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ENGEL CURVES LEADING TO THE WEAK AXIOM IN THE AGGREGATE

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For every range of admissible incomes we characterize the class of Engel curves with the property that if an economy has, first, a price independent distribution of income and, second, preferences which are identical across consumers and generate Engel curves in the class, then the corresponding aggregate demand function satisfies the Weak Axiom of Revealed Preference. This class is defined by two simple conditions. The no torsion condition says that, in the relevant range of income, the Engel curve is contained in a plane through the origin (which may depend on the price vector). The uniform curvature condition says that in addition, the Engel curve is either convex or concave to the origin. We also study the nonidentical preferences case and give a sufficient condition on Engel curves (called positive association) for the Weak Axiom to be satisfied in the aggregate.

KEYWORDS: Aggregate demand function, Weak Axiom, Engel curve, uniform curvature, no torsion, positive association.

1. INTRODUCTION

IT WOULD BE DESIRABLE if the aggregate demand behavior of an economy satisfied the Weak Axiom of Revealed Preference (WA). This is, after all, the most general condition on the demand side of a (regular) economy which guarantees the uniqueness of equilibrium prices. It is well known, however, that the WA is not generally fulfilled in the aggregate; see Shafer and Sonnenschein (1982) for an excellent survey on the theory of aggregation of demand. The precise delimitation of the class of economies for which the WA obtains is, nevertheless, a task worth devoting effort to. This paper is a contribution to this effort.

The aim of obtaining the WA in the aggregate admits a variety of approaches, their common feature being the imposition of restrictions on the distribution of characteristics of the agents in the economy. Those restrictions could be roughly classified as: (i) conditions on the substitution terms induced by preferences (see Mitjuschin and Polterovich, 1978), (ii) conditions on the Engel curves induced by preferences, (iii) conditions on the distribution of preferences, (iv) conditions on the distribution of income, (v) conditions on the joint distribution of preferences and income. Our work has been stimulated by a recent surprising result of Hildenbrand (see Section 3 for an account) which exploits conditions of types (iv) and (v) but, notably, imposes no strong restriction on the class of permissable preferences. We proceed, however, in the reverse direction. Strong restrictions on preferences (of type (ii)) are imposed but we allow for any (price independent) distribution of income.

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Specifically, for every range of admissible incomes we characterize the class of Engel curves with the property that if an economy has, first, a price independent distribution of income and, second, preferences which are identical across consumers and generate Engel curves in the class, then the corresponding aggregate demand function satisfies the Weak Axiom.

It turns out that this class is defined by two very simple conditions. We call them uniform curvature (UC) and no torsion (NT). Given an Engel function $f(w) \in R^l_+$ parametrized by income, the UC condition says that every component $f_i(w)$ is, linearly extended to the origin, either convex or concave. With normal goods an interpretation is that commodities admit the same classification as necessities or luxuries at any level of admissible income.

The NT condition, vacuously satisfied in the two-commodity case, says that, in the relevant range of income, the Engel curve is contained in a plane through the origin (i.e., a linear space of dimension two), which may depend on the price vector. Thus, the NT notion coincides with Muellbauer's concept of Generalized Linearity (1975). It is a restrictive condition. Its economic interpretation is that for any given price system the consumption bundles at any level of admissible income are obtained by combining the same two composite commodities.

On account of the NT condition we regard our result as being rather negative. Nevertheless, two remarks are in order. First, our conditions strictly include the linear Engel curve case and so they are perhaps weaker than one would have expected. Second, the UC and NT conditions are satisfied by some of the functional specifications used, with various motivations, in the econometrically oriented literature on demand (see references in Section 5). Because of this positive side we devote the last section of the paper (Section 6) to a brief exploration of the sufficiency theory for the nonidentical preferences case.

We would like to call attention to the recent work of Jerison (1982, 1984) which, among other things, spells out the conditions to be satisfied by the family of budget shares and income terms of an arbitrary economy in order for the WA to hold in the aggregate. Although we do not proceed this way (our research and Jerison's were done independently), there is no doubt that one could establish our results by a line of proof that took Jerison's as point of departure.

It is clear that our research is related to the theory of the representative consumer. However, we want to emphasize that, first, we have not concerned ourselves with the Strong Axiom of Revealed Preference (see Lenninghaus (1984) for the case considered by Hildenbrand (1983), also Jerison (1984)) and second, that except for the linear Engel curve case, our aggregate demand is not invariant to redistributions of income.

The plan of the paper is as follows. A preliminary result on a differential version of the WA is presented in Section 2. In Section 3, we motivate our work by analyzing the situation where the stringent hypotheses of Hildenbrand (1983) are not satisfied. The heart of the paper is in Sections 4 and 5 where we analyze first the two- and then the l-commodity case and deduce, respectively, the UC and the NT conditions. Section 6 goes on to the nonidentical preferences case. For convenience our study is placed in a smoothness framework.

2. MODEL AND PRELIMINARIES

We consider a population of consumers $t \in [0, 1]$ described by their preference relation $\succsim_t$ on R^l_+ and their income (wealth, expenditure, ...) $w_t \geq 0$.

Every t has an *individual demand function* $f(p, w_t, \succsim_t)$ which we assume is well defined and continuously differentiable for any $p \gg 0$. Also, we take $p \cdot f(p, w_t, \succsim_t) = w_t$ to be satisfied for all t and $p \gg 0$. Hypotheses on preferences guaranteeing these properties of individual demand functions are well known.

We assume that the assignment $t \mapsto (w_t, \succsim_t)$ satisfies the weak measurability and boundedness conditions which are required for the *aggregate demand function*:

$$F(p) = \int_0^1 f(p, w_t, \succsim_t)\, dt$$

to be well defined and C^1 for all $p \gg 0$.

We normalize so that $\int_0^1 w_t\, dt = 1$. We then have $p \cdot F(p) = 1$ for all $p \gg 0$.

DEFINITION: The aggregate demand function F satisfies the *Weak Axiom of Revealed Preference* (WA) if $F(p_1) \neq F(p_2)$ and $p_1 \cdot F(p_2) \leq 1$ implies $p_2 \cdot F(p_1) > 1$.

When, as in our case, F is C^1 one can state differential necessary and sufficient conditions for the WA to hold. The fact we will repeatedly use in this paper is contained in the following lemma:

LEMMA 1: *Suppose that, for all p, $v \cdot \partial F(p) v > 0$ whenever $v \cdot F(p) = 0$ and $v \neq 0$, i.e., the Jacobian matrix $\partial F(p)$ is negative definite on the subspace orthogonal to $F(p)$. Then F satisfies the WA. Conversely, if F satisfies the WA then, for all p and v with $v \cdot F(p) = 0$, we have $v \cdot \partial F(p) v \leq 0$.*

PROOF: We prove first the sufficiency part and argue by contradiction. So, suppose the WA is not satisfied.

We use Lemma 2 in Kihlstrom, Mas-Colell, and Sonnenschein (1976) which tells us that if the WA fails then there will always be violations of a particularly convenient form. Namely (see Figure 1), there is p_0 and $v \neq 0$ such that, letting $p(\alpha) = p_0 + \alpha v$ we have: $p(\alpha) \cdot F(p_0) = 1$ and $p_0 \cdot F(p(\alpha)) > 1$ for $\alpha > 0$.

Note that $p(\alpha) \cdot F(p_0) = 1$ is equivalent to $v \cdot F(p_0) = 0$.

Define $\psi(\alpha) = 1 - p_0 \cdot F(p(\alpha))$. Then $\psi(\alpha) > 0$ for $\alpha > 0$ and $\psi(0) = 0$. Also, $\psi'(\alpha) = -p_0 \cdot \partial F(p(\alpha)) \cdot v$. Because $p(\alpha) \cdot F(p(\alpha)) = 1$ is always satisfied, we have $p(\alpha) \cdot \partial F(p(\alpha)) v + v \cdot F(p(\alpha)) = 0$. Hence, at $\alpha = 0$, $\psi'(0) = -p_0 \cdot \partial F(p_0) \cdot v = 0$. Thus, the positivity of ψ implies, with $\psi(0) = \psi'(0) = 0$, that $\psi''(0)$, if it exists, is nonnegative.

Using $p_0 = p(\alpha) - \alpha v$, we can rewrite

$$\psi'(\alpha) = -p(\alpha) \cdot \partial F(p(\alpha)) \cdot v + \alpha v \cdot \partial F(p(\alpha)) v.$$

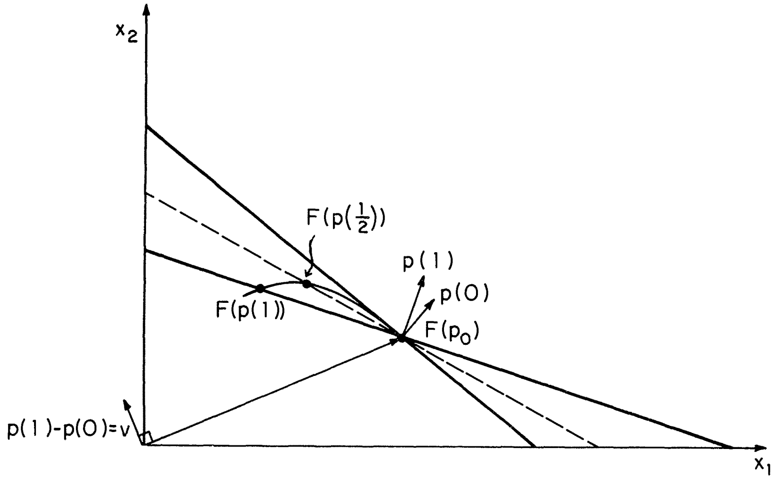

FIGURE 1

As we saw above, $-p(\alpha)\cdot \partial F(p(\alpha))v = v\cdot F(p(\alpha))$. Therefore,

$$\psi'(\alpha) = v\cdot F(p(\alpha)) + \alpha v\cdot \partial F(p(\alpha))v \qquad \text{and so}$$

$$\psi''(0) = \lim_{\alpha\to 0} \frac{\psi'(\alpha)-\psi'(0)}{\alpha} = v\cdot \partial F(p(0))v + \lim_{\alpha\to 0} v\cdot \partial F(p(\alpha))v$$

$$= 2v\cdot \partial F(p(0))v.$$

We conclude $v\cdot \partial F(p(0))v \geqslant 0$ which constitutes the desired contradiction.

For the necessity part, consider arbitrary p_0 and v with $v\cdot F(p_0)=0$. Let $p(\alpha)=p_0+\alpha v$. Then, by the *WA*, $1-p_0\cdot F(p(\alpha))\leqslant 0$ and, except for a change of sign, the proof proceeds as in the two previous paragraphs. *Q.E.D.*

REMARK 2.1: It is clear from the proof of the Lemma that the result is closely related to Lemma 1 and Theorem 2 of Kihlstrom, Mas-Colell, and Sonnenschein (1976), and it is likely that our Lemma could be obtained as a corollary of the earlier one. Nevertheless, we thought it helpful to give a direct proof.

3. UNRESTRICTED, IDENTICAL PREFERENCES AND THE WEAK AXIOM

In this and the next two sections we shall assume that consumers have identical preferences. Thus, the economy is described by the common preference relation $\succsim$ and the distribution of income whose density, which we assume to exist, we denote by ρ:

$$\rho:\mathbb{R}_+\to\mathbb{R}_+, \quad \int_0^\infty \rho(w)\,dw = 1 \quad \text{and} \quad \int_0^\infty w\rho(w)\,dw = 1.$$

The aggregate demand is then $F(p)=\int_0^\infty e(p, w, \succsim)\rho(w)\,dw$. A recent important result of Hildenbrand (1983) shows that if ρ is a nonincreasing function defined on an interval $[0, b]$, then the aggregate demand function has everywhere negative definite Jacobian matrix and so both the "law of demand," i.e., decreasing in own price demand functions, and the WA are satisfied. In this paper we concentrate only on the WA implication.

As a motivation for our work we begin by pointing out that if no restrictions are imposed on preferences, Hildenbrand's hypothesis of nonincreasing density cannot be weakened in order to obtain the WA result. This will lead us in the next two sections to reexamine the matter when restrictions are placed on the form of Engel curves.

As it is well known, for any p, w we can write, suppressing reference to the fixed $\succsim$, $\partial f(p, w)=s(p, w)-f(p, w)(\partial_w f(p, w))^T$ where $s(p, w)$ is the symmetric, negative semidefinite matrix of substitution terms and the T denotes vector transposition. Hence, setting $S(p)=\int_0^\infty s(p, w)\rho(w)\,dw$, we have:

$$v \cdot \partial F(p)v = v \cdot S(p)v - \int_0^\infty g'_{p,v}(w)g_{p,v}(w)\rho(w)\,dw,$$

where $g_{p,v}(w)=v \cdot f(p, w)$. Whenever p or v are clear from the context they will be suppressed as subindices.

From now on we let p be given. Then $v \cdot \partial F(p)v = v \cdot S(p)v - G(v)$ where $G(v)=\frac{1}{2}\int_0^\infty \partial_w[g(w)]^2\rho(w)\,dw$. Of course, if ρ is C^1 and concentrated on [0, b], we have $2G(v)=[g_v(b)]^2\rho(b)-\int_0^b [g(w)]^2\rho'(w)\,dw$. If $\rho'(w)\leq 0$ for all w, then $G(v)\geq 0$ and so, $v \cdot \partial F(p)v \leq 0$ (remember that S is negative semidefinite). If in fact $v \cdot S(p)v<0$, then $v \cdot \partial F(p)v<0$. This is a variation of Hildenbrand's result. We shall now show that if $\rho'(w)>0$ somewhere then we can find a preference relation $\succsim$, a price vector p, and $v\neq 0$ such that $v \cdot F(p)=0$ and $v \cdot \partial F(p)v>0$. By Lemma 1 this implies that F does not satisfy the WA.

So let ρ be C^1 and $\rho'(w)>0$ on the interval (a, b). Without loss of generality we take $l=2$ and construct a $\succsim$ by letting the indifference curves be L-shaped and defining the Engel function $f(w)$ associated to the price vector $p=e=(1, 1)$. We choose $f(w)$ to be C^1 and satisfy:

(i) if $w\notin(a, b)$ then $f(w)=(w/2, w/2)$,

(ii) $f(w)\neq(w/2, w/2)$ for some $w\in(a, b)$,

(iii) $\int_a^b f(w)\rho(w)\,dw=(\lambda/2, \lambda/2)$ for some $\lambda>0$.

Given (i), (iii) is equivalent to $F(p)$ being on the forty-five degree line. See Figure 2 from where it is obvious that such an f exists.

Now let $v=(1, -1)$ and $g(w)=v \cdot f(w)$. By (i) to (iii), $v \cdot F(p)=0$, $g(w)=0$ for $w\notin(a, b)$ and $g(w)\neq 0$ for some $w\in(a, b)$. Then,

$$-2G(v)=-\int_a^b \partial[g(w)]^2\rho(w)\,dw=\int_a^b [g(w)]^2\rho'(w)\,dw>0.$$

Because of the L-shaped indifference curves, $v \cdot S(p)v=0$. Hence, $v \cdot F(p)=0$ and $v \cdot \partial F(p)v>0$, which by Lemma 1 yields the desired violation of the WA. To summarize:

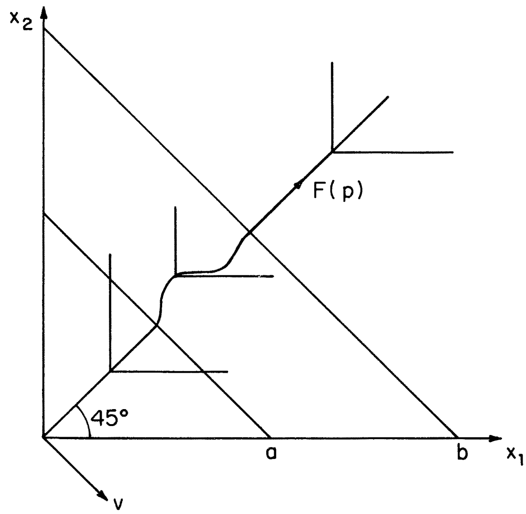

FIGURE 2

PROPOSITION 1: *If ρ is C^1 and increasing somewhere, then there is a preference relation such that the WA is not satisfied for the aggregate demand function F.*

We remark that the differentiability of ρ has only been resorted to for simplicity. The result of the next sections builds on the observation that the wiggly shape of the Engel curve of Figure 2 is an essential feature of the construction leading to a violation of the WA.

4. THE TWO COMMODITIES, IDENTICAL PREFERENCES, CASE

Let $(\alpha, \beta) \subset R^l_{++}$ be a given nontrivial interval. In this and the next section, we shall characterize the class of Engel curves on (α, β) having the property that the WA holds for every economy with distribution of income on the interval (α, β) and with identical preferences compatible with the Engel curves in the class. By Engel curve on (α, β) we mean, precisely, the image of an arbitrary function $f(p, \cdot): (\alpha, \beta) \to R^l_+$ such that $p \cdot f(p, w) = w$. This is also known as income expansion path. We refer to $f(p, \cdot)$ as the *Engel function.* Given an Engel function $f(p, \cdot)$ on (α, β) denote by $\hat{f}(p, \cdot)$ the linear extension to $(0, \beta)$, i.e., $\hat{f}(p, w) = (w/\alpha) f(p, \alpha)$ for $w \leq \alpha$ and $\hat{f}(p, w) = f(p, w)$ for $w \geq \alpha$.

As suggested in the previous section, a natural condition is to eliminate Engel curves with inflection points. We will refer to this as the *uniform curvature condition* (UC):

DEFINITION: The Engel function $f(p, \cdot) = (f_1(p, \cdot), \ldots, f_l(p, \cdot))$ on (α, β) is said to satisfy the *UC condition* if and only if every $\hat{f}_i(p, \cdot): (0, \beta) \to R_+$ is either concave or convex.

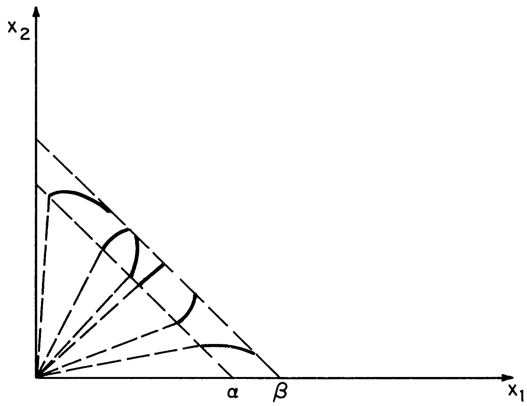

FIGURE 3

For the case $l=2$ and $p=(1,1)$, Figure 3 (resp. Figure 4) provides specimens of admissible (resp. inadmissible) Engel curves on (α, β).

The concavity (resp. convexity) of $\hat{f}_i$ has the implication that if the ith good is not inferior then it is a necessity (resp. a luxury) at all levels of income between α and β. This is clear enough since then

$$0 \leqslant \frac{\partial f_i(p, w)}{\partial w} \leqslant \frac{f_i(p, w)}{w}$$

for $a < w < \beta$; see Figure 3.

We will now proceed to show that for $l=2$ and identical preferences the UC condition, assumed for every p, is sufficient and, in a certain sense, necessary for the WA to obtain whatever the distribution of income μ with support in (α, β). Of course, for $l=2$, the concavity of $\hat{f}_1(p, \cdot)$ implies the convexity of $\hat{f}_2(p, \cdot)$ and vice versa.

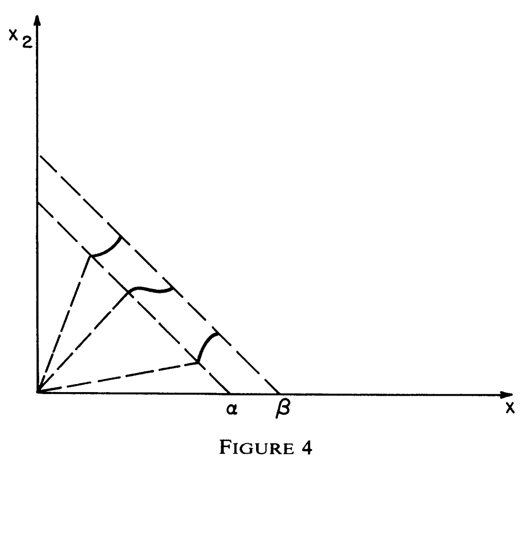

FIGURE 4

PROPOSITION 2: *Let $l=2$, the preferences of all agents be identical, and the distribution of income be supported in (α, β). Suppose that, for all p, the aggregate substitution matrix has maximal rank and the Engel function $f(p, \cdot):(\alpha, \beta) \rightarrow R_{+}^{l}$ satisfies the UC condition. Then the WA holds in the aggregate.*

Conversely, if a C^1 Engel function $f(p, \cdot)$ on (α, β) does not satisfy the UC condition then there is a preference relation $\succsim$, compatible with the Engel function $f(p, \cdot)$, and a distribution of income μ supported in (α, β) such that the aggregate demand function does not satisfy the WA.

PROOF: We first establish sufficiency. Let μ be a given distribution income supported in (α, β). As in Section 3, we need to show that for any v if we define $g(w)=v \cdot f(p, w)$, then $\int_0^{\infty} g(w)\, d\mu(w)=0$ implies $G=\int_0^{\infty} g(w) g'(w)\, d\mu(w) \geqslant 0$. Because the UC condition holds, the function $\hat{g}(w)=v \cdot \hat{f}(p, w)$ is either concave or convex on $(0, \beta)$. Indeed, v_1 and v_2 are of opposite signs and if $\hat{f}_1(p, \cdot)$ is concave, $\hat{f}_2(p, \cdot)$ is convex. Without loss of generality we assume that $\hat{g}$ is convex (replace v by $-v$ if necessary). Let $\bar{w}=\sup \{w \in(0, \beta): g(w) \leqslant 0\}$. If $\bar{w}=\beta$ then $\int_0^{\infty} g(w)\, d\mu(w)=0$ implies that $\mu(\{w: g(w)=0\})=1$ and so $G \geqslant 0$. Let $\bar{w}<\beta$. Then because of the convexity of $\hat{g}$ we have

$$g(w_1) \leqslant 0, \qquad g'(w_1) \leqslant g'(w\bar{w}) \quad \text{for} \quad \alpha<w_1 \leqslant \bar{w}, \quad \text{and}$$

$$g(w_2)>0, \qquad g'(w_2) \geqslant g'(\bar{w}) \quad \text{for} \quad \bar{w}<w_2<\beta .$$

Therefore, $g(w) g'(w) \geqslant g(w) g'(\bar{w})$ for any $w \in(\alpha, \beta)$. Integrating, we get

$$G=\int_0^{\infty} g(w) g'(w)\, d\mu(w) \geqslant g'(\bar{w}) \int_0^{\infty} g(w)\, d\mu(w)=0$$

which is what we wanted to prove.

Before proceeding with the necessity part, we state a key lemma:

LEMMA 2: *Let g be a C^1 function $g:(\alpha, \beta) \rightarrow R$. Suppose that for every measure μ on (α, β) such that $\int_0^{\infty} g(w)\, d\mu(w)=0$ we have $G=\int_0^{\infty} g(w) g'(w)\, d\mu(w) \geqslant 0$. Then for any $w_1, w_2 \in(\alpha, \beta)$ such that $g(w_1)<0<g(w_2)$ we have $g'(w_1) \leqslant g'(w_2)$.*

PROOF OF LEMMA 2: Let w_1 and w_2 satisfy $g(w_1)<0<g(w_2)$ and take a μ concentrated on w_1 and w_2 and such that $g(w_1) \mu(w_1)+g(w_2) \mu(w_2)=0$, i.e.,

$$\mu(w)=\frac{g(w_2)}{g(w_2)-g(w_1)} .$$

The nonnegativity of G evaluated according to this μ yields:

$$\frac{g(w_2)}{g(w_2)-g(w_1)}[g'(w_1) g(w_1)-g'(w_2) g(w_2)]+g'(w_2) g(w_2) \geqslant 0$$

which simplifies to $g'(w_1) \leqslant g'(w_2)$. *Q.E.D.*

Suppose now that the WA holds for any μ on (α, β) and that the C^1 Engel function $f(p, \cdot)$ on (α, β) does not satisfy the UC condition (to save notation we drop reference to p from now on).

Assume first that $f_1'(\cdot)$ is nonmonotonic on (α, β). Equivalently, we can choose $\alpha < w_1 < w_2 < \beta$ such that $f'(w_1) = f'(w_2)$ and $f'(w) \neq f'(w_1)$ for all $w \in (w_1, w_2)$.

We can pick a $\bar{w} \in (w_1, w_2)$ such that $f'(\bar{w}) \neq (1/\bar{w})f(\bar{w})$. To see this suppose that $f'(w) = (1/w)f(w)$ for all $w \in [w_1, w_2]$. Then on $[w_1, w_2]$ f is C^2 and $wf''(w) + f'(w) = f'(w)$, i.e., $f''(w) = 0$. Hence $f'(w)$ should be constant on $[w_1, w_2]$ and this contradicts our hypothesis.

Choose a v such that $v \cdot f(\bar{w}) = 0$. Denote $g(w) = v \cdot f(w)$. Because v is not collinear to p we have that for any s the equation system $v \cdot f'(w) = s$, $p \cdot f'(w) = 1$ has at most one solution $f'(w)$. Hence $f'(w_1) \neq f'(\bar{w})$ implies $g'(w_1) \neq g'(\bar{w})$. Also, $(1/\bar{w})v \cdot f(\bar{w}) = 0$ and $(1/\bar{w})p \cdot f(\bar{w}) = 1$. Therefore, $g'(\bar{w}) = v \cdot f'(\bar{w}) = 0$ would yield $f'(\bar{w}) = (1/\bar{w})f(\bar{w})$ against our hypothesis. Hence $g'(\bar{w}) \neq 0$. Without loss of generality (replace v by $-v$ if necessary) we can take $g'(\bar{w}) > 0$.

The conclusion of Lemma 2 must hold for the g defined in the previous paragraph. Otherwise, we can obtain a violation of the WA by applying Lemma 1 to an economy with preferences constructed from f by means of L-shaped indifference curves (as was done in Section 3). Let $g(w) > 0$. Because $g'(\bar{w}) \neq 0$ we can find $w_n \to \bar{w}$ with $g(w_n) < 0$. By the conclusion of Lemma 2, $g'(w_n) \leq g'(w)$. Since $g'(w_n) \to g'(\bar{w})$ we get:

(i) $g'(\bar{w}) \leq g(w)$ whenever $g(w) > 0$, $w \in (\alpha, \beta)$.

Approaching $\bar{w}$ by a sequence w_n satisfying $g(w_n) > 0$ we obtain:

(ii) $g'(w) \leq g'(\bar{w})$ whenever $g(w) < 0$, $w \in (\alpha, \beta)$.

We reach a contradiction by spelling out the implications of (i) and (ii) for the values $s = g'(w_1) = g'(w_2)$ and $g(w_1)$, $g(w_2)$.

Because of (i), $g'(\bar{w}) > 0$ and $w_2 > \bar{w}_2$, we have $g(w_2) > 0$ and $sg'(w_2) \geq g'(\bar{w})$. Since $s \neq g'(\bar{w})$, we conclude $s > g'(\bar{w}) > 0$. Because of this, $s = g'(w_1)$ and (ii) we cannot have $g(w_1) < 0$. On the other hand, if $g(w_1) \geq 0$ then the fact that $g'(w_1) > 0$ and $g(\bar{w}) = 0$ implies that g reaches a strictly positive local maximum at some $w \in (w_1, \bar{w})$. But then $g'(w^*) = 0$ and $g(w^*) > 0$ which contradicts (i).

We must conclude therefore that $f_1'(w)$ is monotonic (say, nondecreasing) on (α, β). If the UC condition is not satisfied this means that $f_1(\alpha)/\alpha > f_1'(w_1)$ and $f_1'(w_1) < f_1'(w_2)$ for some $\alpha < w_1 < w_2 < \beta$; see Figure 5. Pick $v \neq 0$ such that $v \cdot f(w_1) = 0$ and $v \cdot f(\alpha) \leq 0$. Define $g(w) = v \cdot f(w)$. Then on (α, β) the function g must be as in Figure 6, i.e., $g(\cdot)$ is concave, $g(\alpha) < 0$ and $g'(w') < g(w_1)$ for some $w' > w_1$ with $g(w') > 0$. Therefore, this g violates the conclusion of Lemma 2 which, again, leads to a contradiction of the hypothesis that the WA holds for any μ on (α, β). Hence, the UC condition must be satisfied. *Q.E.D.*

REMARK 4.1: For the sake of convenience we have placed our analysis in a smoothness setting. It seems to us that by the use of approximation techniques all the results could be extended to the continuous case. Also, the nonsingularity hypothesis on the substitution matrix is made in Proposition 2 only for convenience. It can be dispensed with.

REMARK 4.2: The income distribution by which we have established, via Lemma 2, the necessity of the UC condition is not unimodal. It should be possible,

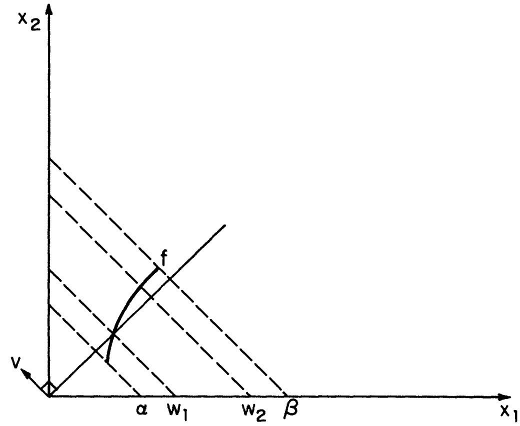

FIGURE 5

but less simple, to carry out the construction with unimodal distribution. This is a direction of improvement of our result.

REMARK 4.3: It should be emphasized that we obtain the fulfillment of the WA, i.e., the negative definiteness of $\partial F(p)$ on $\{v: v \cdot F(p)=0\}$, but we do not obtain, as Hildenbrand does, that $\partial F(p)$ be negative definite on the entire $\mathbb{R}^l$. In fact, it is easy to verify, using the techniques of the proof of Proposition 2, that unless we restrict ourselves to the class of linear Engel curves there is always a distribution of income for which a violation of the negative definiteness of $\partial F(p)$ obtains. Notice also that the negative definiteness of $\partial F(p)$ rules out Giffen goods and that the UC condition allows for them (see Figure 3).

REMARK 4.4: The UC condition allows a good to be a necessity at some price vector and a luxury at a different one. The uniformity is with respect to w. This is not a minor point. Suppose, for example, that we have a C^4, strictly concave, symmetric utility function of the form $u(x_1, x_2)=v(x_1)+v(x_2)$. Let $\gamma(t)=-v'(t)/v''(t)$. A sufficient condition for the UC condition to be satisfied on $(0, \infty)$ is that $\gamma(\cdot)$ be either concave or convex. The family of Engel curves for the two cases are represented in Figure 7. From the figure we can see that a good being a necessity or a luxury depends on the relative prices.

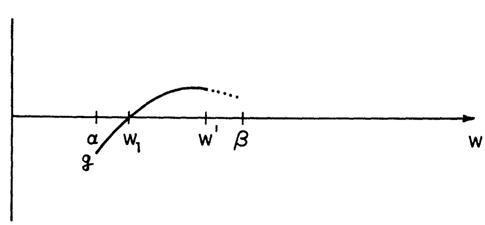

FIGURE 6

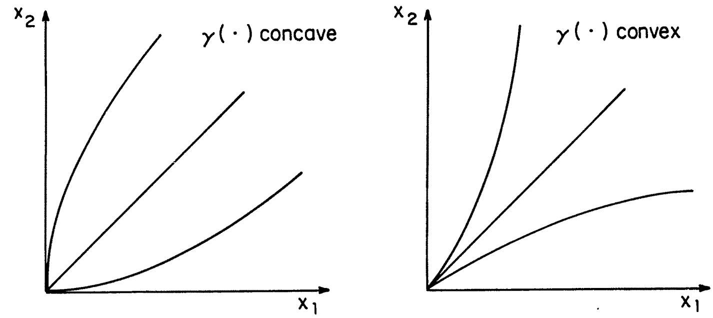

FIGURE 7

REMARK 4.5: For a model where commodities are to be thought of as broad aggregates (as should be the case if l is small), we view the UC condition, even on $(0, \infty)$, as reasonably weak. It should be pointed out, however, that the condition is untenable on $(0, \infty)$ for very disaggregated models (see Houthakker (1953)). Inferior goods, for example, are then the rule rather than the exception, but the UC hypothesis on $(0, \infty)$ does not allow for them.

REMARK 4.6: The requirement that the distribution of income be price-independent cannot be dispensed with. Thus, the situation where income arises from the selling of initial endowments is covered only in the very special case where all the individual initial endowments are collinear. The preferences of the example in Figure 8 satisfy the UC condition, but instead of a distribution of income we have a distribution of (noncollinear) endowments. In the figure we

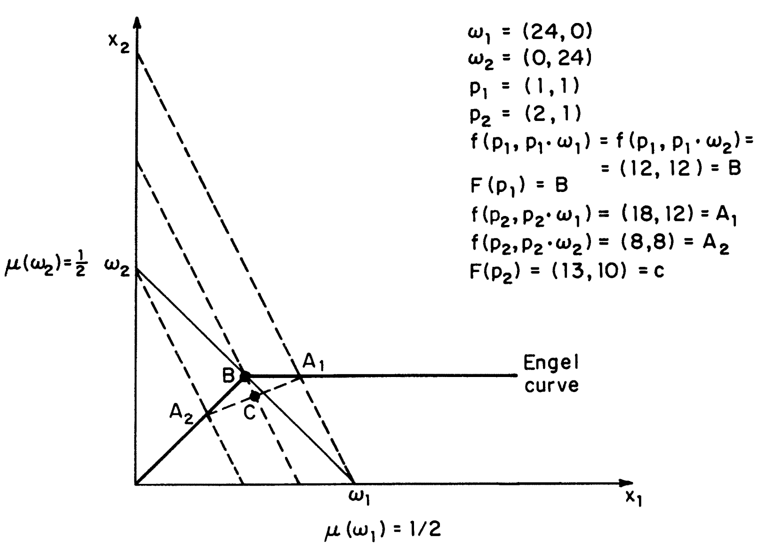

FIGURE 8

implicitly assume that indifference curves are L-shaped. (A smoothed out version can be easily obtained.) When $(1, 1)$ is the price system, the two types of individuals have the same income. They both consume the commodity bundle B, which is, therefore, also the aggregate demand. But for the price system $(2, 1)$, individuals of Type 1 have an income of 48 and will consume A_1, while those of Type 2 have an income equal to 24 and consume A_2. The aggregate demand is thus the consumption bundle C, and a violation of the WA is obtained.

REMARK 4.7: It is well known that if preferences are identical and Engel curves linear, then the WA holds for any distribution of income, price dependent or not. This is a consequence of Gorman's aggregation theorem (see Deaton and Muellbauer, 1980a, Ch. VI) and it seems to us that the result cannot be improved. What we have shown (so far for $l=2$) is that if the distribution of income is price-independent the result can be generalized to allow for certain types of nonlinear Engel curves, namely, those satisfying the UC condition.

REMARK 4.8: The term "uniform curvature" may be slightly misleading. Geometrically, besides the uniform curvature proper of the Engel curve on (α, β), we also require a more global condition, namely, that the uniform curvature be preserved when the Engel curve is extended linearly to the origin. Of course, this second condition holds vacuously when $\alpha = 0$.

5. THE l COMMODITIES, IDENTICAL PREFERENCES CASE

As in Section 4 for the case of two commodities, we shall now characterize for the general l commodity case, the class of Engel curves on (α, β) with the property that for any distribution of income supported on (α, β) among consumers with identical preferences having Engel curves in the class, the aggregate demand function satisfies the WA. The class of permissible Engel curves turns out to be defined by quite restrictive conditions. Indeed, we shall now have to require that, whatever the number of commodities, the Engel curve must be contained in a plane through the origin, i.e., in a two-dimensional linear space. The plane may, however, depend on the particular price system. Formally, we have the following definition.

DEFINITION: The Engel function $f(p, \cdot)$ on (α, β) satisfies the *no torsion* (NT) condition if $f(p,(\alpha, \beta))$ is contained in a plane through the origin.

For the geometric ideas behind the term *torsion* see, for example, do Carmo (1976). The term *generalized linearity* has been used by Muellbauer (1975).

We have the following proposition.

PROPOSITION 3: *Given a C^1 Engel function $f(p, \cdot)$ on (α, β) if the WA is satisfied in the aggregate for every economy with distribution of income supported on (α, β) and identical preferences compatible with $f(p, \cdot)$, then the Engel curve must satisfy the NT condition on (α, β).*

PROOF: Let $f(w)=f(p, w)$ be the given Engel function. Fix $\alpha<\bar{\alpha}<\bar{\beta}<\beta$. Without loss of generality it suffices to prove that $f([\bar{\alpha}, \bar{\beta}])$ is contained in a plane. To this effect, denote by M the linear space spanned by $\{f(\bar{\alpha}), f(\bar{\beta})\}$. Of course, M is at most two-dimensional (i.e., either a straight line or a plane through the origin), and so we are done if we show that $f([\bar{\alpha}, \bar{\beta}]) \subset M$.

Suppose, by way of contradiction, that this was not the case, i.e., $f(w) \notin M$ for some $\bar{\alpha}<w<\bar{\beta}$. Take then a vector v with $v \cdot f(w)>0$, $v \cdot f(\bar{\alpha})<0$, $v \cdot f(\bar{\beta})<0$. (Such a vector exists: let v' be such that $v' \cdot x=0$ for all $x \in M$ and $v' \cdot f(w)=1$; put then $v=v'-\varepsilon f(\bar{\alpha})$.) We are now in a familiar situation. The function $g(w)=v \cdot f(w)$ is as in Figure 9. Hence it violates the conclusion of Lemma 2. As in the proof of Proposition 2, this leads to a violation of the WA by applying Lemma 1 to an economy constructed from f by means of L-shaped indifference curves (as was done in Section 3). Therefore, we must conclude that $f([\bar{a}, \bar{\beta}]) \subset M$.
Q.E.D.

The combination of NT and UC yields the sufficiency result. Observe to that effect that if any particular Engel curve $f(p, \cdot)$ defined on (α, β) is contained in a plane through the origin, then it can be written as $f(p, w)=f_1^*(p, w) e_1(p)+f_2^*(p, w) e_2(p)$ where $e_1(p)$, $e_2(p)$ are positive vectors (i.e., composite commodities) and $f_1^*(p, w)$, $f_2^*(p, w)$ are nonnegative real numbers. Then it is simply verified that if the UC condition is satisfied on (α, β) for f then it is also satisfied for f_1^* and f_2^*. Therefore, with the same proof as that of Proposition 2, and taking into account Proposition 3, we get the following proposition.

PROPOSITION 4: *Let the preferences of all agents be identical and the distribution of income be supported on (α, β). Suppose that, for all p, the aggregate substitution matrix has maximal rank and the Engel function $f(p, \cdot):(\alpha, \beta) \rightarrow \mathbb{R}_+^l$ satisfies the NT and UC conditions. Then the WA holds in the aggregate.*

Conversely, if a C^1 Engel function does not satisfy both the NT and UC conditions on (α, β), then there is a preference relation $\succsim$, inducing the Engel function $f(p, \cdot)$, and a distribution of income μ supported on (α, β) such that the aggregate demand function does not satisfy the WA.

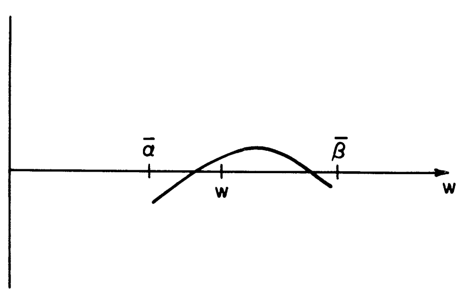

FIGURE 9

REMARK 5.1: As with *uniform curvature* when $\alpha > 0$, there is some abuse of language in the use of the term *no torsion.* In its strict mathematical sense, it only imples that the Engel curve is contained in a plane while our necessary condition also requires that the plane passes through the origin.

REMARK 5.2: The class of preferences with Engel curves satisfying the NT and the UC conditions is limited but it includes more than the linear Engel curve, or Gorman, class. As a trivial example, the utility function $U(x_1, x_2, x_3) = \log x_1 + \log x_2 + x_3^{1/2}$, which we borrow from Shafer and Sonnenschein (1982), is permissible albeit the Engel curves are not linear. A procedure to construct examples consists in letting $f(w)$ be an arbitrary increasing curve satisfying NT and UC and define then the preference relation by specifying that indifference surfaces be "L-shaped." A more interesting source of examples can be found in the econometrically oriented demand analysis literature. Thus, our conditions are satisfied by preferences generated by the transcendental logarithmic indirect utility functions of Christensen, Jorgenson, and Lau (1975), or belonging to Muellbauer's PIGL class (1975), or compatible with Deaton and Muellbauer's Almost Ideal Demand System (1980b). The analysis of Gorman (1981) helps one to understand why the NT condition may often be a consequence of the aggregation requirements imposed in the empirical research.

REMARK 5.3: In an economy with identical preferences and price-independent distribution of income, if the WA is satisfied in the aggregate and the NT condition holds, then the Strong Axiom is also satisfied. That the NT condition has this implication follows from the work of Jerison (1982). The example of Lenninghaus (1984), where the Weak but not the Strong Axiom is satisfied in the aggregate, is based on an indirect addilog utility function. Except for trivial cases this class of utility functions violates the NT condition for any price vector.

REMARK 5.4: It is a notable fact that a wide variety of demand aggregation problems lead to the NT condition. For example, the aggregation of the WA (this paper), the aggregation of symmetry properties of the substitution matrix (Jerison, 1982, Shafer, 1978) or the aggregation of certain budget shares properties (Muellbauer, 1975). Thus, the NT condition must be very close to the heart of the matter.

6. THE GENERAL CASE: NONIDENTICAL PREFERENCES

In this section we argue that the analysis of the previous sections has implications for the case where consumers' preferences are not identical. We should begin by remarking that our results do not generalize easily because, as is well known, the WA is not additive across groups of consumers and our previous analysis does not yield the negative definiteness of $\partial F(p)$ on $\mathbb{R}^l$ (see Remark 4.3) which would be the property preserved under addition. So we shall settle in this section for obtaining a rather coarse sufficient condition.

We first hypothesize that income and preferences are independently distributed with distributions μ and ν. For the technical aspects of the preference distribution concept, see Hildenbrand (1974). Preferences are assumed smooth. For every p, we than have a (cross-section) aggregate Engel function $\bar{f}(p, w) = \int f(p, w, \succsim)\, d\nu(\succsim)$. It goes without saying that the independence hypothesis is very strong. It may be possible to weaken it at the cost of a more complex formulation.

Once we have a concept of aggregate Engel function $\bar{f}(p, w)$, it is clear that we shall require that it satisfy, on the support of μ, the no torsion and uniform curvature conditions of the previous sections. Restrictive as this is, we may mention that *aggregate* Engel curves fulfilling NT and UC have been hypothesized in econometric studies on consumer demand (Leser, 1963; Deaton and Muellbauer, 1980a).

The independence, NT and UC conditions do not yet imply the WA. Consider the example of Figure 10 where the two consumers have the same income $w = 24$ (hence, independence is satisfied) and, as usual, we take preferences to be L-shaped. The UC condition is obviously satisfied. In the figure, with the price system $(6, 6)$, individual demands are at points A_1 and A_2 and the aggregate demand is $F(p_1) = (2, 2)$. With price system $(8, 4)$, individual choices are B_1 and B_2. Since $F(p_1)$ is the average of B_1 and B_0, the aggregated demand $F(p_2)$, which is the average of B_2 and B_1, is between $F(p_1)$ and B_2. Hence a violation of the Weak Axiom is obtained.

Heuristically, the reason for the failure of the WA in the example is that while, given independence, the NT and UC conditions can be trusted to insure a "nonperverse" relation between average consumptions and average marginal propensities to consume across levels of income, the conditions are mute on what

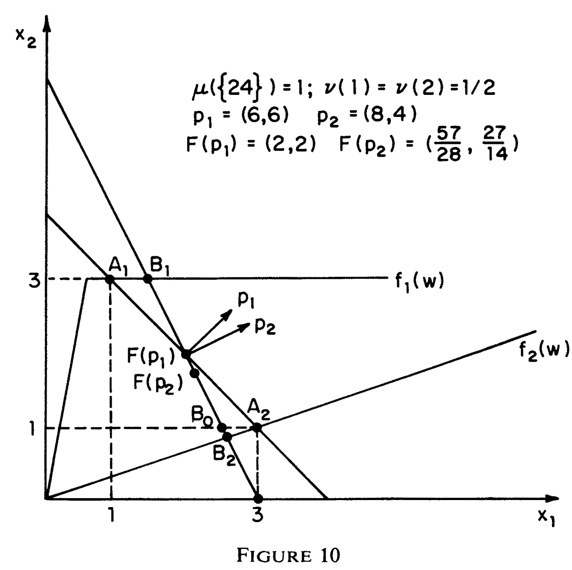

FIGURE 10

happens for a fixed level of income across different preferences. Indeed, we see in the figure that for the level of income $w=24$ the association is negative. Agent 1 consumes more of good 2 than agent 2, but he is "saturated" with this good and so his marginal propensity to consume it is zero while that of agent 2 is positive.

This will prompt our last assumption.

DEFINITION: We say that *positive association* (PA) holds at p, w if the "covariance" matrix,

$$A(p, w)=\int (f(p, w, \succsim)-\bar{f}(p, w))\,(\partial_w f(p, w, \succsim)-\partial_w \bar{f}(p, w))^T d\nu(\succsim),$$

is positive semidefinite.

Notice that if preferences are homothetic positive association is always satisfied. This should help interpret the PA condition. Roughly, the condition says that, given p and w, on average for any commodity (simple or composite) agents with larger than average marginal propensities to consume tend to consume more than the average. As in the example, the condition will not hold if there are too many consumers close to saturation in some particular goods. A general form of the PA condition has, independently, been used by Jerison (1982) with a purpose similar to ours.

The aggregate demand function is $F(p)=\int \bar{f}(p, w)\, d\mu(w)$. By simple manipulations we can write:

$$\partial F(p)=\int s(p, w, \succsim)\, d(\mu(w)\times \nu(\succsim))-A(p, w)\, d\mu(w)$$

$$-\int \bar{f}(p, w)(\partial_w \bar{f}(p, w))^T\, d\mu(w).$$

The first term of this sum is a substitution matrix and so, negative definite. If the PA condition is satisfied for all p and w, then the second term is negative semidefinite. Finally, if the NT and UC conditions hold for the aggregate Engel curve $\bar{f}(p, w)$, then as in Propositions 2 and 4, the third will also be negative semidefinite on $\{v\colon\ v\cdot F(p)=0\}$. We conclude, therefore, with the following proposition:

PROPOSITION 5: *Suppose that the distribution of preferences and income are independent and that the positive association condition holds for all p, w. If for all p the aggregate Engel curve fulfills, on the range of possible incomes, the no torsion and uniform curvature conditions, then the WA holds for the aggregate demand function.*

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