Two Propositions on the Global Univalence of Systems of Cost Function

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I. INTRODUCTION AND STATEMENT OF RESULTS

1. *Preliminaries and Definitions*

Consider a C^1 function $F\colon Q \to R^l_+$ where Q is either R^l_{++} or $R^l_+\backslash\{0\}$. We interpret F as a system of unit cost functions, the arguments being input prices. We assume that F is *linear homogeneous* [i.e., $F(\lambda w) = \lambda F(w)$ for all $\lambda \geq 0$] and call the Jacobian matrix of F at w, denoted $JF(w)$, the *input demand matrix*. If defined, the elasticity matrix $SF(w)$, i.e., the matrix with generic entry

$$s_{ij} = \frac{w}{F^i(w)} \frac{\partial F^i(w)}{\partial w^j},$$

is called the *input share matrix*.

We note that the interpretation of F as a system of cost functions demands that every component function be concave, but formally we will not impose this requirement.

2. *The Problem*

The factor price equalization problem in the pure theory of international trade (see Samuelson [11, 12], McKenzie [7], and Chipman [2]) prompted in an incidental way the following mathematical question: *Under which hypothesis has the equation $F(w) = v$ at most one solution for all $v \in R^l_+$, i.e., under which hypothesis is F globally univalent?*

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With some disrespect for chronology and completeness, the rather complicated history of the analytical treatments of the univalence problem in the economic literature can be summarized thus. Samuelson [11] posed the question and showed that for $l = 2$ the everywhere nonsingularity of the input demand matrix (i.e., the Jacobian matrix) of F was a sufficient condition for univalence. Any hope that nonsingularity could suffice in the general case (Pierce [10]) was quickly dashed by an example of McKenzie [8]). One suggestion of Samuelson [12] involving the positive signedness of principal minors of the Jacobian of F was amended and finally led to the Gale–Nikaido theorem [4] of relevance for general, not just cost, functions. Sticking to cost functions, Samuelson [13, p. 908] suggested that if there existed a nested sequence of principal minor matrices of the input share matrices $SF(w)$ with determinants bounded away from zero uniformly on R^l_{++}, then F would be globally univalent on R^l_{++}. This was later established by Nikaido [9].

In this chapter we reexamine the global univalence problem for cost functions and provide improvements in the following two directions:

(1) In the Samuelson–Nikaido theorem the restrictions on the principal minors of the input share matrix are irrelevant; all that matters is that the determinant of $SF(w)$ be uniformly bounded away from zero (Proposition 1). This provides a global univalence theorem for systems of cost functions with domain R^l_{++}.

(2) In the context of cost functions with domain R^l_+, the Gale–Nikaido conditions can be substantially weakened (Proposition 2). The weakened conditions are of the same nature that the ones obtained in Mas-Colell [8] for general C^1 functions defined on compact polyhedra.

3. *Conditions on the Input Share Determinant*

PROPOSITION 1 *Let $F: R^l_{++} \to R^l_{++}$ be C^1. If for some $\varepsilon > 0$ the absolute value of $|SF(w)|$ is larger than ε for all $w \in R^l_{++}$, then F is a homeomorphism onto.*

That is, for all $v \in R^l_{++}$ the equation $F(w) = v$ has a unique solution which in addition depends continuously on v.

Remark 1 Of course, $|SF(w)| \neq 0$ if and only if $|JF(w)| \neq 0$. Therefore McKenzie's example [7] proves that we cannot take $\varepsilon = 0$ in Proposition 1.

Remark 2 Proposition 1 generalizes the Samuelson–Nikaido theorem [9, 13] by dropping all the conditions on minors.

The strong conclusion of Proposition 2 is also its weakness. It yields not only that F is globally univalent but also that $F(R^l_{++}) = R^l_{++}$. This means that the ε restriction on the input share matrix determinant imposes stringent restrictions on the form of the isocost surfaces near the boundary of R^l_{++}.

For example, for no $1 \leq i \leq l$ can an isocost surface be bounded. See Fig. 1, where $(2, 3) \notin F(R^2_{++})$. Essentially, isocost surfaces must be as in Fig. 2.

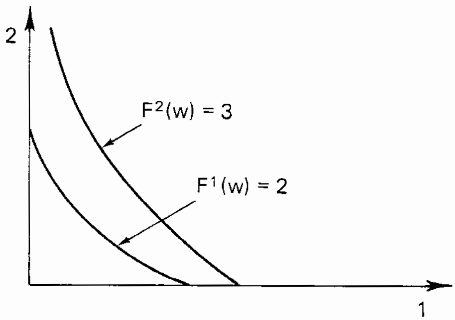

FIGURE 1

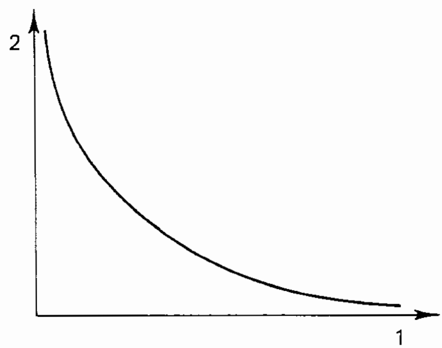

FIGURE 2

Summing up: Proposition 1 can be applied only if isocost surfaces are unbounded. This is an unduly strong requirement.

If isocost surfaces are bounded, then it is natural to assume that F is defined in R^l_{+}, (see Fig. 3). We can then appeal to the global univalence theorems based on weakened Gale–Nikaido conditions to be discussed in the next section.

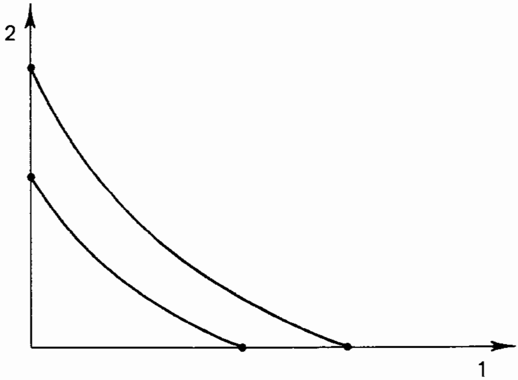

FIGURE 3

4. *Generalized Gale–Nikaido Conditions*

Let the system of cost functions be defined on R^l_+, i.e., $F: R^l_+ \to R^l_+$.

For every subset $N \subset \{1, \ldots, l\}$ of variables, we can define a reduced system of cost functions $F_N: R^{l-n}_+ \to R^{l-n}_+$, where $n = |N|$, by indentifying R^{l-n}_+ with the inputs with index not in N, letting the inputs with index in N be free, i.e., $w_i = 0$ for $i \in N$, and disregarding the cost of variables with index in N. For example, if $l = 3$ and $N = \{3\}$, then $F_N: R^2_+ \to R^2_+$ is defined by $F^i_N(w^1, w^2) = F^i(w^1, w^2, 0)$ for $i = 1, 2$.

Note that for any $w \in R^{l-n}_+ \backslash \{0\}$, if we let $\hat{w}^i = 0$ for $i \in N$ and $\hat{w}^i = w^i$ otherwise, then $|JF_N(w)|$ is nothing but the principal minor determinant of $JF(\hat{w})$ obtained by deleting the rows and columns with indices in N.

PROPOSITION 2 *Let $F: R^l_+ \to R^l_+$ be C^1 on $R^l_+ \backslash \{0\}$. Suppose that for all $N \subset \{1, \ldots, l\}$, $|JF_N(w)|$ is positive for every $w \in R^{l-n}_+ \backslash \{0\}$. Then F is a homeomorphism.*

Remark 3 It is clear that Proposition 2 generalizes the Gale–Nikaido (GN) theorem as applied to systems of homogeneous functions on R^l_+. In particular we may note: (i) only infinitesimal conditions are imposed (as in the GN theorem); (ii) for $w \gg 0$ the only condition on $JF(w)$ is that $|JF(w)| > 0$. For an analogous generalization of the GN theorem for the case of arbitary C^1 functions on a compact convex polyhedron, see Mas-Colell [6].

Remark 4 One could perhaps wonder if Proposition 2 remains valid with the word "positive" replaced by "negative." It does, but in a vacuous manner. No system of homogenous cost functions $F: R^l_+ \to R^l_+$ can satisfy the negative version of the condition of the theorem.

II. PROOFS

1. *Proof of Proposition 1*

The euclidean norm in R^l is $\| \ \|$. Given a linear map $T: R^l \to R^l$, its norm is $\|T\| = \max_{\|x\|=1} \|Tx\|$.

Proposition 1 is a consequence of the following theorem, which goes back to Hadamard [5]. It is taken from Berger [1, p. 222], where it is proved for maps on general Banach spaces.

THEOREM 1 *Let $G: R^l \to R^l$ be C^1. Suppose there is a number k such that, for all $x \in R^l$, $JG(x)$ is nonsingular and $\|[JG(x)]^{-1}\| \leq k$. Then G is a homeomorphism onto.*

It follows from Cramer's rule that sufficient conditions for the uniform boundedness above of the norm of the inverse of $JG(x)$ are (i) $|JG(x)|$ is bounded away from zero uniformly; (ii) all the entries of the matrix $JG(x)$ are uniformly bounded in absolute value.

Let $F: R^l_{++} \to R^l_{++}$ satisfy the hypothesis of Proposition 1. Define then $G: R^l \to R^l$ by letting $G^i(x) = \ln F^i(e^{x^1}, \ldots, e^{x^l})$. At any $x \in R^l_{++}$ we have $JG(x) = SF(w(x))$ where $w(x) = (e^{x^1}, \ldots, e^{x^l})$. Because of the homogeneity of F, all the entries of $S(w(x))$ are less than or equal to one. Also, by hypothesis, the absolute value of $|S(w(x))| > \varepsilon > 0$. Therefore, by the Theorem, $G: R^l \to R^l$ is a homeomorphism onto and consequently so is $F: R^l_{++} \to R^l_{++}$.

Remark 5 Let $G: R^l \to R^l$ satisfy the hypothesis of the Theorem. Take any $y \in R^l$; then we know that $G(x) = y$ has a unique solution $\bar{x}$. In fact, $\bar{x} = x(1)$ where for some arbitrary x_0, $x(t)$ is the solution to the differential equation $\dot{x} = [JG(x)]^{-1}\,(y - G(x_0))$ with initial condition $x(0) = x_0$.

2. *Proof of Proposition 2*

Although the content of Proposition 2 is analogous to the one in Mas-Colell [6], we will take a different line of proof. While the demonstration in [6] was based on index-theoretic arguments, we will rely here on the following well-known topological fact (see, for example, Berger [1, p. 221]):

THEOREM 2 *Let U be homeomorphic to R^l and $G: U \to U$ be given. Then G is a homeomorphism onto if and only if*

(i) *G is proper (i.e., if $K \in U$ is compact, then $G^{-1}(K)$ is compact).*

(ii) *G is a local homeomorphism (i.e., for every $x \in U$, G is a homeomorphism on some neighborhood of x).*

Let us assume we have given a system of cost functions $F: R^l_+ \to R^l_+$ satisfying the hypothesis of Proposition 2.

Because of the homogeneity hypothesis, F is a homeomorphism of R^l_+ if and only if it is a homeomorphism of the unit simplex. Hence F is a homeomorphism if and only if it is one to one.

Note that if F is a homeomorphism of $R^l_+ \backslash \{0\}$, then it is a homeomorphism of R_+ because if $F(w) = 0$ and $w \neq 0$, then $F(\frac{1}{2}w) = 0$ by homogeneity; hence F would not be one to one on $R^l_+ \backslash \{0\}$.

For any $x \in R^l$ let $\Pi(x)$ be the point of R^l_+ closest to x in the euclidean norm. More explicitly, $\Pi^i(x) = \max\{x^i, 0\}$.

Define a map $G: R^l \backslash R^l_- \to R^l$ by $G(x) = F(\Pi(x)) + x - \Pi(x)$. Of course $G | R^l_+ \backslash \{0\} = F$. So it suffices to show that G is a homeomorphism. The region $R^l \backslash R^l_-$ is homeomorphic to R^l. Hence by Theorem 2, our task will be accomplished if we show (i) $G(R^l \backslash R^l_-) \subset R^l \backslash R^l_-$, (ii) $G: R^l \backslash R^l_- \to R^l \backslash R^l_-$ is proper, and (iii) G is a local homeomorphism.

(i) $G(R^l \backslash R^l_-) \subset R^l \backslash R^l_-$.

LEMMA 1 Let $F: R^m_+ \to R^m_+$ be a system of homogeneous functions. Suppose that for all $x \in R^m_+ \backslash \{0\}$, $JF(x)$ is nonsingular. Then $\|x_n\| \to \infty$ implies $\|F(x_n)\| \to \infty$.

Proof Suppose there is $\|x_n\| \to \infty$ such that $\|F(x_n)\| \leq k$ for some k. Then $\|F((1/\|x_n\|)x_n)\|$ $(k/\|x_n\|)$. Without loss of generality we can assume $(1/\|x_n\|)x_n \to x$. By continuity $\|x\| = 1$ and also $\|F(x)\| = 0$, which implies $F(x) = 0$. By hypothesis $JF(x)$ is nonsingular. By the homogeneity of F we have (using Euler's rule) that $JF(x)x = 0$. So $x = 0$, which contradicts $\|x\| = 1$. Therefore no such sequence x_n can exist, and the lemma is proved. ■

Given $\bar{x} \in R^l \backslash R^l_-$ by relabeling variables if necessary, we can assume that $\bar{x}^i > 0$ for $i \leq k$ and $\bar{x}^i \leq 0$ for $i > k$. Let $N = \{k+1, \ldots, l\}$ and Π^* be the linear operator $\Pi^*(x) = (x^1, \ldots, x^k, 0, \ldots, 0)$. Note that $\Pi(\bar{x}) = \Pi^*(\bar{x})$ and that $\Pi^* \circ F(\Pi^*(x))$, if defined, is naturally identified with $F_N(x^1, \ldots, x^k)$. Henceforth, since $\bar{x}^1 > 0$, it follows from Lemma 1 and the homogeneity of F_N that $F_N(\bar{x}^1, \ldots, \bar{x}^k) \neq 0$. On the other hand, suppose that $G(\bar{x}) \leq 0$. Then $\Pi^* \circ G(\bar{x}) \leq 0$. However, $\Pi^* \circ G(\bar{x}) = \Pi^* \circ F(\Pi^*(\bar{x}))$. So $\Pi^* \circ F(\Pi^*(\bar{x})) \leq 0$. But $F(\Pi^*(\bar{x})) \geq 0$ by hypothesis (this is the only point in the proof where the assumption $F(R^l_+) \subset R^l_+$ is appealed to). So $\Pi^* \circ F(\Pi^*(\bar{x})) = 0$. Contradiction.

(ii) $G: R^l \backslash R^l_- \to R^l \backslash R^l_-$ *is proper*. To verify that $G: R^l \backslash R^l_- \to R^l \backslash R^l_-$ is proper, it suffices to check that given a sequence $x_n \in R^l \backslash R^l_-$, one has

(a) if $\|x_n\| \to \infty$, then $\|G(x_n)\| \to \infty$;
(b) if $\max_i x^i_n \to 0$, then $\max_i G^i(x_n) \to 0$.

That (b) holds is obvious; if $\max_i x_n^i \to 0$, then $\Pi(x_n) \to 0$, and by continuity, $F(\Pi(x_n)) \to 0$. So $\max_i G^i(x) = \max_i (F^i(\Pi(x_n)) + x_n^i - \Pi^i(x_n)) \to 0$.

Since there are only a finite number of orthants in R^l, it suffices to prove (a) for a sequence x_n all of which terms belongs to the same orthant. [Indeed, if (a) is not true, then $G(x_n)$ has a bounded subsequence. But any subsequence of x_n has a subsequence all of which terms are in the same orthant. Contradiction.] So, by relabeling variables if necessary, we can assume that for a, $1 \le k \le l$ and all n, $x_n^i \ge 0$ if $i \le k$, and $x_n^i \le 0$ for $i > k$.

Let $N = \{k+1, \ldots, l\}$ and Π^* be the linear operator

$$\Pi^*(x) = (x^1, \ldots, x^k, 0, \ldots, 0).$$

For any x_n, $\Pi(x_n) = \Pi^*(x_n)$. Note, also, that $\Pi^* \circ F(\Pi^*(x_n))$ is naturally identified with $F_N(x_n^1, \ldots, x_n^k)$. In particular, by Lemma 1, if $\{\Pi^*(x_n)\}$ is unbounded, $\{\Pi^* \circ F(\Pi^*(x_n))\}$ must be unbounded.

Letting I be the identity map, we have $G(x_n) = \Pi^* \circ F(\Pi^*(x_n)) + (I - \Pi^*)(F(\Pi^*(x_n)) + x_n)$. So

$$\|G(x_n)\| = \|\Pi^* \circ F(\Pi^*(x_n))\| + \|(I - \Pi^*)(F(\Pi^*(x_n)) + x_n)\|.$$

Either $\{\|\Pi^*(x_n)\|\}$ or $\{\|(I - \Pi^*)(x_n)\|\}$ must be unbounded. If $\{\|\Pi^*(x_n)\|\}$ is bounded, then $\{\|(I - \Pi^*)(x_n)\|\}$ is unbounded, and therefore so is the second term of the previous sum. If $\{\|\Pi^*(x_n)\|\}$ is unbounded, then so is the first term of the sum by the observation of the preceding paragraph. In either case $\{\|G(x_n)\|\}$ is unbounded. Since x_n is arbitrary, this yields (b).

(iii) *G is a local homeomorphism.* To establish that G is a local homeomorphism, we will make use of the following theorem due to Clarke [3]:

Suppose that $G: Q \to Q$, $Q \subset R^l$ open, is a Lipschitzian function. Denote by $U \subset Q$ the region of differentiability of G, i.e., $U = \{x \in Q : JG(x) \text{ exists}\}$. For any $x \in Q$ define $\hat{J}G(x) = \{A : JG(x_n) \to A \text{ for some } x_n \to x,\ x_n \in U\}$. If every $A \in$ closed convex hull $\hat{J}G(x)$ is nonsingular, then G is a local homeomorphism at x.

Our function $G: R^l \backslash R^l_- \to R^l \backslash R^l_-$ is piecewise continuously differentiable, hence Lipschitzian. Also, G is C^1 at any $x \in R^l \backslash R^l_-$ such that $x^i \ne 0$ for all i. Henceforth, by Clarke's theorem, we must verify that, given $A \in$ convex hull $\hat{J}G(x) =$ convex hull $\{A : JG(x_n) \to A$, for some $x_n \to x$, $x_n^i \ne 0$ for all $i, n\}$ is nonsingular. Note that $\hat{J}G(x)$ is a finite set.

So let $x \in R^l \backslash R^l_-$. By relabeling variables if necessary, we can assume that $x^i > 0$ for $i \le k$, $x^i = 0$ for $k < i \le h$, and $x^i < 0$ for $h < i \le l$. Note that $k \ge 1$. In order to compute $\hat{J}G(x)$, it suffices to consider sequences $x_n \to x$ with all x_n belonging to the same orthant; i.e., we can assume $x_n^i > 0$ for $i \le f \le h$ and $x_n^i < 0$ for $i > f \ge k$. Then $JG(x_n)$ exists. In fact, if we denote by $J_f F(\Pi(x))$ the first f columns of $JF(\Pi(x))$, we have

$$\lim JG(x_n) = \begin{bmatrix} J_f F(\Pi(x)) & \begin{matrix} 0 \\ I \\ {\scriptstyle (l-f)\times(l-f)} \end{matrix} \end{bmatrix}.$$

Therefore, denoting by $\sigma_i F(\Pi(x))$ the ith column of $JF(\Pi(x))$, by e_i the ith column unit vector, and by a_i the ith column of a generic matrix A, we have

$$\begin{aligned} \hat{J}G(x) = \{A\colon & \text{for } i \leq k,\ a_i = \sigma_i F(\Pi(x)) \\ & \text{for } i > h,\ a_i = e_i \\ & \text{for } k < i \leq h,\ a_i = e_i \text{ or } a_i = \sigma_i F(\Pi(x))\}. \end{aligned}$$

A set of matrices defined as $\hat{J}G(x)$ has an interesting property. The multilinearity of the determinant function implies that if $A \in$ convex hull $\hat{J}G(x)$, then $|A|$ is a positive linear combination of determinants of matrices belonging to $\hat{J}G(x)$. Therefore, if the determinant of the matrices in $\hat{J}G(x)$ have a uniform sign, A must be nonsingular.

As we have just seen, if $A \in \hat{J}F(\Pi(x))$, then

$$A = \begin{bmatrix} J_f F(\Pi(x)) & \begin{matrix} 0 \\ I \\ {\scriptstyle (l-f)\times(l-f)} \end{matrix} \end{bmatrix}, \qquad \text{for some} \quad f \geq k.$$

Therefore $|A| = |J_{ff}F(\Pi(x))|$, where $J_{ff}F(\Pi(x))$ is the leading $f \times f$ principal minor of $JF(\Pi(x))$. But $J_{ff}F(\Pi(x)) = JF_{\{f+1,\ldots,l\}}(x^1, \ldots, x^f)$. So by hypothesis, all the $|A|$, $A \in JG(x)$ have the same sign and this concludes our proof.

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