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A Selection Theorem for Open Graph Correspondences with Star-shaped Values

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A *correspondence* is a point-to-set map $\phi: X \to Y$ such that $\phi(x) \neq \varnothing$ for all $x \in X$.

A set $A \subset R^n$ is *star-shaped* if there is a $x \in A$ such that, for all $y \in A$ and $t \in [0, 1]$, $ty + (1 - t)\, x \in A$. It is easy to verify that an open, star-shaped set is homeomorphically convex.

It is the purpose of this note to give a proof of the following

THEOREM 1. *Let $K \subset R^m$ be, up to homeomorphism, a compact simplicial complex and $\phi: K \to R^n$ a correspondence such that:*

(i) $\{(x, z) \in R^n \times R^m : z \in \phi(x)\}$ *is open (open graph),*

(ii) *for every $x \in K$, $\phi(x)$ is non-empty and homeomorphically convex,*

then ϕ has a continuous selection, i.e. there is a continuous function $f: K \to R^m$ such that $f(x) \in \phi(x)$ for all $x \in K$.

As an application:

THEOREM 2. *Let $K \subset R^m$ be a non-empty, compact, convex set and $\phi: K \to K$ an open graph correspondence with homeomorphically convex values. Then ϕ has a fixed point, i.e. $x \in \phi(x)$ for some $x \in K$.*

For a proof combine Theorem 1 with Brouwer Fixed Point Theorem. In the convex valued case (an infinite dimensional version of) Theorem 2 has been proved by Ky Fan (2) via the Knaster–Kuratowsky–Mazurkiewicz lemma. The need to investigate the fixed point theory of open graph correspondences has recently been felt in mathematical economics (see, for an example, Gale and Mas-Colell (1)). A natural approach is to reduce it to the continuous functions theory via selection theorems. This is easily accomplished in the convex valued case and Theorem 1 takes care of the homeomorphically convex one. It would be of interest to determine if homeomorphically convex could be weakened to contractible in the statement of Theorem 1 since as a corollary this would yield the open-graph analog of the Eilenberg–Montgomery fixed point theorem. We

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want to emphasize that the star-shaped case is an interesting one; in applications it is not uncommon to encounter open graph mappings ϕ which values $\phi(x)$ are finite unions of open convex sets with non-empty intersection, a particular instance of star-shaped sets (see Gale and Mas-Colell (1), fixed point theorem on p. 10).

A correspondence $\psi: X \to Y$, X, Y topological spaces, is *upper semicontinuous* if every open $U \subset Y$, $\{x \in X: \phi(x) \subset U\}$ is open.

The Selection Theorem follows from the two propositions:

PROPOSITION 1. *Let $K \subset R^m$ be a compact simplicial complex, $\psi: K \to R^n$ an upper semicontinuous, contractible valued correspondence, and $\phi: K \to R^n$ and open graph correspondence such that $\psi(x) \subset \phi(x)$ for all $x \in K$. Then there is a continuous function $f: K \to R^m$ such that $f(x) \in \phi(x)$ for all $x \in K$.*

PROPOSITION 2. *Let $K \subset R^m$ be a compact simplicial complex and $\phi: K \to R^n$ an open graph correspondence such that for all $x \in K$ $\phi(x)$ is homeomorphic to a star-shaped set. Then there is an upper semicontinuous contractible valued correspondence $\psi: K \to R^n$ such that $\psi(x) \subset \phi(x)$ for all $x \in K$.*

Proposition 1 was proved in Mas–Colell [3] and there is no point in repeating the proof here. We proceed to demonstrate Proposition 2.

It is obvious that if $\phi: K \to R^n$ satisfies the hypothesis of Proposition 2 then there is an upper semicontinuous correspondence $\psi: K \to R^n$ such that $\psi(x) \subset \phi(x)$ for all $x \in K$. Therefore Proposition 2 is implied by

(I) Let $K \subset R^m$ and $\phi: K \to R^n$ be as in Proposition 2. Let $\psi: K \to R^n$ be an upper semicontinuous correspondence such that $\psi(x) \subset \phi(x)$ for all $x \in K$. Then there is an upper semicontinuous contractible valued correspondence $\Phi: K \to R^n$ such that $\psi(x) \subset \Phi(x) \subset \phi(x)$ for all $x \in K$.

Statement (I) shall be proved by induction on the dimension of the simplicial complex K. Clearly (I) holds if dimension $K = -1$ (i.e. if K is empty). Suppose that (I) holds for dimension $K \leqslant m - 1$.

Let K, ψ, ϕ satisfy the hypothesis of (I).

For $\epsilon > 0$ let $B_\epsilon \subset R^n$ be the open ϵ-ball. For every $x \in K$ let ϵ_x be such that $\overline{\psi(x) + B_{\epsilon_x}} \subset \phi(x)$. We assert that there is a compact, contractible set $A_x \subset R^n$ such that $\psi(x) + B_{\epsilon_x} \subset A_x \subset \phi(x)$. Indeed, let $\eta_x: \phi(x) \to R^n$ be a homeomorphsim between $\phi(x)$ and a star-shaped set $\eta_x(\phi(x))$; if $z \in \eta_x(\phi(x))$ is an admissible center of the star let $F: \eta_x(\phi(x)) \times [0, 1] \to \eta_x(\phi(x))$ be the map $F(y, t) = ty + (1 - t) z$. Note that for any set $C \subset \eta_x(\phi(x))$, $F(C \times [0, 1])$ is contractible and $C \subset F(C \times [0, 1])$. Hence, let $A_x = \eta_x^{-1}(F(\eta_x(\overline{\psi(x) + B_{\epsilon_x}}) \times [0, 1]))$.

For every $x \in K$ let $\delta_x > 0$ be such that:

(i) $\psi(B_{\delta_x}(x)) \subset \psi(x) + B_{\epsilon_x}$,

(ii) if $x' \in B_{\delta_x}(x)$ then $A_x \subset \phi(x')$.

This δ_x exists because ψ is upper semicontinuous, ϕ has an open graph and A_x is compact.

Let $\mathscr{B} = \{B_{\delta_{x_1}}, \ldots, B_{\delta_{x_H}}\}$ be such that $K = \bigcup_{j=1}^{H} B_{\delta_{x_j}}$ and take a triangulation K' of K subordinated to the covering $\mathscr{B}$ (i.e. every simplex of K' is a subset of a set in $\mathscr{B}$).

Consider any full dimensional simplex Δ of the triangulation K'. Pick any $1 \leqslant j \leqslant H$ such that $\Delta \subset B_{\delta_{x_j}}$. Define then the correspondence $\Phi_\Delta : \Delta \to R^n$ by $\Phi_\Delta(y) = A_{x_j}$. We have, for all $y \in \Delta$:

(i) $\psi(y) \subset \Phi_\Delta(y)$ because $\psi(y) \subset \psi(x_j) + B_{\epsilon x_j}$,

(ii) $\Phi_\Delta(y)$ is contractible,

(iii) $\Phi_\Delta(y) \subset \phi(y)$ because $y \in B_{\delta_{x_j}}(x_j)$.

Define then the correspondence $\bar{\Phi}: K \to R^n$ by $\bar{\Phi}(y) = \{z \in R^n$: for some simplex Δ of the triangulation K' containing y, $z \in \Phi_\Delta(y)\}$. The correspondence $\bar{\Phi}: K \to R^n$ is upper semicontinuous and $\bar{\Phi}(y) \subset \phi(y)$ for all $y \in K$. Further, if $H \subset K$ denotes the $m - 1$ skeleton of the triangulation K', $\bar{\Phi}$ is contractible valued on $K \backslash H$. By the induction hypothesis there is an upper semicontinuous, contractible correspondence $\bar{\bar{\Phi}}: H \to R^n$ such that, for all $y \in H$, $\bar{\Phi}(y) \subset \bar{\bar{\Phi}}(y) \subset \phi(y)$. Define $\Phi: K \to R^n$ by

$$\begin{aligned} \Phi(y) &= \bar{\Phi}(y) \quad &\text{if} \quad & y \in K \backslash H \\ &= \bar{\bar{\Phi}}(y) \quad &\text{if} \quad & y \in H. \end{aligned}$$

This Φ satisfies the conclusion of Proposition 2.

References

1. D. Gale and A. Mas-Colell, An equilibrium existence theorem for a general model without ordered preferences, *J. Math. Econ.* **2**, 1 (1975), 9–15.
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