method to compute, by linear programming techniques, a piecewise linear concave utility function in which demand is predetermined on any finite number of budgets; the only requirement is that the *a priori* given budget-choice combinations satisfy the SA. This is a very appealing result and prompts one to inquire to what extent its use as a practical method of ascertaining consumer's preferences could be justified; we will see that indeed it can under very mild conditions.

In this paper we attempt to exploit the just mentioned general existence results to make the determinateness problem an easier and more tractable one, or, in other words, we try to bridge the approaches of the last two paragraphs by starting from the latter. It will be seen that the income Lipschitz property, which is a weak regularity condition, is the key for the successful accomplishment of this task. Considerable sharpenings of the available existence-uniqueness results will be obtained; in particular, no indirect demand conditions will be appealed to. Unrelated (at least explicitly) to revealed preference considerations the problem of the uniqueness of continuous preference rationalizing given demand functions has been studied elsewhere [15] and, of course, the results there will be used here.

Consider any demand function h and a sequence of continuous, convex, monotone preferences $\succsim_n$ rationalizing h in a finite set of budgets $C_n = \{(p_n, w_n)\}$; $\succsim_n$ may be obtained from Afriat's method. If C_n increases and becomes eventually " dense " and if h is continuous, then we show that one can naturally associate with the sequence $\succsim_n$ a, not necessarily unique, limit $\succsim^*$ which turns out to be an upper-semicontinuous, convex, monotone preference relation generating h (Theorem 1). Moreover, and this is the main result of the paper, if h is income Lipschitzian (in a certain sense) and either it satisfies a boundary condition or it is onto and non-inferior, then: (i) $\succsim^*$ is in fact continuous; (ii) this $\succsim^*$ is representable by a Lipschitzian, regular (this will be defined) utility function (this result is quoted from [15]), and (iii) $\succsim^*$ is the unique upper semicontinuous, convex, monotone preference relation generating h (Theorems 2, 2′, 3 and 4).

Precise definitions, many more details, discussion, and references to the literature will be given in the main body of the text. We may mention here that to avoid boundary complications our consumption set will be $P = \{x \in R^l: x \gg 0\}$.

2. DEFINITIONS AND STATEMENT OF RESULTS

We let $P = \{x \in R^l: x \gg 0\}$, $V = P \times (0, \infty)$.[1]

Definition 1. A *demand function* h: $V \to P$ is a continuous function such that, for all $(p, w) \in V$: (i) $ph(p, w) = w$ and (ii) $h(\lambda p, \lambda w) = h(p, w)$ for every real $\lambda > 0$.

Definition 2. A demand function h satisfies the *strong axiom of revealed preference* (SARP) if the relation $W \subset P \times P$ defined by " xWy if $x \neq y$ and $x = h(p, w)$, $py \leqq w$ for some $(p, w) \in V$" is acyclic (i.e. $xWyW...zWx$ is impossible).

Definition 3. A demand function h is *income Lipschitzian* if for every $p \gg 0$, $w > 0$ there are positive reals $r > 0$, $\varepsilon > 0$ such that if $(p', w') \in V$ and $\| p - p' \| < \varepsilon$, $| w - w' | < \varepsilon$, then $\| h(p', w) - h(p', w') \| \leqq r | w - w' |$.[2]

Definition 4. A demand function h is *non-inferior* if $h(p, w') \geqq h(p, w)$ for all $p \in P$ and $w' \geqq w > 0$.

Definition 5. A demand function h satisfies the *boundary condition* (BC) if

$$(p_n, w_n) \to (p, w) \notin V, p \neq 0, w \neq 0, (p_n, w_n) \in V$$

implies $\| h(p_n, w_n) \| \to \infty$.

Definition 6. A relation $\succsim \subset P \times P$ is a *preference relation* if it is *reflexive* (i.e. $x \succsim x$ for all $x \in P$), *complete* (i.e. for all $x, y \in P$, either $x \succsim y$ or $y \succsim x$) and *transitive* (i.e. for all $x, y, z \in P$, $x \succsim y$ and $y \succsim z$ implies $x \succsim z$). For a relation $\succsim \subset P \times P$, $x \succ y$ means $x \succsim y$ and $\neg(y \succsim x)$ (i.e. no $y \succsim x$). A relation $\succsim \subset P \times P$ is: *monotone* if $x \gg y$ implies $x \succ y$; *strictly monotone* if $x > y$ implies $x \succ y$; *convex* if $x \succsim y$ implies $tx + (1-t)y \succsim y$ for $t \in [0, 1]$; *strictly convex* if $x \succsim y$, $x \neq y$, implies $tx + (1-t)y \succ y$ for $0 \leqq t \leqq 1$; *continuous* if it closed in $P \times P$; *upper semicontinuous* (u.s.c.) if, for all $x \in P$, $\{y \in P: y \succsim x\}$ is closed; *lower semicontinuous* (l.s.c.) if, for all $x \in P$, $\{y \in P: x \succsim y\}$ is closed.

Definition 7. A *preference relation* $\succsim$ *generates a demand function* h if, for all $(p, w) \in V$, " $px \leqq w$ and $x \neq h(p, w)$ " implies $h(p, w) \succ x$.

Definition 8. A function u: $P \to R$ is a utility function for a continuous preference relation $\succsim$ if it is continuous and " $u(x) \geqq u(y)$ if and only if $x \succsim y$ ".

Definition 9. A utility function u for a continuous, strictly monotone preference relation $\succsim$ is:

(i) *Lipschitzian* if for every $x \in P$ there are $r > 0$, $\varepsilon > 0$ such that if $\| x-y \| \leqq \varepsilon$, $y \in P$, then $| u(x)-u(y)| \leqq r \| x-y \|$;

(ii) *regular* if for every $x \in P$ there is $s > 0$ such that if $y \geqq x$, then

$$u(y)-u(x) \geqq s \| y-x \|.$$

For the definition of the topology of closed convergence and its use to formalize the notion of closeness of preference relations see Hildenbrand [9], especially B.II and 1.2.

Definition 10. Let $\succsim_n$, $\succsim$ be continuous, monotone preference relations. We say that $\succsim_n$ *converges to* $\succsim$, written $\succsim_n \to \succsim$, if when regarded as closed subsets of $P \times P$, $\succsim_n$ converges to $\succsim$ in the closed convergence (which defines a topology on the space of closed subsets of $P \times P$).

The concepts described in the above definitions are for the most part well known. The relation W was introduced by Samuelson [20], who postulated its asymmetry (the weak axiom of revealed preference); the SARP was first proposed by Houthakker [10], who formulated as well the income Lipschitz condition. Uzawa [22] noted that non-inferior demand functions were income Lipschitzian. The interest of preferences representable by Lipschitzian, regular utility functions (Definition 9) has been suggested in Mas-Colell [15]; it is also shown there that the class of such preferences can be intrinsically characterized without reference to utility functions.

The first result is a revealed preference theorem for continuous demand functions:

Theorem 1. *Every continuous demand function which satisfies the strong axiom of revealed preference can be generated by an upper semi-continuous, convex, monotone preference relation.*

Next, a revealed preference theorem for income Lipschitzian demand functions.

Theorem 2. *Every continuous income Lipschitzian demand function which satisfies the strong axiom of revealed preference and the boundary condition can be generated by a continuous, monotone, strictly convex preference relation.*

It is possible to verify that if h satisfies the boundary condition, then $h(V) = P$.[3] An example in the Appendix (a modification of one due to Hurwicz and Richter [12]) shows that in Theorem 2 the boundary condition cannot be replaced by the weaker $h(V) = P$. If " income Lipschitzian " is strengthened to " non-inferior ", however, the replacement can be made (or, what *a posteriori* is the same, under non-inferiority the boundary condition and $h(V) = P$ are equivalent):

Theorem 2′. *Every continuous, non-inferior demand function which satisfies the strong axiom of revealed preference and the condition $h(V) = P$ can be generated by a continuous, monotone, strictly convex preference relation.*

We state now a uniqueness result which sharpens considerably the conclusion of Theorems 2 and 2′. It has been proved elsewhere (Mas-Colell [15]).

Theorem 3. *If a continuous, monotone, strictly convex preference relation $\succsim$ generates an income Lipschitzian demand function h, then:*

(i) *$\succsim$ is the unique upper semicontinuous, monotone, convex preference relation generating h, and*

(ii) *$\succsim$ is representable by a Lipschitzian, regular utility function.*

The combination of Theorems 2 and 3 yields the conclusion that underlying every demand function which satisfies SARP and is income Lipschitzian (and fulfils the BC) there is a unique, continuous (i.e. a " true ") preference relation. The next theorem implies that those preferences can be (approximately) constructed from a finite number of price-income observations. More specifically, Afriat [1] (see also, Diewert[5]) has shown that if $C \subset V$ is finite and $h \mid C$ satisfies the SARP, then there is a concave, monotone utility function generating demand choices, which, for every $(p, w) \in C$, include $h(p, w)$; moreover, this utility function can be computed as the solution of a certain simple linear programme. Our result adds to this that, provided the choice observations are taken from an income Lipschitzian h, the sequence of preferences induced by Afriat's utility functions as one keeps increasing (in a regular manner) the number of observations, will have a well-defined unique limit which is nothing but the theoretical true preference underlying h. Hence, we may conclude that the income-Lipschitz condition is the appropriate theoretical assumption (one obviously very weak) on which the attachment of operational significance to revealed preference analysis as a method of ascertaining preferences should rest.[4]

Theorem 4. *Let h be a continuous demand function satisfying the strong axiom of revealed preference and the boundary condition. Let $C_n \subset V$ be an increasing sequence of sets $(C_n \subset C_{n+1} \subset \ldots)$ such that $\bigcup_n C_n$ is dense in V. Suppose that, for every n, $\succsim_n$ is a continuous, convex, monotone preference relation with the property that for every*

$$(p, w) \in C_n,\ h(p, w) \subset \{y \in P\colon py \leqq w \text{ and if } pv \leqq w, \text{ then } y \succsim_n v\}.$$

Then, if h is income Lipschitzian, $\succsim_n \to \succsim$ where $\succsim$ is the unique continuous, monotone, strictly convex preference relation generating h.

Remark 1. Theorem 1 is new, but it is in the same spirit as a result of Hurwicz and Richter (Theorem 1 in [12]) which yields analogous conclusions without the continuity assumption on h but postulates a certain convexity condition in the range $h(V)$. It should be clear that the $\succsim$ one gets from Theorem 1 needs not be unique.

Remark 2. Consider the condition, denoted U for brevity, " $h(V) = P$ and for every $x \in P$, $h^{-1}(x)$ is a single ray ", it is easily seen that it implies the boundary condition and so Theorem 2 remains *a fortiori* valid if the latter condition is replaced by the former. In this particular case where U holds Theorem 2 (with uniqueness) is a quite old one; it is the form that Houthakker gave to his theorem and it has been proved, with increasing rigour, by Houthakker [10], Uzawa [22], [23], Stigum [21] and, no doubt, others. Under the general condition of its statement (no " invertibility " assumption on demand), Theorem 2 (combined with the uniqueness given by Theorem 3) seems to be new.[5]

Remark 3. Observe that even for the case where condition U (Remark 2) holds, Theorems 2 and 4 yield a stronger conclusion than the one obtained by Houthakker and Uzawa. Preferences are not only continuous but also " Lipschitzian " in the sense introduced in [15] (i.e. they are representable by Lipschitzian and regular utility functions).[6]

Remark 4. An attempt to derive some result analogous to Theorem 2′ from Uzawa's [22] lemmata has been made by v. Moeseke [16]. It is our feeling, however, that no such thing is possible.

Remark 5. The attentive reader can verify that the proof of Theorem 2 goes through if the boundary condition is dropped and the Lipschitzian condition strengthened to: " for every compact $K \subset P$ there are reals $\varepsilon > 0$ and $r > 0$ such that if $(p, w) \in h^{-1}(K)$ and $| w' - w | < \varepsilon$, then $\| h(p, w) - h(p, w') \| \leqq r \mid w - w' \mid$ ". This condition has been introduced by Hurwicz and Uzawa [13], who, with its help, established the analog of Theorem 2 in the integrability approach to the theory of " rational " demand (smoothness assumptions are made and the rationality axioms on demand are local ones, such as the symmetry and negative semidefiniteness of Slutzky matrices, rather than global ones as the SARP; see Hurwicz [11]). The condition is not as innocent as it appears; with the SARP it implies the conclusion of Theorem 2 and so the boundary condition should hold.

Remark 6. If the hypothesis $h(V) = P$ is dropped in Theorem 2′ it can still be proved that h can be generated by a $\succsim$ which is monotone, convex, and continuous on $h(V)$ (i.e. $\succsim \cap h(V)\times h(V)$ is closed relative to $h(V)\times h(V)$). No similar result for a case where $h(V) \neq P$ holds with respect to Theorem 2.

A few words on the strategy of the proofs. Theorem 1 is established by combining the referred to result of Afriat with a limit argument. Theorems 2 and 2′ are obtained by taking the preferences given by Theorem 1 and arguing by contradiction with a little help from convex analysis (the key fact is provided by the rules of differentiation of support functions). Theorem 4 follows, essentially, by combination of Theorems 1 and 3. We do not use (neither here nor in [15]) the differential equations-type techniques, which have been popular in revealed preference analysis from its founding (we refer to the price-income sequences used by Samuelson [20], Houthakker [10], Uzawa [22], Stigum [21] and others) in order to construct preferences from demand functions. While local methods are appropriate for the integrability approach, which is based on local assumptions, it is probably more efficient, in revealed preference theory, to take advantage from the beginning of the global nature of the SARP and proceed by getting at once *some* preference relation generating the demand function under consideration; this is precisely what the very direct theorems of Richter [18] and, in a more concrete setting, Afriat [1] allow one to do.

3. LEMMATA

Let $\mathscr{F}$ be the space of non-empty, closed subsets of $P\times P$ endowed with the closed convergence topology (see Hildenbrand [9]). The subspace of $\mathscr{F}$ formed by the continuous, monotone, convex preference relations is denoted $\mathscr{P}$. The closure of $\mathscr{P}$ in $\mathscr{F}$ will be $\bar{\mathscr{P}}$.

For any $\succsim \in \bar{\mathscr{P}}$ define $\succsim^* \subset P\times P$ by letting $y \succsim^* x$ if and only if for all $v \in P$, $x \succsim v$ implies $y \succsim v$.

Lemma 1. *For any* $\succsim \in \bar{\mathscr{P}}$, $\succsim^*$ *is an upper semicontinuous, convex, monotone preference relation.*

Proof. It is immediately verified that $\succsim \in \bar{\mathscr{P}}$ is closed, monotone, reflexive, and complete; moreover, for any $x \in P$, $\{y\colon y \succsim x\}$ is convex (see, for example, B. Grodal [8]). Since $\{y\colon y \succsim^* x\} = \bigcap_{x \succsim^* v} \{y\colon y \succsim^* v\}$ it is clear that $\succsim^*$ is reflexive, transitive, upper semicontinuous, convex, and monotone. To check that $\succsim^*$ is complete we argue by contradiction: let $x \succsim v$, $v \succ y$, $y \succsim u$, $u \succ x$ (implying $\neg(x \succsim^* y)$ and $\neg(y \succsim^* x)$). By definition, there are $x_n \to x$, $y_n \to y$, $u_n \to u$, $v_n \to v$, such that $x_n \succsim_n v_n$ and $y_n \succsim_n u_n$; for $\bar{n}$ large enough $v_{\bar{n}} \succ_{\bar{n}} y_{\bar{n}}$, $u_{\bar{n}} \succ_{\bar{n}} x_{\bar{n}}$ and so, we get $x_{\bar{n}} \succ_{\bar{n}} x_{\bar{n}}$, which is impossible. ||

N.B. For the rest of this section h is a fixed, continuous demand function.

Lemma 2 (Afriat). *Let* $C \subset V$ *be a finite set; then there is* $\succsim \in \mathscr{P}$ *such that for every* $(p, w) \in C$, $h(p, w)$ *is a maximizer of* $\succsim$ *on* $\{y\colon py \leqq w\}$.

Proof. Afriat [1] see also, Diewert [5]. ||

Lemma 3. *Let* $C_n \subset C_{n+1} \subset \ldots \subset V$ *be an increasing sequence such that* $C = \bigcup_n C_n$ *is dense in V. Let* $\succsim_n \in \mathscr{P}$ *be such that for every* $(p, w) \in C_n$, $h(p, w)$ *is a maximizer of* $\succsim_n$ *on* $\{y\colon py \leqq w\}$. *If* $\succsim_n \to \succsim \in \bar{\mathscr{P}}$, *then* $\succsim^*$ *generates h.*

Proof. Let $(p, w) \in V$ and denote $y = h(p, w)$. Pick $x \neq y$ with $px \leqq w$. We need to show $y \succ^* x$.

By the continuity of h there is a sequence $(p_m, w_m) \in C$ such that $(p_m, w_m) \to (p, w)$ and, letting $y_m = h(p_m, w_m)$, one has $p_m x < w_m$, $py_m < w$. (Figure 1 indicates how the sequence can be constructed.) Obviously, $y_m \to y$ and, letting $(p_m, w_m) \in C_{n_m}$, $y_m \succsim_{n_m} x$. So, $y \succsim x$. (Note: x being arbitary, we can conclude here that $pv \leqq w$ implies $y \succsim v$.) Now take an arbitrary m and let $z \gg y_m$, $pz \leqq w$ if $x \succsim z$ there is $x_n \to x$, $z_n \to z$ with $x_n \succsim_n z_n$, but, for sufficiently large n, $z_n \succ_n y_m \succsim_n x_n$ (remember $C_n \subset C_{n+1} \subset \ldots$) and we have a contradiction. Therefore, $z \succ x$. Since $y \succsim z$ (because $pz \leqq w$) we conclude $\neg(x \succsim^* y)$, or $y \succ^* x$. Hence $\succsim^*$ generates h. ||

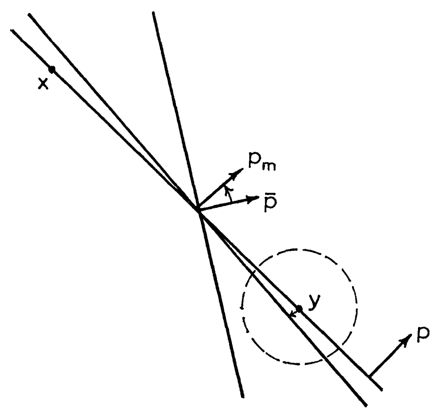

FIGURE 1

N.B. For the rest of this section $\succsim$ *is a fixed upper semicontinuous, monotone preference relation generating h.*

Define μ: $P \times \bar{P} \to R$ by $\mu(x, p) = \inf\{py: y \succsim x\}$. By the monotonicity of $\succsim$, $\mu(x, p) > 0$ whenever $p \gg 0$. For every $(x, p) \in P \times \bar{P}$ let

$$\sigma(x, p) = \{y \in P: y \succsim x, py = \mu(x, p)\} \subset P.$$

Lemma 4. *For all* $(p, w) \in V$, $w = \mu(h(p, w), p)$.

Proof. Immediate. ||

Lemma 5. *For every* $p \gg 0$ *and* $x \in P$, *if* $x_n \to \bar{x} \in \bar{P}$, $x_n \succsim x$, *and* $p\bar{x} = \mu(x, p)$, *then* $x = h(p, \mu(x, p))$.

Proof. Suppose not. Denote $y = h(p, \mu(x, p))$ and pick $z = \bar{x}/2 + \bar{y}/2$, $\bar{p} \gg 0$ such that $\bar{p}\bar{y} > \bar{p}\bar{x}$. Let $p_t = tp + (1-t)\bar{p}$, $0 \leqq t \leqq 1$. By the continuity of h if $\bar{t}$ is close enough to 1, then $ph(p_{\bar{t}}, p_{\bar{t}}z) < \mu(x, p)$ (see Figure 1) which is impossible because $p_{\bar{t}}\bar{x} < \bar{p}_{\bar{t}}z$ and this implies $h(p_{\bar{t}}, p_{\bar{t}}z) \succsim x$. ||

Lemma 6. $\sigma(x, p) = \{h(p, \mu(x, p))\}$ for every $p \gg 0$, $x \in P$.

Proof. Denote $y = h(p, \mu(x, p))$. By the definition of μ there is $x_n \to \bar{x} \in \bar{P}$ such that $x_n \succsim x$ and $p\bar{x} = \mu(x, p)$. By Lemma 5, $\bar{x} = y$ and so, by the u.s.c. of $\succsim$, $y \succsim x$. Hence, $y \in \sigma(x, p)$; $\sigma(x, p) \subset \{y\}$ follows trivially from Lemma 5. ||

Lemma 7. *For every* $x \in P$ *and* $p > 0$, $\mu(x, p)$ *is continuous on p; moreover, if* $p \gg 0$, *then* $\mu(x, p)$ *is (continuously) differentiable on p and* $\nabla_p \mu(x, p) = \sigma(x, p) = h(x, \mu(x, p))$.

Proof. This is a standard result on the continuity and differentiability of support functions (see H. Nikaido [17, p. 298]); note that, by Lemma 6, $\sigma(x, p)$ is a singleton for every $p \gg 0$. ||

Lemma 8. *Let h satisfy the boundary condition. If* $x \in P$, $p > 0$, *and* $\sigma(x, p) \neq \phi$, *then* $p \gg 0$. *If* $\gtrsim$ *is continuous, convex, then it is strictly monotone.*

Proof. Easy. ||

Lemma 9. *If* $\gtrsim$ *is continuous, convex, and strictly monotone, then it is strictly convex.*

Proof. Obvious. ||

Lemma 10. *Let* $\gtrsim$ *be convex. If, for every* $x, y \in P$ *and* $p > 0$, $\sigma(x, p) \cap \sigma(y, p) \neq \phi$ *implies* $\mu(x, \bar{p}) = \mu(y, \bar{p})$ *for all* $\bar{p} \in \bar{P}$, *then* $\gtrsim$ *is continuous.*

Proof. Let $x_n \to y$, $x \gtrsim x_n$, $y \succ x$. Then, on account of monotonicity, y belongs to the boundary of the convex set $\{v \in P\colon\ v \gtrsim x\}$; take $p > 0$ such that if $v \gtrsim x$, then $pv \geqq py$. We have $y \in \sigma(x, p) \cap \sigma(y, p)$. Now take $x' \gg x$ such that $y \succ x'$ (remember $\gtrsim$ is u.s.c.) and let $\bar{p} > 0$ separate x' and $\{v \in P\colon\ v \gtrsim y\}$ (i.e. $v \gtrsim y$ implies $\bar{p}v \geqq \bar{p}x'$). Then

$$\mu(x, \bar{p}) \leqq \bar{p}x < \bar{p}x' \leqq \mu(y, \bar{p}),$$

which contradicts the assumption of the lemma. Therefore, "$x_n \to y$, $x \gtrsim x_n$" implies $x \gtrsim y$ and $\gtrsim$ is lower semicontinuous; since $\gtrsim$ is an u.s.c. preference relation, it is continuous. ||

4. PROOF OF THE THEOREMS

Proof of Theorem 1

Let $C_n \subset C_{n+1} \subset \ldots \subset V$ be a sequence of finite sets such that $\bigcup_n C_n$ is dense in V. For every n pick $\gtrsim_n \in \mathscr{P}$ according to Lemma 2. Since $\bar{\mathscr{P}}$ is closed convergence compact (see Hildenbrand [9]) we can assume $\gtrsim_n \to \gtrsim \in \bar{\mathscr{P}}$. By Lemmas 1 and 3, $\gtrsim^*$ is as desired. ||

Proof of Theorem 2

Let $\gtrsim$ be an upper semicontinuous, monotone, convex preference relation generating h (Theorem 1). We apply Lemma 10 and argue by contradiction. Define μ and σ as in the lemmata and suppose that $v \in \sigma(x, \bar{p}) \cap \sigma(y, \bar{p})$, $\mu(x, p) > \mu(y, p)$ for some $x, y \in P$ and $\bar{p}, p > 0$. By the boundary condition (Lemma 8) we have $\bar{p} \gg 0$. Let $w = \bar{p}v$.

For $0 \leqq t \leqq 1$ take $p(t) = tp + (1-t)\bar{p}$ and define $\xi\colon [0, 1] \to R$ by

$$\xi(t) = \mu(x, p(t)) - \mu(y, p(t)).$$

By Lemma 7, ξ is continuous hence (replacing, if necessary, $\bar{p}$ by $p(\bar{t})$, $\bar{t} = \inf\{t\colon\ \xi(t) = 0\}$, and v by $h(p(\bar{t}), \mu(p(\bar{t}), x))$) we can assume that $\xi(t) > 0$ for $0 < t \leqq 1$. By Lemma 7 ξ is continuously differentiable on $0 < t < 1$; denote by $\xi'(t)$ the derivative of t. If $\sup_{0<t<1} \frac{\xi'(t)}{\xi(t)} < \infty$, then $\int_0^1 \frac{\xi'(t)}{\xi(t)} = \log \xi(1) - \log \xi(0) < \infty$, which is impossible, since $\xi(0) = 0$ and $\xi(1) > 0$. Hence there is a sequence t_n, $0 < t_n \leqq 1$ such that $\frac{\xi'(t_n)}{\xi(t_n)} \to \infty$. Since $\xi(t) > 0$ for $t > 0$, we can assume by the same argument that $t_n \to 0$. Denote

$$p_n = p(t_n),\ w_n = \mu(x, p(t_n)),\ w'_n = \mu(y, p(t_n)),\ z_n = h(p_n, w_n),\ z'_n = h(p_n, w'_n);$$

then $w_n \neq w'_n$, $z_n \neq z'_n$. By Lemma 7, $\xi'(t_n) = (\bar{p} - p)(z_n - z'_n)$. Therefore, $\frac{(p-\bar{p})(z_n - z'_n)}{| w_n - w'_n |} \to \infty$ and since $\left\{(p - \bar{p}) \frac{z_n - z'_n}{\| z_n - z'_n \|}\right\}_{n=1}^{\infty}$ is bounded, we have $\frac{\| z_n - z'_n \|}{| w_n - w'_n |} \to \infty$, or, rewriting, $\frac{\| h(p_n, w_n) - h(p_n, w'_n) \|}{| w_n - w'_n |} \to \infty$, $p_n \to \bar{p}$, $w_n \to w$, and $w'_n \to w$, which contradicts the fact that h is income Lipschitzian. Hence we conclude (Lemma 10) that $\gtrsim$ is continuous, strictly monotone, and strictly convex (Lemmas 8 and 9). ||

Proof of Theorem 2'.

Let $\succsim$ be an upper semicontinuous, convex, monotone preference relation generating h (Theorem 1). Define μ and σ as in the lemmata. We show that $x \in P, p>0$, and $\sigma(x, p) \neq \phi$ implies $p \gg 0$. Once this is established, we can apply the proof of Theorem 2 without modification (the BC was only used to get $p \gg 0$) and obtain the continuity of $\succsim$. But, together with $h(V) = P$, this is easily seen to imply strict monotonicity and convexity.

Let $x \in P, p>0, x' \in \sigma(x, p), J = \{j: p^j = 0\}$. The proof is divided into three parts.

(1) There is $y \in P$ and $\bar{p} \gg 0$ such that, letting $p_t = tp+(1-t)\bar{p}$, $y = h(p_t, p_t y)$ for every $0 \leqq t<1$.

To see this let $x' = h(p', p'x')$, $p'_t = tp+(1-t)p'$; for every $0 \leqq t<1$, $h(p'_t, p'_t x') \succsim x'$ and so $ph(p'_t, p'_t x') \geqq px'$, which implies $p'h(p'_t, p'_t x') \leqq p'x'$. Therefore,

$$\{h(p'_t, p'_t x'): 0 \leqq t<1\}$$

has a limit point $x'' \in \bar{P}$; then $px'' = px'$ and, denoting $y = \frac{1}{2}x'+\frac{1}{2}x'' \in P$, we have, by the convexity and the upper semicontinuity of $\succsim$, that $py = px'$ and $y \succsim x'$. Take $\bar{p} \gg 0$ such that $y = h(\bar{p}, \bar{p}y)$. Then (defining p_t as in (1)), if $p_t v \leqq p_t y$, $t<1$, either $\bar{p}v \leqq \bar{p}y$ or $pv<px'$; in both cases $y \succsim v$. So, $y = h(p_t, p_t y)$ for $0 \leqq t<1$.

(2) If the $\bar{p} \gg 0$ in (1) is chosen close enough to p there is a real $\varepsilon>0$ such that if $py-\varepsilon \leqq w \leqq py$, then $h^j(p_t, w) \geqq \varepsilon$ for every t and $j \notin J$.

To see this let $0<\gamma< \min_{j \notin J} y^j$, $\delta = \inf \{p^j_t(y^j-\gamma): 0 \leqq t \leqq 1, j \notin J\}>0$. We can assume, without loss of generality, that $\max_{0 \leqq t \leqq 1} | p_t y-py |<\delta/2$. If $py-\delta/2 \leqq w \leqq py, 0 \leqq t \leqq 1$, and $j \notin J$, then $| w-p_t y |<\delta$ and, by non-inferiority, $| p^j_t(y^j-h^j(p_t, w))| \leqq | p_t y-w |<\delta$. Hence $h^j(p_t, w) \geqq \gamma$. So, take $\varepsilon = \min \{\delta/2, \gamma\}$.

Finally,

(3) $p \gg 0$.

Let $t_n \to 1$ and denote $p_n = p_{t_n}$. Take $py-\varepsilon<\bar{w}<py$ and a decreasing sequence $w_m \to \bar{w}$ with $w_1<py$. For every m, $\{h(p_n, w_m)\}_{n=1}^{\infty}$ has a limit point in $\bar{P}$ because h is non-inferior and, for large enough n, $w_m<p_n y$ (i.e. $h(p_n, w_n) \leq y$). Let it be v_m, then $pv_m = w_m$, $v_m \leqq y$, and $v^j_m \geqq \varepsilon$ for $j \notin J$. Let $v \in \bar{P}$ be a limit point of $\{v_m\}_{m=1}^{\infty}$. We have $pv = \bar{w}$ and $v^j>0$ for $j \notin J$. Define $\bar{v} \in P$ by $\bar{v}^j = v^j$ if $j \notin J$ and $\bar{v}^j = v^j+1$ if $j \in J$; then $p\bar{v} = \bar{w}$. Now, for every m and N_m large enough, if $n>N_m$, then $w_m>p_n\bar{v}$; so, by the upper semicontinuity of $\succsim$, $v_m \succsim \bar{v}$ for every m and $v \succsim \bar{v}$. If $\bar{v} \neq v$, then, since $h(V) = P$ and $\bar{v} \geqq v$, $\bar{v} \succ v$, a contradiction. Therefore, $v = \bar{v}$, which implies $p \gg 0$. ||

Proof of Theorem 4.

Since $\mathscr{P}$ compact, $\{\succsim_n\}_{n=1}^{\infty}$ has an accumulation point $\succsim \in \mathscr{P}$. By Lemma 3, $\succsim^*$ generates h and by Lemma 1, Theorem 2 and Theorem 3, $\succsim^*$ is the *unique* relation *in* $\mathscr{P}$ which generates h. From the definitions, $x \succsim^* y$ implies $x \succsim y$; by the completeness of $\succsim$ and $\succsim^*$, $x \succ y$ implies $x \succsim^* y$. Hence, by the monotonicity of $\succsim$, $\succsim$ is the closure of $\succsim^*$; but $\succsim^*$ is closed, so $\succsim = \succsim^*$. Hence $\succsim \in \mathscr{P}$ is the unique element of $\mathscr{P}$ which generates h. Therefore, since $\{\succsim_n\}_{n=1}^{\infty}$ has a unique accumulation point, $\succsim_n \to \succsim$. ||

APPENDIX

The purpose of this appendix is to show how the example in Hurwicz and Richter [12] can be easily modified in order to obtain an income Lipschitzian and continuous demand function h satisfying the SARP and $h(V) = P$ but which cannot be generated by any continuous preference relation.

In Figure 2 the curve I is the graph of:

$$f(x) = \begin{cases} ((x-1)^2/2+1, & 0 \leqq x \leqq 1 \\ 1, & 1 \leqq x. \end{cases}$$

In region A (above I) the preference map of the Hurwicz-Richter example is drawn. The Engel curves are indicated. The preference relation is (so far) u.s.c., but it will necessarily fail to be continuous along I from v to the right (line I′).

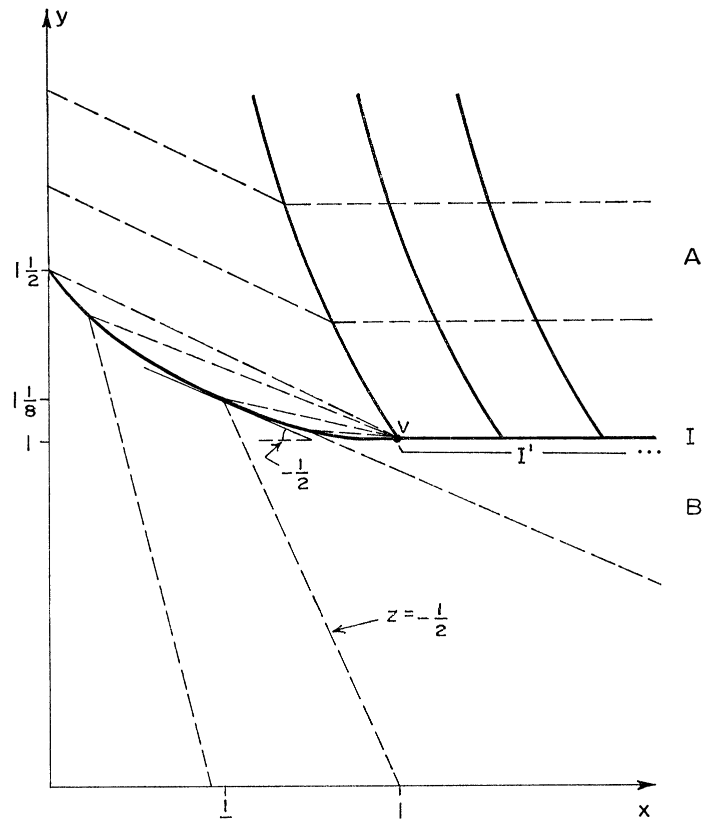

FIGURE 2

In region B (below I) we want to find an indifference map which associates with every $z \in (-1, 0)$ the Engel curve determined by the segment connecting $(z+1, (z^2/2)+1)$ and $(-(z+1)/z, 0)$. Two distinct segments of those do not intersect, and as z ranges over $(-1, 0)$ the segments cover the whole of B. Hence a function ϕ: B$\to(-1, 0)$ is well defined. Since the slope of $f(x)$ at $x = z+1$ is z, this should be the indifference-curve-slope associated with the Engel curve passing through $(z+1, (z^2/2)+1)$. We can write $-\dfrac{dy}{dx} = \phi(x, y)$.

If the preference map on A can be so extended to an u.s.c. preference relation $\succsim$ it is easily checked that the generated demand function h is income Lipschitzian (the Engel curves are piecewise linear and, when defined, the gradient vector of h at (p, w) is continuous with respect to p and w), that $\succsim$ is the only u.s.c. preference relation generating h and that $h(V) = P$.

Now, $\phi(x, y)$ can be expressed as the unique (and regular) solution in $(-1, 0)$ of a fourth-degree equation.[7] Hence it is C^1, and it can be integrated locally. Moreover, since $\phi(x, y) < 0$ for all $x, y \in B$ the solution can be uniquely defined in the large (see Debreu [4]). It remains to show that the indifference curves define a strictly convex preference relation. This is clear from Figure 2; otherwise, some indifference curve would have to be tangent to an Engel curve, which is obviously not the case.

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NOTES

1. Usual notational conventions will be followed; $x \gg y$ means $x^i > y^i$ for all i, $x \geqq y$ means $x^i \geqq y^i$ for all i; $x > y$ means $x \geqq y$ but no $x = y$.

2. We shall use the euclidean norm in R^l throughout; the definitions, however, are independent of the particular *norm* used.

3. This requires a fixed-point argument. Let $x \in P$ and define the excess demand function $h(p, px) - x$; it satisfies the boundary condition in Debreu [3], and so, by his Theorem 2, there is a $\bar{p}$ with $h(\bar{p}, \bar{p}x) = x$.

4. It is worth while to remark that even if the " true " preferences underlying h are continuous, Afriat's sequences of preferences may fail to converge if h is not income Lipschitzian. The demand function generated by the preferences of Example 1 in [15] provides an instance.

5. The example in the Appendix and Theorems 2 and 2′ settles a question left open by Uzawa [23, p. 19]. The example shows that it may *not* be possible to get a continuous $\succsim$ if, while maintaining $h(V) = P$, the assumption that $h^{-1}(x)$ is a single ray for every $x \in P$ is dropped. However, a continuous $\succsim$ is obtained if either $h(V) = P$ is strengthened to the boundary condition (Theorem 2) or the income Lipschitz hypothesis is replaced by non-inferiority (Theorem 2′).

Recently, Gordon [7] asserted the existence of a continuous, generating $\succsim$ under the assumptions that h is continuous, $h(V) = P$, and h fulfils the SARP and an income Lipschitz condition weaker than the one in our Definition 3. However, as the example in the Appendix shows, this may not be possible even with the Lipschitz condition of Definition 3. The proof of his Proposition 3 seems to be seriously flawed.

There is an inconsequential difference between the set-up here and in Uzawa [22]: In order to avoid inessential complications we have taken our consumption set to be P rather than $\bar{P}$.

6. So, referring to a question raised by Stigum [21, Remark 6, p. 413] the income Lipschitz condition does yield further restrictions on utility (i.e. preferences) besides continuity. They, however, do not make the income Lipschitz condition a necessary one. Lipschitzian preferences may yield non-income Lipschitzian demands (Example 3 in [15]).

7. To every $z \in (-1, 0)$ there is associated the linear function

$$y = (x + (z+1)/z) \frac{z((z^2/2)+1)}{(z+1)^2}, \quad z+1 \leqq x \leqq -(z+1)/z,$$

which, for given $x > 0$, $y \leqq f(x)$, generates a fourth-degree equation in z.

REFERENCES

[1] Afriat, S. " The Construction of Utility Functions from Expenditure Data ", *International Economic Review*, **8** (1967).

[2] Chipman, J. S., Hurwicz, L., Richter, M. and Sonnenschein, H. *Preferences, Utility, and Demand* (New York: Harcourt, Brace, Jovanovich, Inc., 1971).

[3] Debreu, G. " Economies with a Finite Set of Equilibria ", *Econometrica*, **38** (1970), 387-392.

[4] Debreu, G. " Smooth Preferences ", *Econometrica*, **40** (1972), 603-615.

[5] Diewert, W. E. " Afriat and Revealed Preference Theory ", *Review of Economic Studies*, **40** (1973), 419-425.

[6] Gale, D. " A Note on Revealed Preference ", *Economica*, **27** (1960), 348-354.

[7] Gordon, E. " Uzawa's Preference Axioms ", *Review of Economic Studies*, **38** (1971), 319-331.

[8] Grodal, B. " A Note on the Space of Preference Relations ", *Journal of Mathematical Economics*, **1, 3** (1974), 279-295.

[9] Hildenbrand, W. *Core and Equilibria of a Large Economy* (Princeton, N.J.: Princeton University Press, 1974).

[10] Houthakker, H. " Revealed Preference and the Utility Function ", *Economica*, **NS 17** (1950), 159-174.

[11] Hurwicz, L. " On the Problem of Integrability of Demand Functions ", Ch. 9 in [2] (1971).

[12] Hurwicz, L. and Richter, M. " Revealed Preference without Demand Continuity Assumptions ", Ch. 3 in [2] (1971).

[13] Hurwicz, L. and Uzawa, H. " On the Integrability of Demand Functions ", Ch. 6 in [2] (1971).

[14] Kihlstrom, R., Mas-Colell, A. and Sonnenschein, H. " The Demand Theory of the Weak Axiom of Revealed Preference ", *Econometrica*, **43** (1975).
[15] Mas-Colell, A. " On the Recoverability of Consumer's Preferences from Market Demand Behavior ", Working Paper IP-184 (Center for Research in Management Science, University of California, Berkeley, 1973); to appear in *Econometrica* **44** (1976).
[16] v. Moeseke, P. " Revealed Preference: Equivalence Theorem and Induced Preorder ", in *Economic Models, Estimation and Risk Programming* (Fox, K., Sengupta, J., Narasimhan, G. *eds.*) (New York: Springer-Verlag, 1969), 92-110.
[17] Nikaido, H. *Convex Structures and Economic Theory* (Academic Press, New York, 1968).
[18] Richter, M. " Revealed Preference Theory ", *Econometrica*, **34** (1966), 635-645.
[19] Samuelson, P. " A Note on the Pure Theory of Consumer's Behavior ", *Economica*, **NS 5** (1938), 61-71.
[20] Samuelson, P. " Consumption Theory in Terms of Revealed Preference ", *Economica*, **NS 15** (1948), 243-253.
[21] Stigum, B. " Revealed Preference: A Proof of Houthakker's Theorem ", *Econometrica*, **41** (1973), 411-425.
[22] Uzawa, H. " Preference and Rational Choice in the Theory of Consumption ", in *Mathematical Methods in the Social Sciences* (Arrow, K., Karlin, S. and Suppes, P. ed.) (California: Stanford University Press, Stanford, 1959), 129-148.
[23] Uzawa, H. " Preference and Rational Choice in the Theory of Consumption "; revised version of [22], Chapter 1 in [2] (1974).

LINKED CITATIONS

On Revealed Preference Analysis
Andreu Mas-Colell
The Review of Economic Studies, Vol. 45, No. 1. (Feb., 1978), pp. 121-131.
Stable URL:
http://links.jstor.org/sici?sici=0034-6527%28197802%2945%3A1%3C121%3AORPA%3E2.0.CO%3B2-N

Notes

[3] **Economies with a Finite Set of Equilibria**
Gerard Debreu
Econometrica, Vol. 38, No. 3. (May, 1970), pp. 387-392.
Stable URL:
http://links.jstor.org/sici?sici=0012-9682%28197005%2938%3A3%3C387%3AEWAFSO%3E2.0.CO%3B2-1

[5] **Uzawa's Preference Axioms**
E. Gordon
The Review of Economic Studies, Vol. 38, No. 3. (Jul., 1971), pp. 319-329.
Stable URL:
http://links.jstor.org/sici?sici=0034-6527%28197107%2938%3A3%3C319%3AUPA%3E2.0.CO%3B2-5

[6] **Revealed Preference--A Proof of Houthakker's Theorem**
Bernt P. Stigum
Econometrica, Vol. 41, No. 3. (May, 1973), pp. 411-423.
Stable URL:
http://links.jstor.org/sici?sici=0012-9682%28197305%2941%3A3%3C411%3ARPPOHT%3E2.0.CO%3B2-L

References

LINKED CITATIONS

[1] **The Construction of Utility Functions from Expenditure Data**
S. N. Afriat
International Economic Review, Vol. 8, No. 1. (Feb., 1967), pp. 67-77.
Stable URL:
http://links.jstor.org/sici?sici=0020-6598%28196702%298%3A1%3C67%3ATCOUFF%3E2.0.CO%3B2-K

[3] **Economies with a Finite Set of Equilibria**
Gerard Debreu
Econometrica, Vol. 38, No. 3. (May, 1970), pp. 387-392.
Stable URL:
http://links.jstor.org/sici?sici=0012-9682%28197005%2938%3A3%3C387%3AEWAFSO%3E2.0.CO%3B2-1

[4] **Smooth Preferences**
Gerard Debreu
Econometrica, Vol. 40, No. 4. (Jul., 1972), pp. 603-615.
Stable URL:
http://links.jstor.org/sici?sici=0012-9682%28197207%2940%3A4%3C603%3ASP%3E2.0.CO%3B2-W

[5] **Afriat and Revealed Preference Theory**
W. E. Diewert
The Review of Economic Studies, Vol. 40, No. 3. (Jul., 1973), pp. 419-425.
Stable URL:
http://links.jstor.org/sici?sici=0034-6527%28197307%2940%3A3%3C419%3AAARPT%3E2.0.CO%3B2-9

[6] **A Note on Revealed Preference**
David Gale
Economica, New Series, Vol. 27, No. 108. (Nov., 1960), pp. 348-354.
Stable URL:
http://links.jstor.org/sici?sici=0013-0427%28196011%292%3A27%3A108%3C348%3AANORP%3E2.0.CO%3B2-7

[7] **Uzawa's Preference Axioms**
E. Gordon
The Review of Economic Studies, Vol. 38, No. 3. (Jul., 1971), pp. 319-329.
Stable URL:
http://links.jstor.org/sici?sici=0034-6527%28197107%2938%3A3%3C319%3AUPA%3E2.0.CO%3B2-5

NOTE: *The reference numbering from the original has been maintained in this citation list.*

LINKED CITATIONS

[10] **Revealed Preference and the Utility Function**
H. S. Houthakker
Economica, New Series, Vol. 17, No. 66. (May, 1950), pp. 159-174.
Stable URL:
http://links.jstor.org/sici?sici=0013-0427%28195005%292%3A17%3A66%3C159%3ARPATUF%3E2.0.CO%3B2-J

[18] **Revealed Preference Theory**
Marcel K. Richter
Econometrica, Vol. 34, No. 3. (Jul., 1966), pp. 635-645.
Stable URL:
http://links.jstor.org/sici?sici=0012-9682%28196607%2934%3A3%3C635%3ARPT%3E2.0.CO%3B2-G

[19] **A Note on the Pure Theory of Consumer's Behaviour**
P. A. Samuelson
Economica, New Series, Vol. 5, No. 17. (Feb., 1938), pp. 61-71.
Stable URL:
http://links.jstor.org/sici?sici=0013-0427%28193802%292%3A5%3A17%3C61%3AANOTPT%3E2.0.CO%3B2-H

[20] **Consumption Theory in Terms of Revealed Preference**
Paul A. Samuelson
Economica, New Series, Vol. 15, No. 60. (Nov., 1948), pp. 243-253.
Stable URL:
http://links.jstor.org/sici?sici=0013-0427%28194811%292%3A15%3A60%3C243%3ACTITOR%3E2.0.CO%3B2-0

[21] **Revealed Preference--A Proof of Houthakker's Theorem**
Bernt P. Stigum
Econometrica, Vol. 41, No. 3. (May, 1973), pp. 411-423.
Stable URL:
http://links.jstor.org/sici?sici=0012-9682%28197305%2941%3A3%3C411%3ARPPOHT%3E2.0.CO%3B2-L

NOTE: *The reference numbering from the original has been maintained in this citation list.*