PARETO OPTIMA AND EQUILIBRIA: THE FINITE DIMENSIONAL CASE

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1. INTRODUCTION

We aim at a succint presentation of the finite-dimensional mathematical theory associated with the so-called fundamental theorems of welfare economics. Very roughly these assert that, under some conditions, every price equilibrium is an optimum in the sense of Pareto and, conversely, under other (typically stronger) hypotheses, every optimum is a price equilibrium. This is a classical area of the theory of general economic equilibrium and it has been the object of extensive mathematical economic research. We refer to Debreu (1959, Ch. 7), Arrow and Hahn (1971, Ch. 4) and Mas-Colell (1985, Ch. 4) for systematic accounts.

Since we can hardly claim to be original some justification is needed for these pages. There are two reasons which, we hope, will make them useful. The first is that the emphasis throughout is on fine structure and minimal assumptions. The second is more forward looking. Although we only handle the finite-dimensional case everything is done with an eye to the infinite-dimensional generalization. Thus, for example, a hypothesis such as the non-emptiness of the interior of the set of feasible productions is avoided as much as possible.

2. DEFINITIONS

The commodity space is $\mathbb{R}^\ell$. There is a *non-empty, closed* production set $Y \subseteq \mathbb{R}^\ell$. There are m consumers indexed by i.

Every consumer i has a *non-empty, closed* consumption set $X_i \subseteq \mathbb{R}^\ell$ and a complete, transitive and reflexive preference relation $\succsim_i$ on X_i. Induced strict preferences are denoted by $\succ_i$.

An allocation x is a list $x = (x_1, \ldots, x_m) \in X_1 \times \cdots \times X_m$ such that $\bar{x} = \sum_i x_i \in Y$. Note that we are imbedding the initial endowments in the def-

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inition of the aggregate production set.

DEFINITION 1. *An allocation* x *is a weak optimum if there is no allocation* x' *such that* $x'_i \succ_i x_i$ *for all* i.

DEFINITION 2. *An allocation* x *is an optimum if there is no allocation* x' *such that* $x'_i \succsim_i x_i$ *for all* i *and* $x'_i \succ x_i$ *for some* i.

Obviously, an optimum is always a weak optimum. A price system is an arbitrary vector $p \in \mathbb{R}^\ell$.

DEFINITION 3. *An allocation* x *is a quasiequilibrium with respect to* $p \in \mathbb{R}^\ell$ *if:*
(a) $p \cdot \bar{x} \geq p \cdot z$ *for all* $z \in Y$ *(profit maximization), and*
(b) "$z \succsim_i x_i \Longrightarrow p \cdot z \geq p \cdot x_i$" *for all* i *(cost minimization).*

Obviously, the quasiequilibrium concept is of interest only if we can take $p \neq 0$. Even more, except in trivial cases we may want a stronger definition to be satisfied.

DEFINITION 4. *An allocation* x *is a proper quasiequilibrium with respect to* $p \in \mathbb{R}^\ell$ *if it is a quasiequilibrium with respect to* p *and* $p \cdot \bar{x} \neq p \cdot \bar{x}'$ *for some allocation* x'.

DEFINITION 5. *An allocation* x *is an equilibrium if there is* $p \in \mathbb{R}^\ell$ *such that:*
(a) $p \cdot \bar{x} \geq p \cdot z$ *for all* $z \in Y$ *(profit maximization), and*
(b) "$z \succ_i x_i \Longrightarrow p \cdot z > p \cdot x_i$" *for all* i *(preference maximization).*

We say that x *is an equilibrium with respect to* p.

3. HYPOTHESES

We now state a long list of hypotheses to be used in different combinations in the next three sections.

[a] For some i preferences $\succsim_i$ are such that for all $x_i \in X_i$ and $\varepsilon > 0$ there is some $z \in X_i$ with $\|z - x_i\| < \varepsilon$ and $z \succ_i x_i$ (local non-satiation for one consumer).

[a'] The property in [a] holds for all i (local non-satiation for all consumers).

[b] For some i preferences are such there is at most one satiation consumption (i.e., $\#\{z\colon z \succsim_i x_i \text{ for all } x_i \in X_i\} \leq 1$) and for all non-satiation consumptions $x_i \in X_i$ and $\varepsilon > 0$ there is some $z \in X_i$ with $\|z - x_i\| < \varepsilon$ and $z \succ_i x_i$ (local non-satiation, except possibly at a single bliss point, for some consumer).

[b'] The property in [b] holds for all i.

[c] Every X_i is convex.

[d] Every $\succ_i$ is continuous (i.e., closed as a subset of $X_i \times X_i$).

[e] $\sum_i X_i \cap \text{Int}Y \neq \emptyset$.

[f] For every i, $X_i = \mathbb{R}^\ell_+$ and $\succ_i$ is strictly monotone, i.e.,

$$\text{"} z \geqslant x_i,\ z \neq x_i \text{"} \implies \text{"} z \succ_i x_i \text{"}.$$

[g] Y is convex.

[h] Every $\succ_i$ is convex (i.e., for all $x_i \in X_i$ the set $\{z: z \succ_i x_i\}$ is convex).

[k] There is a non-trivial open cone $\Gamma \subseteq \mathbb{R}^\ell$ such that

$$(\{x_i\} + \Gamma) \cap \{z_i \in X_i: z_i \succ_i x_i\} = \emptyset$$

for all i and $x_i \in X_i$. Moreover, there are allocations x, x' with $\bar{x} \neq \bar{x}'$ and $\bar{x} - \bar{x}' \in \Gamma$.

Property [k] is less familiar than the previous ones. It has two parts. The first has been called *properness* elsewhere (Mas-Colell,1983) and its interpretation is that there is a commodity (possibly composite) which is very desirable in the sense that its marginal rates of substitution with respect to any other commodity are uniformly bounded above; see Figure 1. It is always satisfied by monotone preferences (take $\Gamma = -\mathbb{R}^\ell_+$) and it can be equivalently formulated as: "there is a non-trivial open cone Γ such that

$$(\{x_i\} + \Gamma) \cap X_i \subseteq \{z_i: z_i \succ_i x_i\}$$

for all i and $x_i \in X_i$". The second part simply says that the desirable commodity can in fact be produced from some initial allocation.

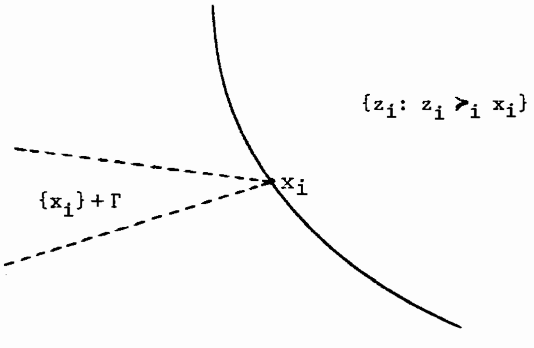


FIGURE 1

So far the hypotheses have been on the data of the economy. The next two are of a different nature because they involve allocations and prices.

[P] The allocation x and the price vector p are such that, for some i we have

$$p \cdot x_i > \text{Inf}\{p \cdot z : z \in X_i\}.$$

[P'] The property in [P] holds for all i.

4. OPTIMALITY PROPERTIES OF EQUILIBRIA

[I] *An equilibrium allocation x is a weak optimum.*

PROOF. Let p be the price system. If $x_i' \succ_i x_i$ for all i, then $p \cdot x_i' > p \cdot x_i$ for all i and so, $p \cdot \bar{x}' > p \cdot \bar{x}$. By profit maximization $p \cdot \bar{x} \geqslant p \cdot z$ for all $z \in Y$. Hence, $\bar{x}' \notin Y$, i.e., x' is not an allocation.■

Example 1 shows that an equilibrium allocation need not be an optimum.

EXAMPLE 1. See Figure 2, where the allocation x is an equilibrium but $x_2' \succ_2 x_2$ and, of course, $x_1' \succcurlyeq_1 x_1$.

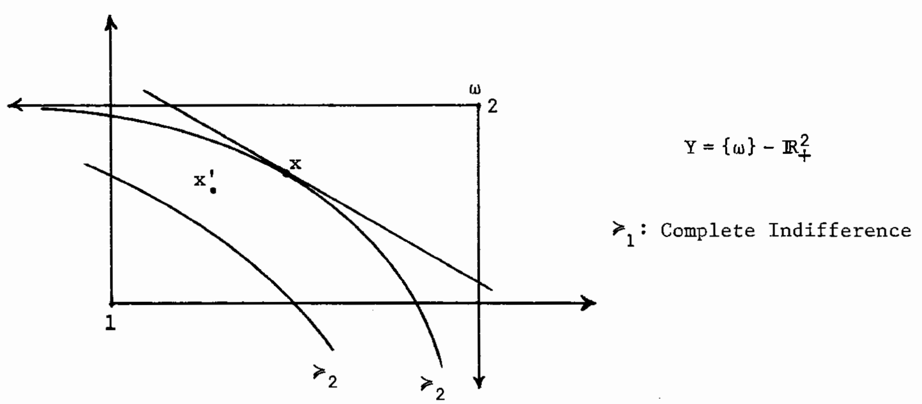


FIGURE 2

[II] *If the allocation x is both an equilibrium and a quasiequilibrium with respect to p, then x is an optimum.*

PROOF. Let $x_i' \succcurlyeq_i x_i$ for all i and $x_i' \succ_i x_i$ for some i. Because of the quasiequilibrium property $p \cdot x_i' \geqslant p \cdot x_i$ for all i and, by the equilibrium property, $p \cdot x_i' > p \cdot x_i$ for some i. Therefore, $p \cdot \bar{x}' > p \cdot \bar{x}$ and the rest of the proof is as in [1].■

[III] *Under [b'] an allocation x which is an equilibrium with respect to p is a quasiequilibrium with respect to p.*

PROOF. Suppose that $z \succ_i x_i$ and $p \cdot z < p \cdot x_i$. If z is a unique satiation point of $\succ_i$, then $z \succ_i x_i$. If not, then by [b'] we can find $z' \succ_i z$ with $p \cdot z' < p \cdot x_i$. In any case we have a $y \succ_i x_i$ with $p \cdot y < p \cdot x_i$ which contradicts the fact that x is an equilibrium with respect to p. Therefore, for all i, $z \succ_i x_i$ implies $p \cdot z \geq p \cdot x_i$.■

Example 1 shows the need of [b'] in [III]. Combining [II] and [III]:

[IV] *Under [b'] every equilibrium allocation is an optimum.*

5. OPTIMALITY PROPERTIES OF QUASIEQUILIBRIA

Example 2 shows that a quasiequilibrium x with respect to $p \neq 0$ need not be a weak optimum.

EXAMPLE 2. See the single consumer economy represented in Figure 3 and note that $\omega \succ x$.

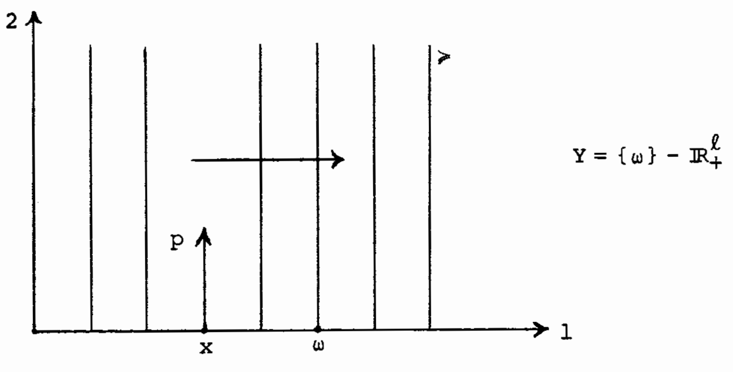


FIGURE 3

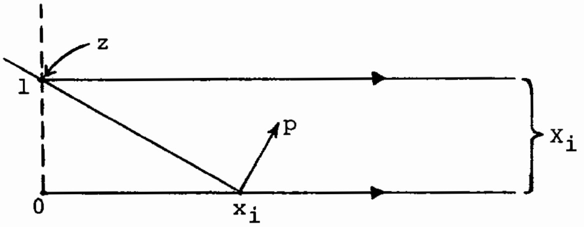
In order to obtain sufficient conditions for a quasiequilibrium to be a weak optimum or an optimum we first establish an important preliminary lemma.

LEMMA. *Consider a single i and suppose that X_i is convex and $\succsim_i$ is continuous. Suppose that $x_i \in X_i$ and $p \in \mathbb{R}^\ell$ are such that "$z \succ_i x_i$ implies $p \cdot z \geqslant p \cdot x_i$" and $p \cdot x_i > \mathrm{Inf}\{p \cdot z\colon z \in X_i\}$. Then "$z \succ_i x_i$ implies $p \cdot z > p \cdot x_i$".*

PROOF. Suppose not, i.e., $z \succ_i x_i$ and $p \cdot z = p \cdot x_i$. Pick $y \in X_i$ such that $p \cdot y < p \cdot x_i$. By the convexity of X_i, $\alpha y + (1-\alpha)z \in X_i$ for arbitrarily small $\alpha > 0$. By continuity, if $\alpha > 0$ is small, then $(\alpha y + (1-\alpha)z) \succ_i x_i$. But $p \cdot (\alpha y + (1-\alpha)z) = \alpha p \cdot y + (1-\alpha) p \cdot z < p \cdot x_i$ and we obtain a contradiction to our hypothesis.∎

The next three examples show, respectively, that the assumptions "X_i is convex," "$\succsim_i$ is continuous" and "$p \cdot x_i > \mathrm{Inf}\{p \cdot z\colon z \in X_i\}$" cannot be dispensed from the lemma.

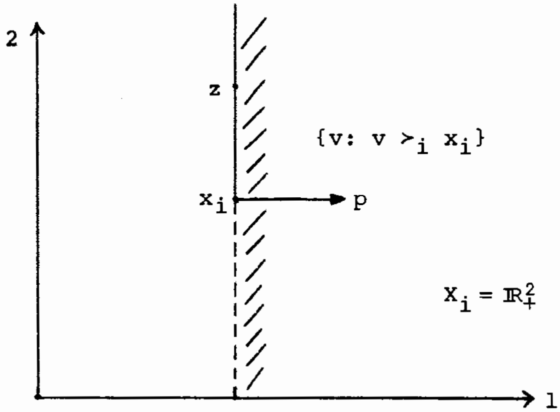
EXAMPLE 3. See Figure 4, where $z \succ_i x_i$ but $p \cdot z = p \cdot x_i$.



$\succsim_i$: Everything in the line $\{1\} \times \mathbb{R}$ is preferred to anything in the line $\{0\} \times \mathbb{R}$. Within the lines more of the continuous commodity is preferred to less.

FIGURE 4

EXAMPLE 4. See Figure 5.



$\succsim_i$: Lexicographically ordered, i.e., $v \succsim_i z$ if either $v^1 \geqslant z^1$ or $v^1 = z^1$ and $v^2 > z^2$.

FIGURE 5

EXAMPLE 5. See Figure 6.

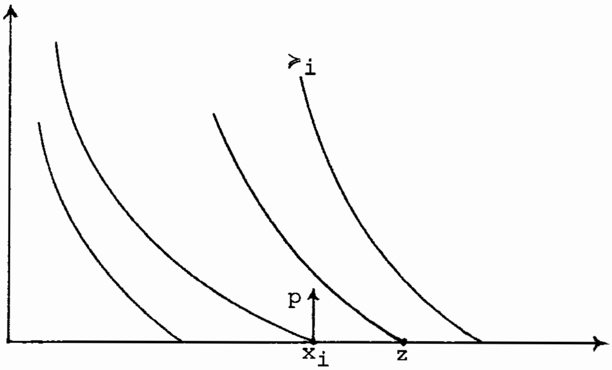


FIGURE 6

We can now prove:

[V] *Under* [c] *and* [d] *if the allocation* x *is a quasiequilibrium with respect to* p *and* [P] *holds for* x *and* p, *then* x *is a weak optimum.*

PROOF. Let $x_i' \succ_i x_i$ for all i. Then $p \cdot x_i' \geq p \cdot x_i$ for all i. Because of [P] and the Lemma we have $p \cdot x_i' > p \cdot x_i$ for some i. Therefore, $p \cdot \bar{x}' > p \cdot \bar{x}$ which, by profit maximization, implies $\bar{x}' \notin Y$, i.e., x' is not an allocation.■

[VI] *Under* [c] *and* [d] *if the allocation* x *is a quasiequilibrium with respect to* p *and* [P'] *holds for* x *and* p, *then* x *is an equilibrium with respect to* p *(hence, by* [II], *it is an optimum).*

PROOF. Obvious consequence of the Lemma.■

We note that:

[VII] *If* x *is a proper quasiequilibrium with respect to* p, *then* [P] *holds for* x *and* p.

PROOF. Let x' be an allocation such that $p \cdot \bar{x}' \neq p \cdot \bar{x}$. By profit maximization, $p \cdot \bar{x}' < p \cdot \bar{x}$. Therefore, $p \cdot x_i' < p \cdot x_i$ for some i which is, precisely, property [P].■

The results [V] and [VI] are not very satisfactory because properties [P], [P'] depend on x and p and not exclusively on the original data of the economy. The next three results yield sufficient conditions on the economy for [P] or [P'] to hold. The conditions for [P'] are quite strong (essentially, strict monotonicity of preferences).

[VIII] *Under* [e] *if* x *is a quasiequilibrium with respect to* $p \neq 0$*, then it is a proper quasiequilibrium (hence, by* [VII]*,* [P] *holds).*

PROOF. Obvious.■

Example 2 shows the need of condition [e] in [VIII].

[IX] *Under* [d], [e] *and* [f] *if* x *is a quasiequilibrium with respect to* $p \neq 0$*, then* $p \gg 0$*.*

PROOF. By [VIII], $p \cdot x_i > 0$ for some i. If $p^h = 0$, then

$$p \cdot (x_i + e^h) = p \cdot x_i \quad \text{and} \quad x_i + e^h \succ_i x_i$$

(e^h has a zero in every component except the h^{th}) which contradicts the Lemma. Therefore, $p \gg 0$.■

Examples 4 and 5 show, respectively, the need of hypothesis [d] and [e] in [IX]. (In the two examples take $Y = \{z\} - \mathbb{R}^2_+$.) As for [f], Example 6 (which has monotone preferences) will do:

EXAMPLE 6. See Figure 7. Both preference relations are monotone.

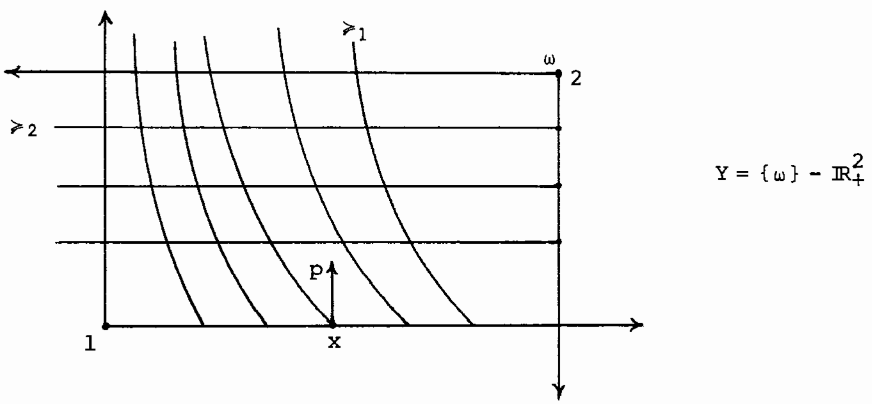


FIGURE 7

Under the conditions of [IX] if, for every i, $x_i \neq 0$ then $p \cdot x_i > 0$ for all i and so, by [VI], x is an equilibrium (and, therefore, an optimum). Obviously, this conclusion is not altered if $x_i = 0$ for some i (if $z \succ_i x_i$ then $z \neq 0$ and so $p \cdot z > 0 = p \cdot x_i$). Summing up:

[X] *Under* [*d*],[*e*] *and* [*f*] *if* x *is a quasiequilibrium with respect to* $p \neq 0$, *then* x *is an equilibrium with respect to* p *(hence, by* [*II*], x *is an optimum).*

6. EQUILIBRIA PROPERTIES OF OPTIMA

To what an extent can we now reverse the direction of our conclusions and assert equilibrium properties for arbitrary optima? The first obvious remark (consider the one consumer case) is that the answer will require some sort of application of the separating Hyperplane Theorem (see, e.g. Debreu, 1959) and that, therefore, the proper setting for the question is one where the convexity hypotheses are made throughout. Hence:

N. B.: *In all of this section we assume* [*c*], [*g*] *and* [*h*], *i.e., convexity of consumption sets, production sets and preferences.*

The next example shows that an optimum need not be an equilibrium.

EXAMPLE 7. See Figure 8, where x is an optimum but not an equilibrium.

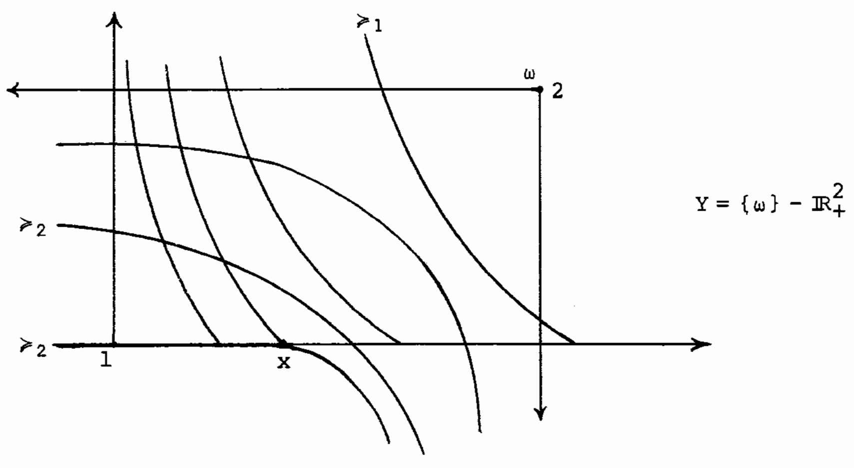


FIGURE 8

A simple, but nevertheless fundamental, theorem is:

[XI] *Under* [*a*] *every optimum* x *is a quasiequilibrium with respect to some* $p \neq 0$.

PROOF. Suppose that the local non-satiation condition [a] holds for i=1.

Define $V = \{z\colon z \succ_1 x_1\} + \sum_{i=2}^{m} \{z\colon z \succcurlyeq_i x_i\}$. By the convexity hypothesis on preferences ([h]) V is convex. Because x is an optimum, $Y \cap V = \emptyset$. Since the production set Y is also convex ([g]) we can find, by the Separating Hyperplane Theorem, a $p \neq 0$ and $\alpha \in \mathbb{R}$ such that $p \cdot z \leqslant \alpha$ for all $z \in Y$ and $p \cdot z \geqslant \alpha$ for all $z \in V$. However, $\bar{x} \in Y$ and, because $i = 1$ is locally non-saturated, $\bar{x} \in \bar{V}$. Hence $\alpha = p \cdot \bar{x}$ which allows us to conclude $p \cdot z \leqslant p \cdot \bar{x}$ for all $z \in Y$ (profit maximization condition). Let now $v \succcurlyeq_1 x_1$ and pick $v' \succ_1 v$ arbitrarily close to v. Then $v' + x_2 + \cdots + x_m \in V$ and so, $p \cdot (v' + x_2 + \cdots + x_m) \geqslant p \cdot \bar{x}$, i.e., $p \cdot v' \geqslant p \cdot x_1$. Letting $v' \longrightarrow v$ we conclude $p \cdot v \geqslant p \cdot x_1$.

Similarly, let $v \succcurlyeq_i x_i$, $i > 1$. Say $i = 2$. Take $z \succ_1 x_1$ arbitrarily close to x_1. Then $p \cdot (z + v + x_3 + \cdots + x_m) \geqslant p \cdot \bar{x}$, i.e., $p \cdot (z + v) \geqslant p \cdot (x_1 + x_2)$. Letting $z \longrightarrow x_1$ we conclude $p \cdot v \geqslant p \cdot x_2$. Therefore, x is a quasiequilibrium with respect to p. ∎

Example 8 shows that hypothesis [a] cannot be dispensed with (or weakened to [b]) in [XI].

EXAMPLE 8. See Figure 9, where the optimum x can only be supported by $p = 0$.

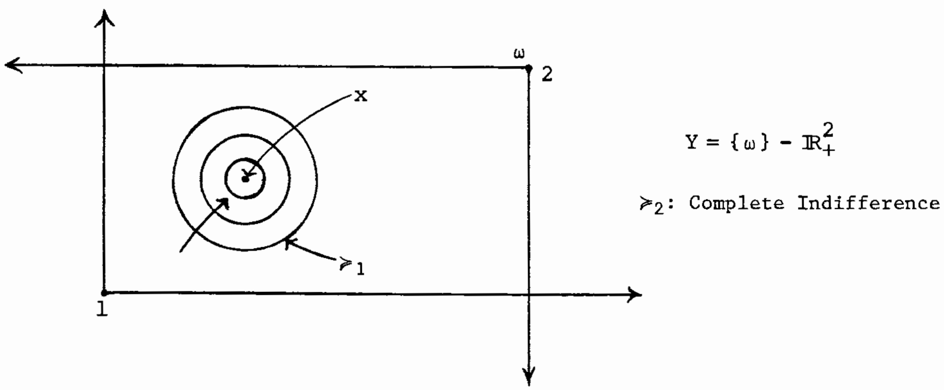


FIGURE 9

Allocation x in Example 1 shows that [XI] cannot be strengthened to replace optimum by weak optimum. However if [a] is required to hold for all i (i.e., replaced by [a']), then we get a corresponding result for weak optima.

[XII] *Under [a'] every weak optimum* x *is a quasiequilibrium with respect to some* $p \neq 0$.

PROOF. Entirely similar to the proof of [XI]. The only difference is that we now should define $V = \sum_{i=1}^{m} \{z: z \succ_i x_i\}$.■

The results [VI] and [X] (resp. the result [VIII]) in section 5 gave conditions for a quasiequilibrium to be an equilibrium (resp. a proper quasiequilibrium). Example 9 shows that even under [a'] an optimum need not be a proper quasiequilibrium.

EXAMPLE 9. Consider the one consumer economy described by Figure 10, where x is an optimum supported by p but $p \cdot \omega = 0$.

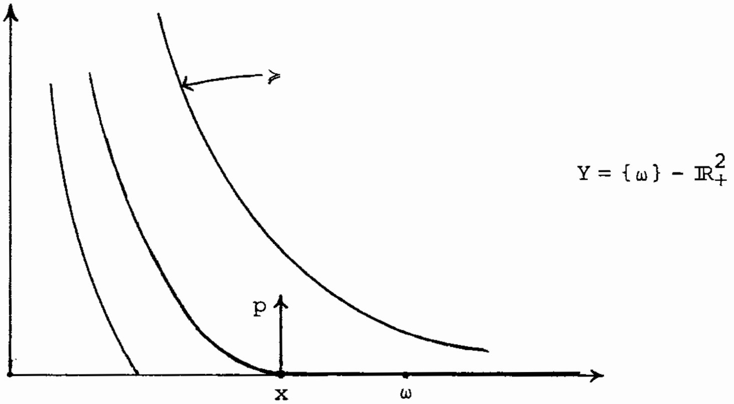


FIGURE 10

For the exchange case, i.e. $Y = \{\omega\} - \mathbb{R}_+^\ell$, the next result gives a sufficient condition for a weak optimum to be a proper quasiequilibrium.

[XIII] *Under [k], $X_i = \mathbb{R}_+^\ell$ and monotone preferences for all i and $Y = \{\omega\} - \mathbb{R}_+^\ell$, every weak optimum is a proper quasiequilibrium.*

PROOF. Let Γ be the cone given by [k]. We must have $\omega \geqslant 0$. The proof will proceed under the further condition that $-\omega \in \Gamma$. There is no conceptual loss in making this desirability assumption. But there is no mathematical loss either. It can be shown that it is implied by [k]. The proof, however, is delicate and technical and it is, thus, scarcely worth spending time on it.

Without loss of generality we take $\|\omega\| = 1$. Let $\varepsilon > 0$ be such that the open cone spanned by a ball of radius ε centered at $-\frac{1}{m}\omega$ is entirely contained in Γ. Let then Γ' be the open cone spanned by a ball of radius ε centered at $-\omega$.

Define an expanded production set $Y' = \{\omega\} + \Gamma' - \mathbb{R}^{\ell}_{+}$. If we can show that any weak optimum of the original economy is also a weak optimum under Y' then, by [VIII] and [XII], the conclusion of [XIII] follows. So, suppose, by way of contradiction, that there is a weak optimum x for Y which does not remain one for Y', i.e. there is a $x' \geqslant 0$ such that $x'_i \succ_i x_i$ for every i and $\bar{x}' \leqslant \omega + z$ for $z \in \Gamma'$. Without loss of generality we assume $\bar{x} = \omega$ and $\bar{x}' = \omega + z$. By the convexity of preferences we can choose x' so that $\|z\| < 1$. Because $z \in \Gamma'$ we can write $z = -\alpha\omega + y$ where $0 \leqslant \alpha \leqslant \|z\| < 1$ and $\|y\| \leqslant \alpha\varepsilon$. Note that $y = z + \alpha\omega \leqslant z + \omega = \bar{x}'$ and so, $y^+ \leqslant \bar{x}'$. Take $0 \leqslant y_i \leqslant x'_i$ such that $\sum_i y_i = y^+$ and define

$$x''_i = x'_i + \frac{\alpha}{m}\omega - y^+_i + \frac{1}{m}y^- \geqslant 0.$$

Observe that

$$\overline{x''} = \overline{x'} + \alpha\omega - y = \overline{x'} - z = \omega.$$

Hence x'' is an allocation for the original economy. Also, for every i, $x''_i \succcurlyeq_i x'_i + \frac{\alpha}{m}\omega - y^+_i \succcurlyeq_i x'_i$. The first inequality follows by monotonicity and the second by property [k] (we have $\|y^+_i\| \leqslant \|y\| \leqslant \alpha\varepsilon$ and so, $\frac{\alpha}{m}\omega - y^+_i \in -\Gamma$). Because $x'_i \succ_i x_i$ for all i we have found the desired contradiction to the weak optimality of x.■

It is not clear to us how far the consumption set restriction in [XIII], i.e. $X_i = \mathbb{R}^{\ell}_{+}$ for all i, can be relaxed. That something is needed is shown by Example 10.

EXAMPLE 10. Let $\ell = 3$, $m = 2$, $Y = \{(1,1,0)\} - \mathbb{R}^3_+$,

$$X_1 = \{(z^1,0,z^3):\ z^1,z^3 \geqslant 0\},\quad X_2 = \{(0,z^2,z^3):\ z^2,z^3 \geqslant 0\}$$

and preference relations be representable by

$$u_1(z) = \ln z^1 + \ln z^3,\quad u_2(z) = \ln z^2 + \ln z^3.$$

Condition [k] is satisfied by the cone $\Gamma = -\mathbb{R}^3_{++}$. The allocation $x_1 = (1,0,0)$, $x_2 = (0,1,0)$ is an optimum but to make it a quasiequilibrium we should put $p^1 = 0$ and $p^2 = 0$. Because then every allocation has a null value the quasiequilibrium cannot be proper. Note that the example can be easily modified to make preference relations strictly monotone in their consumption sets.

It would also be highly desirable to have some version of [XIII] allowing for more general production sets. At a minimum this requires a strengthening of the desirable hypothesis [k] to cover the production set. Of course, if the strenghtening gets as far as adding $Y + \Gamma \subseteq Y$ to the conditions in the statement of [k], then [e] would be satisfied and there would no longer be a distinction between quasiequilibria and proper quasiequilibria, see [VIII].

7. PRICE SYSTEMS AND UTILITY WEIGHTS

A deeper analysis of the optimality concept will be possible if we resort to utility functions. To be specific we assume for the rest of this section that: *for every* i, $X_i = \mathbb{R}_+^\ell$ *and* $\succcurlyeq_i$ *is monotone (i.e.* $z \geqslant x_i$ *implies* $z \succcurlyeq_i x_i$ *for all* z, x_i*) and representable by a continuous utility function* $u_i : \mathbb{R}_+^\ell \longrightarrow \mathbb{R}$ *(we can put* $u_i(0) = 0$*). Moreover,* $Y - \mathbb{R}_+^\ell \subseteq Y$ *(free disposal hypothesis) and* $Y \cap \mathbb{R}_+^\ell$ *is non-empty and compact.*

DEFINITION 6. *The utility set* $U \subseteq \mathbb{R}_+^m$ *is the set of utility vectors* $u = (u_1, \ldots, u_m)$ *that can be formed by allocations, i.e.*

$$U = \{u \in \mathbb{R}_+^m : u_i(x_i) = u_i \text{ for all allocations } x\}.$$

See Figure 11 for some specimens. Abusing notation slightly, we will write $u(x) = (u_1(x_1), \ldots, u_m(x_m))$. Clearly, U is non-empty and compact. Also, by the continuity of the u_i's as well as the free disposal hypothesis on Y we have "$u \in U$ and $0 \leqslant u' \leqslant u$ implies $u' \in U$."

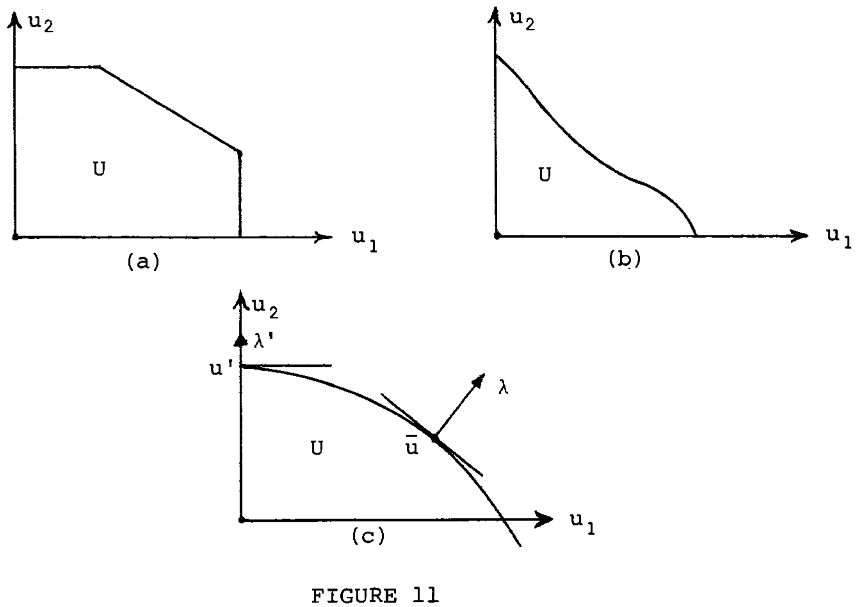


FIGURE 11

By definition, an allocation x is a weak optimum if and only if $u(x)$ belongs to the upper boundary of U, i.e., if and only if $u' \gg u(x)$ implies $u' \notin U$ (we call these u *weak utility optima*), while x is an optimum if and only if $u' > u(x)$ implies $u' \notin U$ (we call these u *utility optima*). As Figure 11(a) illustrates, both sets need not coincide, although, as in Figures 11(b) and (c), they will if preferences are strictly monotone.

In analogy with the concept of price equilibrium we will now consider vectors of *utility weights* $\lambda \in \mathbb{R}^m_+$ and linear expressions on $\mathbb{R}^m$ of the form $\sum_{i=1}^{m} \lambda_i u_i$.

DEFINITION 7. *A vector* $\bar{u} \in U$ *is supported by* $\lambda > 0$ *if* $\bar{u}$ *maximizes* $\sum_{i=1}^{m} \lambda_i u_i$ *subject to* $u \in U$. *(See Figure 11(c).)*

It is obvious that any $u \in U$ supported by a $\lambda > 0$ (resp. $\lambda \gg 0$) is a weak utility optimum (resp. a utility optimum). As in Section 6 to obtain a converse we need some convexity hypothesis.

[XIV] *If each* u_i *is concave, then* U *is convex and every weak utility optimum can be supported by a* $\lambda > 0$.

PROOF. Let $u, u' \in U$ and $0 \leqslant \alpha \leqslant 1$. Take allocations x, x', with $u(x) = u$, $u(x') = u'$. Then $x'' = \alpha x + (1-\alpha)x'$ is also an allocation, i.e., $u'' \in U$, and by the concavity of each u_j, $u'' = u(x'') \geqslant \alpha u + (1-\alpha)u' \geqslant 0$. Hence, $\alpha u + (1-\alpha)u' \in U$. Once it is known that U is convex, the second claim follows immediately from the Supporting Hyperplane Theorem.■

The previous Proposition is illustrated in Figure 11(c). Note that, as u' in the figure, an optimum may not be supportable by a $\lambda \gg 0$, even in the strictly monotone case. Observe also that the set of supporting weights for a given u is always a convex cone.

Taking stock of developments so far, we see that in the case where preferences admit concave utility representations, any weak optimum can be supported by prices (results [XII] and [XIII]) and by a vector of utility weights (result [XIV]). This suggests a deeper connection between the two notions of supportability. It will be worthwhile to briefly explore the matter.

DEFINITION 8. *The pair* $(p, \lambda) \in \mathbb{R}^\ell_+ \times \mathbb{R}^m_+$ *supports the allocation* x *if* $p \neq 0$, $\lambda \neq 0$ *and:*

(a) $p \cdot \bar{x} \geqslant p \cdot y$ *for all* $y \in Y$;

(b) $p \cdot (x'_i - x_i) \geqslant \lambda_i [u_i(x'_i) - u_i(x_i)]$ *for all* i *and* $x'_i \in \mathbb{R}^\ell_+$.

We have:

[XV] *If* (p,λ) *supports the allocation* x, *then:*

(a) x *is a price quasiequilibrium with respect to* p; *and*

(b) λ *is a vector of supporting utility weights for* U *at* $u(x)$.

PROOF. (a) If $x_i' \succ_i x_i$, then $u_i(x_i') \geqslant u_i(x_i)$. So, $p \cdot x_i' \geqslant p \cdot x_i$.
(b) Let x' be an allocation. Then

$$0 \leqslant p \cdot (\bar{x} - \bar{x'}) \leqslant \sum_{i=1}^{m} \lambda_i [u_i(x_i) - u_i(x_i')].$$

So,

$$\sum_{i=1}^{m} \lambda_i u_i(x_i) \geqslant \sum_{i=1}^{m} \lambda_i u_i(x_i')$$

for all allocations x'.■

Either of the conclusions of [XV] implies that x is a weak optimum. Under the hypotheses that every u_i is concave and x is a weak optimum we offer now two approaches to the converse of [XV], namely to the existence of a supporting pair (p,λ) for x. Because of [XII], plus the monotonicity assumption, we know that x is a price quasiequilibrium with respect to some $p > 0$. In [XVI] we show that for this p we can find $\lambda > 0$ such that (p,λ) is supporting. As a bonus we get an interpretation of the λ vector. By [XIV] we also know that U can be supported at $u(x)$ by a $\lambda > 0$. In [XVII] we show that for this λ we can find $p > 0$ such that (p,λ) is supporting. As a bonus we will get an interpretation of the p vector.

[XVI] *Suppose that every* u_i *is concave and let* x *be a proper quasiequilibrium with respect to* $p > 0$. *For each* i *with* $p \cdot x_i > 0$ *define* $v_i : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *by*

$$v_i(w_i) = \mathrm{Sup}\{u_i(z_i) : p \cdot z_i \leqslant w_i\}.$$

Then:

(a) *Each* v_i *is well defined, concave, and increasing.*

(b) *For each* i *with* $p \cdot x_i > 0$, *let* μ_i *be a supergradient of* v_i *at* $p \cdot x_i$, *i.e.,* $v_i(p \cdot x_i) + \mu_i(w_i - p \cdot x_i) \geqslant v_i(w_i)$ *for all* i *and* w_i. *Put* $\lambda_i = 1/\mu_i$ *for* i *with* $p \cdot x_i > 0$ *and* $\lambda_i = 0$ *for* i *with* $p \cdot x_i = 0$. *Then* (p,λ) *is a supporting pair for* x.

(c) *Let* λ *be such that* (p,λ) *supports* x. *If* $p \cdot x_i > 0$, *then* $1/\lambda_i$ *is a is a supergradient of* v_i *at* $p \cdot x_i$.

PROOF. (a) We show that v_i is well defined, i.e., $v_i(w) < \infty$ for all w. That v_i is increasing and concave is then immediate. If $p \cdot x_i > 0$, then consumer i is utility maximizing (by the Lemma of Section 5). Hence, $v_i(w) \leqslant u_i(x_i)$ for $w \leqslant p \cdot x_i$. Let $w = \alpha p \cdot x_i$, $\alpha > 1$, $p \cdot x_i' \leqslant w$. Since $p \cdot (\frac{1}{\alpha} x_i') \leqslant p \cdot x_i$ and u_i is concave, we have

$$u_i(x_i) \geq u_i(\frac{1}{\alpha}x_i') \geq \frac{1}{\alpha}u_i(x_i').$$

So, $u_i(x_i') \leq \frac{m}{p \cdot x_i} u_i(x_i)$ for all x_i' with $p \cdot x_i' \leq w$. Therefore,

$$v_i(w) \leq \frac{w}{p \cdot x_i} u_i(x_i).$$

(b) If $p \cdot x_i = 0$, then $p \cdot (x_i' - x_i) \geq 0$ for all $x_i' \in \mathbb{R}_+^\ell$ and so, $\lambda_i = 0$ does the trick. Let $p \cdot x_i > 0$ (which implies $u_i(x_i) = v_i(p \cdot x_i)$) and μ_i be a supergradient of v_i at $p \cdot x_i$. Then for any $x_i' \in \mathbb{R}_+^\ell$,

$$u_i(x_i) + \mu_i p \cdot (x_i' - x_i) = v_i(p \cdot x_i) + \mu_i(p \cdot x_i' - p \cdot x_i) \geq v_i(p \cdot x_i') \geq u_i(x_i').$$

So, $\lambda_i = 1/\mu_i$ satisfies condition (b) of Definition 8.

(c) If $p \cdot x_i > 0$, then we should have $\lambda_i > 0$. Let $\mu_i = 1/\lambda_i$ and $w \geq 0$. Take any x_i' with $p \cdot x_i' = w$. Then

$$v_i(p \cdot x_i) + \mu_i(w - p \cdot x_i) = u_i(x_i) + \mu_i p \cdot (x_i' - x_i) \geq u_i(x_i').$$

So, $v_i(p \cdot x_i) + \mu_i(w - p \cdot x_i) \geq v_i(w)$ and we conclude that μ_i is a subgradient of v_i at $p \cdot x_i$.■

For the given p, $v_i(w_i)$ is the so-called indirect utility of wealth. Hence, for a given supporting pair (p, λ), [XVI] provides an interpretation of $\lambda_i > 0$ as the reciprocal of the marginal utility of wealth.

[XVII] *Suppose that every* u_i *is concave and let* $u(x) \in U$ *be supported by the utility weights* $\lambda > 0$. *Take* $A = \{z \in \mathbb{R}^\ell : (\{z\} + Y) \cap \mathbb{R}_+^\ell \neq \emptyset\}$ *and define the function* $V: A \longrightarrow \mathbb{R}_+$ *by letting* $V(z)$ *be the maximum value of* $\sum_{i=1}^m \lambda_i u_i(x_i)$ *subject to* $\bar{x} - z = \sum_{i=1}^m x_i - z \in Y$. *Then:*

(a) V *is concave.*

(b) If p *is a supergradient of* V *at* 0, *i.e.,* $V(0) + p \cdot z \geq V(z)$ *for all* $z \in A$, *then the pair* (p, λ) *supports* x.

(c) If p *is such that* (p, λ) *is a supporting pair for* x, *then* p *is a supergradient of* V *at* 0.

PROOF. To prove (a) note that if the allocation x (resp. x') is attainable for $z \in A$ (resp. $z' \in A$), then $\alpha x + (1 - \alpha)x'$, $0 < \alpha < 1$, is attainable for $\alpha z + (1 - \alpha)z'$. Therefore, by the concavity of the u_i's,

$$V(\alpha z + (1 - \alpha)z') \geq \alpha[\sum_{i=1}^m \lambda_i u_i(x_i)] + (1 - \alpha)[\sum_{i=1}^m \lambda_i u_i(x_i')]$$

which yields $V(\alpha z + (1 - \alpha)z') \geq \alpha V(z) + (1 - \alpha)V(z')$.

To prove (b) let p be a supergradient of V at 0. Since V is increasing, $p \neq 0$. Take any $y \in Y$. Let $z = \bar{x} - y$. Since $z + y = \bar{x} \in Y$, we have $z \in A$ and $V(z) \geq \sum_{i=1}^{m} \lambda_i u_i(x_i) = V(0)$. Therefore,

$$V(0) + p\cdot(\bar{x} - y) = V(0) + p\cdot z \geq V(z) \geq V(0),$$

which yields $p\cdot\bar{x} \geq p\cdot y$. So, profit maximization holds. Take any i and $x_i' \geq 0$. Put $z = x_i' - x_i = x_i' + \sum_{j\neq i} x_j - \bar{x}$. Since $z + \bar{x} \geq 0$, we have $z \in A$ and

$$V(0) + p\cdot z = V(0) + p\cdot(x_i' - x_i) \geq V(z) \geq \sum_{j\neq i} \lambda_j u_j(x_j) + \lambda_i u_i(x_i') .$$

Subtracting $\sum_{j\neq i} \lambda_j u_j(x_j)$ from both sides, we get

$$\lambda_i u_i(x_i) + p\cdot(x_i' - x_i) \geq \lambda_i u_i(x_i')$$

for all $x_i' \geq 0$. Therefore, condition (b) of Definition 8 is satisfied.

Finally, for (c) let $z \in A$ and take $x' \geq 0$ such that $\overline{x'} = z + v'$ for some $v' \in Y$. Then

$$V(0) + p\cdot z = \sum_{i=1}^{m} \lambda_i u_i(x_i) + \sum_{i=1}^{m} p\cdot(x_i' - x_i) + p\cdot(\bar{x} - v') \geq \sum_{i=1}^{m} \lambda_i u_i(x_i').$$

So, maximizing over x', $V(0) + p\cdot z \geq V(z)$.■

Thus, given the supporting pair (p,λ), [XVIII] tells us that p is a vector of marginal social valuations of resources when social utility is formed as a λ-weighted sum of utilities.

A supergradient of V at 0 will exist under hypothesis [e], i.e., under $Y \cap \mathbb{R}^{\ell}_{++} \neq \emptyset$, because then $0 \in \text{Int}\, A$. That something is needed is shown by Example 11 (which also shows that proper quasiequilibrium cannot be replaced by quasiequilibrium in [XVI]).

EXAMPLE 11. Let $\ell = m = 1$, $Y = -\mathbb{R}_+$, $u(x) = \sqrt{x}$. Then $x = 0$ is optimal and u is concave. But because the slope of $\sqrt{x}$ at 0 is not finite there is no supporting pair (p,λ).

For the case of exchange economies, i.e., $Y = \{\omega\} - \mathbb{R}^{\ell}_{+}$, the next and final result gives conditions different from [e] for the existence of a supergradient to V at 0.

[XVIII] *Let* $Y = \{\omega\} - \mathbb{R}^{\ell}_{+}$ *and define* $V\colon A \longrightarrow \mathbb{R}_+$ *as in* [XVII] *with respect to some* x, $u(x)$ *and* $\lambda > 0$. *We have:*

(a) *If every* u_i *can be extended to a concave function defined on* $\mathbb{R}^{\ell}$, *then* V *has a supergradient at* 0.

(b) *Suppose that every* u_i *is continuously differentiable and define* q *as the lattice maximum of* $\{\lambda_1 \nabla u_1(x_1), \ldots, \lambda_m \nabla u_m(x_m)\}$, *i.e.,* $q^h = \text{Max}_i\{\lambda_i \partial_h u_i(x_i)\}$. *Then* q *is a supergradient of* V *at* 0.

PROOF. (a) Because every u_i is extendable there is a $k > 0$ such that for any i, if α_i is a directional derivative of u_i at some $x_i^! \in [0, \|\omega\| + 1]^\ell$ then $|\alpha_i| < k$. Let $z_n \gg 0$, $z_n \longrightarrow 0$. Because z_n is in Int A result [XVII] yields the existence of a supergradient q_n of V at z_n. Let $x_n \geqslant 0$ be such that $V(z_n) = \sum_i \lambda_i u_i(x_n)$ and $\bar{x}_n = \omega + z_n \gg 0$. It is rather simple to see that q_n must also be a supergradient of every u_i at x_{ni}. Because for every commodity h there is a consumer i with $x_{ni}^h > 0$ we must then conclude that $|q_n^h| < k$ for all h and n. Letting q be a limit point of $\{q_n\}$, a straightforward continuity argument shows that q is a supergradient of V at 0.

(b) Proceeding as in (a) for a $z_n \gg 0$, $z_n \longrightarrow 0$ it is straightforward to verify that, for every h and all i, we have $q_n^h \geqslant \lambda_i u_i(x_{ni}^h)$ with equality holding whenever $x_{ni}^h > 0$ (these are the so-called Kuhn-Tucker conditions). Because for every h there is an i with $x_{ni}^h > 0$, it follows that $q_n^h = \operatorname{Max}_i\{\lambda_i u_i(x_{ni}^h)\}$. Assuming, without loss of generality, that every u_i is strictly concave we have $x_n \longrightarrow x$ and so, the conclusion follows by continuity.■

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