Cost Share Equilibria: A Lindahlian Approach*

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Received August 11, 1986; revised June 1, 1988

We offer a new formalization of Lindahl's equilibrium notion for public goods which yields an endogenous theory of profit distribution in line with the benefit approach to taxation. Increasing returns in the production of public goods are not a priori excluded. Our equilibrium notion coincides with Kaneko's "Ratio Equilibrium" for economies with only one public good. By reinterpreting the commodity space, the equilibrium concept can be applied to economies with purely private goods or with externalities. In the pure private good case, our concept singles out allocations that are efficient and where individual net payments agree with average cost. Journal of Economic Literature Classification Numbers: 021, 022, 024, 322.

I. INTRODUCTION

This paper offers a new equilibrium model for the benefit approach to the allocation of public good costs. Our starting point is the classical contribution of Lindahl [8, 13].

We take the desiderata implicit in Lindahl's analysis to be:

(a) the financing scheme should be decentralized;

(b) it should guarantee optimal outcomes;

* This paper originates in a collective project on the economics of public firms supported by the Instituto Nacional de Industria (Ministry of Industry, Madrid). Previous versions were presented at the University of California, Berkeley and San Diego, Stanford University, and at the 5th World Congress of the Econometric Society (MIT, August 1985). We thank our audiences for encouragement and helpful suggestions. We are also indebted to B. Grodal, H. Moulin, J. Ostroy, and an anonymous referee. The first author acknowledges the financial support of the National Science Foundation and a Guggenheim Fellowship.
(c) the contributions of people benefiting from the public goods should exactly cover their costs; moreover, individual contributions should be in line with individual benefits.

Our formulation attempts to satisfy these requirements. We, as did Lindahl, neglect incentive aspects. Foley [3] (and Fabre-Sender [2]) proposed the interpretation of Lindahl Equilibrium that has gained the widest currency. He reinterpreted the public good problem in terms of jointly produced private goods, and then used the "personalized" prices that arise from the Walrasian approach.

Our approach is not radically different. In fact, it coincides with Foley's for the case of convex economies with constant returns (this is the "easy" case where a precise formulation of the third desideratum is straightforward). However, for the general variable return case we see two drawbacks in Foley's idea. First, production must take place at a price-taking, profit maximizing point: this precludes the existence of equilibrium if increasing returns are present. Such an automatic failure will not arise in our model. Second, if profits are positive (i.e., if there are decreasing returns in the production of public goods) they must be distributed ("given back") in accordance with some exogenously given profit shares. The same is true for "losses" if increasing returns are allowed and profit maximization is replaced by marginal cost pricing (see, e.g., Guesnerie [4]). It seems to us that this does not agree well with the third desideratum. We shall aim at an endogenous theory of "profit" distribution that embodies the principle of individual payment according to individual benefit.

For the case of one public good our proposal is not new: it coincides with Kaneko's [7] notion of Ratio Equilibrium. The latter can be easily described for this one good case. There is a cost function $C(y)$ and every agent must pay a share $r_iC(y)$. The shares are in Ratio Equilibrium if, given them, there is unanimity on the desired level of public good production. It is easily seen that this equilibrium generates an optimal allocation. Note that there is no free lunch in theoretical analysis. The need for a profit maximization condition is replaced, relative to Foley's theory, by a stronger informational requirement: individual agents must know the cost function.

Our main equilibrium concept (Balanced Linear Cost Share Equilibrium) generalizes the cost sharing idea to any number of public goods. This is not done, let us say it once and for all, for the sake of generality but because from the general public good result we can derive significant corollaries for a variety of interesting cases (e.g., pure private goods, externalities).

The paper is organized as follows:

Section II presents the basic model and a first notion of equilibrium:
Linear Cost Share Equilibrium. This equilibrium concept does not require profit maximization and it may exist in the increasing returns case (for instance, it will exist if consumers are identical). In the convex case it always exists and it is shown that Linear Cost Share Equilibria are in one-to-one correspondence with the Foley equilibria obtained by varying the profit share parameters. Thus, the notion of Linear Cost Share Equilibrium does not yet contain a theory of profit distribution.

The center of this paper is Section III: it presents our proposal for formalizing Lindahl's idea. It takes the analysis of Section II one step forward and focuses on a special kind of Linear Cost Share Equilibrium that we call Balanced Linear Cost Share Equilibrium. We show that the theory is then determinate (hence it does not even implicitly contain profit distribution parameters). We establish existence for the situation where the technology is convex and there are no public bads.

Section IV spells out some of the implications of the general analysis for environments with pure private goods or with externalities. An interesting fact is that in the pure private goods case the notion of Balanced Linear Cost Share Equilibrium, under differentiability, picks up exactly the allocations which are optimal and compatible with payment at average cost (hence as a corollary we get the existence of such allocations when the technology is smooth and convex).

Finally, Section V mentions some extensions and discusses their degree of difficulty.

II. Linear Cost Share Equilibria

II.1. The Basic Model

We shall study a model with \( N \) public goods \( y = (y_1, ..., y_N) \geq 0 \) producible from a single input. Taking the latter as numéraire the technology is given to us as a single cost function \( C(y) \). We assume throughout that \( C(0) = 0 \) and that \( C \) is strictly increasing, unbounded above, and continuous.

It will be seen in Section IV that, although formally we study a pure public goods situation, the model admits a broad spectrum of interpretations. We mention in Section V several possible generalizations of the basic environment.

There are \( M \) consumers, each endowed with a strictly positive amount of numéraire \( \omega_i > 0 \) and with a utility function \( u_i(y, x_i) \) for amounts \( y \geq 0 \) of public goods and of numéraire \( x_i \geq 0 \). We assume that every \( u_i \) is continuous and increasing in the amount of numéraire and that it satisfies the "convexity of preferences" assumption: \( u_i((1 - t)(y^0, x_i^0) + t(y^1, x_i^1)) > u_i(y^0, x_i^0) \) whenever \( t \in (0, 1) \) and \( u_i(y^1, x_i^1) > u_i(y^0, x_i^0) \).
A state of the economy is a vector \((y, x) \in R^N \times R^M\). A state \((y, x)\) is feasible if \(C(y) \leq \sum_i (\omega_i - x_i)\). A feasible state \((y, x)\) is optimal if there is no other feasible state \((y', x')\) such that \(u_i(y', x_i') > u_i(y, x_i)\) for all \(i\), with strict inequality for at least one \(i\).

II.2. Cost Share Functions and Cost Share Equilibrium

**Definition 1.** A cost share system is a family of \(M\) functions \(g_i: R^N_+ \to R\) such that \(g_i(0) = 0\) and \(\sum_i g_i(y) = C(y)\) for all \(y\).

Note that by requiring \(g_i(0) = 0\) we are ruling out the possibility of lump-sum transfers unrelated to the production of public goods.

Following Lindahl, and as it is customary in normative public good economics, our notion of equilibrium will be one of unanimity.

**Definition 2.** A Cost Share Equilibrium is a pair formed by a feasible state \((\bar{y}, \bar{x})\) and a cost share system \(g = (g_1, \ldots, g_M)\) with the property that, for every \(i\), \(x_i = \omega_i - g_i(\bar{y})\) and \(u_i(\bar{y}, \bar{x}_i) \geq u_i(y, \omega_i - g_i(y))\) for all \(y\).

**Proposition 1.** Any Cost Share Equilibrium yields an optimal state.

*Proof.* Let \((\bar{y}, \bar{x})\) be the Cost Share Equilibrium state. Take any other feasible state \((y, x)\). Then \(u_i(y, \omega_i - g_i(y)) \leq u_i(\bar{y}, \bar{x}_i)\) for all \(i\), and \(\sum_i (\omega_i - x_i) \geq C(y) = \sum_i g_i(y)\). Suppose that \(u_i(y, x_i) > u_i(\bar{y}, \bar{x}_i)\) for some \(i\). Then \(x_i > \omega_i - g_i(y)\) and so, \(x_h < \omega_h - g_h(y)\) for some \(h\). This yields \(u_h(y, x_h) < u_h(y, \omega_h - g_h(y)) \leq u_h(\bar{y}, \bar{x}_h)\). Therefore \((\bar{y}, \bar{x})\) is an optimum.

Q.E.D.

From now on we shall become much more specific and study cost share systems which are linear (in an appropriate sense). This is the natural thing to do since the set of public goods projects is endowed with a linear structure and we are interested in obtaining an equilibrium notion that can be compared to the usual variety of Lindahl equilibrium.

**Remark 1.** We refer to Mas-Colell [9] for a study where the set of public projects has no linear structure. The general cost share equilibria defined here constitute a special case of the “valuation equilibria” defined there (note that by construction any vector of public goods is profit maximizing for the “nonlinear price” \(\sum_i g_i(y) = C(y)\). Indeed, profits are identically zero by construction).

**Remark 2.** When the functions \(g_i\) are nonnegative, the allocation that arises from a Cost Share Equilibrium is not only an optimum, but also it belongs to the core (as defined in Foley [3]). The proof of Proposition 1 applies with almost no modification.
II.3. Linear Cost Share Equilibrium

We shall consider cost share systems of the form \( g_i(y) = a_i \cdot y + b_i C(y) \), where \( b_i \geq 0 \) is a constant and \( a_i \in \mathbb{R}^N \). We put \( b = (b_1, \ldots, b_M) \in \mathbb{R}^M \) and \( a = (a_1, \ldots, a_M) \in \mathbb{R}^{MN} \).

**Remark 3.** Besides the justification preceding Remark 1, we may also point out that one can arrive at the previous linear form from considerations of convexity of budget sets. For example, suppose there is a function \( h_i(y, C) \) such that for any convex \( C(y) \) the functions \( g_i(y) = h_i(y, C(y)) \) are convex and constitute a cost share system. Then the \( g_i \) functions must have the form \( g_i(y) = a_i \cdot y + b_i C(y) \), \( b_i \geq 0 \).

The coefficients \( a_i, b_i \) cannot be unrestricted. Suppose that \( C(y) \) is twice continuously differentiable and that \( (\partial^2 C/\partial y_j \partial y_k)(\tilde{y}) \neq 0 \) for some \( \tilde{y} \) and some \( j, k \) (possibly \( j = k \)). We always have \( \sum_i (a_i \cdot y + b_i C(y)) = C(y) \) for all \( y \). So, if we differentiate first with respect to \( y_j \), then with respect to \( y_k \) and finally, we evaluate at \( \tilde{y} \), we can conclude that \( \sum_i b_i = 1 \) and \( \sum_i a_i = 0 \) for all \( j \). These, in turn, are sufficient conditions for the linear expressions to constitute a cost share system.

**DEFINITION 3.** A cost share system \( g = (g_1, \ldots, g_M) \) is linear if \( g_i(y) = a_i \cdot y + b_i C(y) \), where \( b_i \geq 0 \), \( \sum_i b_i = 1 \), and \( \sum_i a_i = 0 \). A linear cost share equilibrium is defined accordingly.

The interpretation is clear enough. The \( b_i \) parameters are direct cost share parameters while the \( a_i \), which can be positive or negative, are more in the nature of side compensations based on consumption.

**Remark 4.** The restrictions \( \sum_i b_i = 1, \sum_i a_i = 0 \) are not implied by the argument previous to Definition 3 only in the extreme case where \( C(y) \) is linear. But there is no loss of generality in imposing it as a restriction for this case also.

The \( a_i \) coefficients, although similar, should not be confused with the personalized prices appearing in the Lindahl-Foley theory. In the latter the description of the economic environment must be completed by specifying profit shares \( \theta_i \geq 0, \sum_i \theta_i = 1 \). One then has the following notion of equilibrium:

**DEFINITION 4.** The feasible state \( (\tilde{y}, \tilde{x}) \) is a Lindahl-Foley equilibrium with respect to the profit shares \( \theta = (\theta_1, \ldots, \theta_M) \) and the system of personalized prices \( (p_1, \ldots, p_M) \in \mathbb{R}^{NM} \) if, letting \( q = \sum_i p_i \),

\[
\begin{align*}
\text{(a) } & \tilde{y} \text{ maximizes profits } q \cdot y - C(y), \\
\text{(b) } & \text{for every } i, (\tilde{y}, \tilde{x}_i) \text{ maximizes utility } u_i(y, x_i) \text{ subject to } p_i \cdot y + x_i \leq \omega_i + \theta_i [q \cdot \tilde{y} - C(\tilde{y})].
\end{align*}
\]
What is the relationship between Linear Cost Share Equilibria and Lindahl–Foley Equilibria? If for the Linear Cost Share Equilibria we count equations (feasibility and first-order conditions) and unknowns, we will notice that there is an excess of $M - 1$ of the latter. A correspondence of these $M - 1$ degrees of freedom with the profit shares of the Lindahl–Foley theory suggests itself. The next three propositions make this precise.

**Proposition 2.** If $(\tilde{y}, \tilde{x})$ is a Lindahl–Foley equilibrium with respect to profit shares $\theta$ and prices $p$, then it arises from a Linear Cost Share Equilibrium for the parameters $a_i, b_i$, where $b_i = \theta_i$ and $a_i = \pi_i - \theta_i(\sum_j p_j)$.

**Proof.** By assumption, for all $y \in R^N_+$, $u_i(y, \omega_i, p_i, \tilde{y} \mid \theta_i[(\sum_j p_j) \cdot \tilde{y} - C(\tilde{y})]) \geq u_i(y, \omega_i - p_i, \tilde{y} + \theta_i[(\sum_j p_j) \cdot \tilde{y} - C(\tilde{y})])$. Since, again by assumption, $\tilde{y}$ maximizes profits and $u_i$ is increasing in $x_i$, we have $u_i(y, \omega_i - p_i, \tilde{y} + \theta_i[(\sum_j p_j) \cdot \tilde{y} - C(\tilde{y})]) \geq u_i(y, \omega_i - p_i, \tilde{y} + \theta_i[(\sum_j p_j) \cdot \tilde{y} - C(\tilde{y})])$, for all $y \in R^N_+$. Combining these inequalities and writing $b_i = \theta_i$, $a_i = \pi_i - \theta_i(\sum_j p_j)$ we get $u_i(y, \omega_i - a_i, \tilde{y} - b_iC(\tilde{y})) \geq u_i(y, \omega_i - a_i, \tilde{y} - b_iC(\tilde{y}))$, for all $y \in R^N_+$. Q.E.D.

**Proposition 3.** Assume that $C$ is convex and let $(\tilde{y}, \tilde{x})$ arise from a Linear Cost Share Equilibrium. Then $(\tilde{y}, \tilde{x})$ is a Lindahl–Foley Equilibrium for some profit shares $\theta$ and personalized prices $p$.

**Proof.** Consider consumer $i$. Let $a_i, b_i$ be the parameters of the Linear Cost Share Equilibrium. Define the convex sets:

$$C_i = \{(y, x_i) \in R^N_+ \times R \mid x_i \leq \omega_i - a_i \cdot y - b_i C(y)\},$$

$$U_i = \{(y, x_i) \in R^{N+1}_+ \mid u_i(y, x_i) > u_i(\tilde{y}, \tilde{x_i})\}.$$

Using the Separating Hyperplane Theorem, we can find a nonzero vector $(\lambda_i; \mu_i) = (\lambda_i; \mu_{i1}, ..., \mu_{iN}) \in R^{1+N}$ such that

$$\lambda_i x_i + \mu_i \cdot y \leq \lambda_i \tilde{x_i} + \mu_i \cdot \tilde{y}, \text{ for all } (y, x_i) \in C_i,$$

$$\lambda_i x_i + \mu_i \cdot y \geq \lambda_i \tilde{x_i} + \mu_i \cdot \tilde{y}, \text{ for all } (y, x_i) \in U_i.$$

Since $u_i$ is increasing in $x_i$, the last inequality implies that $\lambda_i \geq 0$. We show first that $\lambda_i > 0$. Suppose not, i.e., let $\lambda_i = 0$. For $j = 1, ..., N$, write $e^j = (0, ..., 0, 1, 0, ..., 0) \in R^N$ (with the 1 in the $j$th place). On the one hand, for $\varepsilon > 0$ we have that $(\tilde{y} + \varepsilon e^j, \omega_i - a_i \cdot (\tilde{y} + \varepsilon e^j) - b_i C(\tilde{y} + \varepsilon e^j)) \in C_i$. This implies that $\mu_i \cdot (\tilde{y} + \varepsilon e^j) \leq \mu_i \cdot \tilde{y}$, i.e., $\mu_i \varepsilon \leq 0$. On the other hand $(\tilde{y} + \delta e^j, \tilde{x_i} + \varepsilon) \in U_i$ for some $(\varepsilon, \delta) > 0$. This implies that $\mu_i \cdot (\tilde{y} + \delta e^j) \geq \mu_i \cdot \tilde{y}$, i.e., $\mu_i \delta \geq 0$. Thus, if $\lambda_i = 0$, then $\mu_i = 0$ for all $j$, i.e., $(\lambda_i, \mu_i) = 0$, a contradiction. Hence, $\lambda_i > 0$. 


Define now $p_i = (1/\lambda_i) \mu_i$. Then

$$x_i + p_i \cdot y \leq \bar{x}_i + p_i \cdot \bar{y},$$

for all $(y, x_i) \in C_i$,

and in particular, $\omega_i - a_i \cdot y - b_i C(y) + p_i \cdot y \leq \omega_i - a_i \cdot \bar{y} - b_i C(\bar{y}) + p_i \cdot \bar{y}$ for all $y \in R^N_+$. Adding over consumers and recalling that $\sum_i b_i = 1$ and $\sum_i a_i = 0$, we obtain

$$-C(y) + C(\bar{y}) = \sum_i p_i \cdot y$$

for all $y \in R^N_+$; i.e., $\bar{y}$ maximizes profits at the price vector $\sum_i p_i$. Hence, condition (a) of Definition 4 is satisfied.

If $(p_i, \bar{y}) = C(\bar{y}) = 0$, define $\theta_i = 1/M$; otherwise, define $\theta_i = (p_i \cdot \bar{y} - a_i \cdot \bar{y} - b_i C(\bar{y}))/((\sum_j p_j) \cdot \bar{y} - C(\bar{y}))$. It is clear that, for all $i$, $(\bar{y}, \bar{x}_i)$ satisfies the budget equation for $p_i$ and $\theta_i$. This in turn implies that $x_i + p_i \cdot y = \omega_i + \theta_i[(\sum_j p_j) \cdot \bar{y} - C(\bar{y})] > \omega_i > 0$; i.e., $(\bar{y}, \bar{x}_i)$ does not minimize $x_i + p_i \cdot y$ on $R^N_+$. Together with the fact that $x_i + p_i \cdot y \geq x_i + p_i \cdot \bar{y}$ whenever $u_i(y, x_i) \geq u_i(\bar{y}, \bar{x}_i)$, this yields constrained utility maximization.

Q.E.D.

When the cost function is differentiable, Proposition 3 can be sharpened into an exact converse of Proposition 2.

**Proposition 4.** Assume that $C$ is convex and differentiable and let $(\bar{y}, \bar{x})$ arise from a Linear Cost Share Equilibrium with parameters $a_i, b_i$. Then $(\bar{y}, \bar{x})$ is a Lindahl-Foley Equilibrium for the profit shares $\theta_i = b_i$ and for prices $p_i$ satisfying $p_i = a_i + b_i(\sum_j p_j)$.

**Proof.** Write $\nabla C(\bar{y})$ for the gradient vector of $C$ at $\bar{y}$ and define $p_i = a_i + b_i \nabla C(\bar{y})$. Since $\sum_i a_i = 0$ and $\sum_i b_i = 1$, we have that $\sum_i p_i = \nabla C(\bar{y})$. By the convexity of $C$, $\langle \sum_j p_j, \bar{y} - C(\bar{y}) \rangle \geq (\sum_j p_j) \cdot y - C(y)$ for all $y \in R^N_+$; i.e., profit maximization is guaranteed.

Write $I_i = \omega_i + b_i[\nabla C(\bar{y}) \cdot \bar{y} - C(\bar{y})] > 0$. (Clearly, $\bar{x}_i + p_i \cdot \bar{y} = I_i$.) We want to show that $(\bar{y}, \bar{x}_i)$ maximizes $u$, subject to $x_i + p_i \cdot y \leq I_i$. Assume, contrary to hypothesis, that there exists an $(y', x'_i)$ satisfying this budget constraint and preferred to $(\bar{y}, \bar{x}_i)$. Without loss of generality, $(y', x'_i)$ can be taken to satisfy $x'_i > 0$ and $x'_i + p_i \cdot y' \leq I_i$. By continuity, $(y', x'_i + \epsilon)$ is still preferred to $(\bar{y}, \bar{x}_i)$ for some $\epsilon > 0$. Therefore, by the convexity hypothesis on preferences $(y', x'_i) = (\bar{y} + t(y' - \bar{y}), \bar{x}_i + t(x'_i - \bar{x}_i) - t \epsilon)$ is preferred to $(\bar{y}, \bar{x}_i)$ for any $t \in (0, 1)$. Because $(\bar{y}, \bar{x}_i)$ arises from a Linear Cost Share Equilibrium, we must have that, for $t \in (0, 1)$,

$$x'_i > \omega_i - a_i \cdot y' - b_i C(y').$$
On the other hand, \( x'_i \) can be written as

\[
x'_i = I_i - p_i \cdot \bar{y} - tp_i \cdot (y' - \bar{y}) - te
\]

\[
= \omega_i + b_i [\nabla C(\bar{y}) \cdot \bar{y} - C(\bar{y})] - (a_i + b_i \nabla C(\bar{y})) \cdot (\bar{y} + t(y' - \bar{y})) - te
\]

\[
= \omega_i - b_i C(\bar{y}) - a_i \cdot y' - tb_i \nabla C(\bar{y}) \cdot (y' - \bar{y}) - te.
\]

Hence, the inequality above becomes

\[-b_i C(\bar{y}) - tb_i \nabla C(\bar{y}) \cdot (y' - \bar{y}) - te > -b_i C(y'),\]

i.e.,

\[-b_i \nabla C(\bar{y}) \cdot (y' - \bar{y}) + \frac{1}{t} b_i (C(y') - C(\bar{y})) > \varepsilon.\]

Letting \( t \) tend to zero, the differentiability of \( C \) implies that the left hand side of this inequality tends to zero, a contradiction. Q.E.D.

**Remark 5.** Note that, in the previous proof, the convexity of the cost function is only used to establish profit maximization (i.e., to show that for \( \sum_i p_i = \nabla C(\bar{y}), \bar{y} \) maximizes profits) and that, in turn, \( \bar{x}_i + p_i \cdot \bar{y} \neq 0 \). Thus, if \( C \) is differentiable (but possibly nonconvex) and \( \bar{x}_i + p_i \cdot \bar{y} \neq 0 \), then a \((\bar{y}, \bar{x})\) that arises from a Linear Cost Share Equilibrium with parameters \( a_i, b_i \) can be viewed as a marginal cost pricing equilibrium for the profit (or loss) shares \( \theta_i = b_i \) and the prices \( p_i = a_i + b_i \nabla C(\bar{y}) \).

The differentiability of the cost function is indispensable for the validity of Proposition 4. With convex, nondifferentiable cost functions, a Linear Cost Share Equilibrium will be a Lindahl–Foley Equilibrium for some \( p_i \) and \( \theta_i \) (Proposition 3), but it may be impossible to choose \( p_i \) and \( \theta_i \) such that \( p_i = a_i + b_i \nabla C(\bar{y}) \). This is shown by the following example.

**Example.** It is illustrated in Fig. 1. We let \( N = 1 \) and \( M = 2 \). The cost function is \( C(y) = y \) if \( y \leq 1 \), \( 1 + 4(y - 1) \) if \( y \geq 1 \). For the consumers, we let \( \omega_1 = \omega_2 = \frac{1}{2} \), and we suppose that the marginal rates of substitution at the consumption point \((1, 1)\) are \( (\partial u_1/\partial y)/(\partial u_1/\partial x_1) = \frac{1}{2} \) and \( (\partial u_2/\partial y)/(\partial u_2/\partial x_2) = 2 \). It is clear that the state \((\bar{y}, \bar{x}_1, \bar{x}_2) = (1, 1, 1)\) and the parameters \((a_1, a_2, b_1, b_2) = (0, 0, \frac{1}{3}, \frac{1}{3})\) constitute a Linear Cost Share Equilibrium. Also, \((\bar{y}, \bar{x}_1, \bar{x}_2)\) is a Lindahl–Foley Equilibrium for the prices \((p_1, p_2) = (\frac{1}{3}, 2)\) and profit shares \((\theta_1, \theta_2) = (0, 1)\). Actually, these prices and profit shares are the only ones that can support \((\bar{y}, \bar{x}_1, \bar{x}_2)\) as a Lindahl–Foley Equilibrium (see Fig. 1). But they of course fail to satisfy the equations \( p_i = a_i + b_i (\sum_j p_j) \).
If the technology displays constant returns to scale then profit maximization implies zero profits. Hence, \((\bar{y}, \bar{x})\) arises from a Lindahl–Foley equilibrium for the profit shares \(\theta_i\) if and only if it does so for any other profit shares. Therefore, it follows from Propositions 2 to 3 that for a convex, constant returns cost function the notion of Lindahl–Foley Equilibrium coincides with that of Linear Cost Share Equilibrium.

For the more general case where \(C\) is convex, (i.e., nonincreasing returns to scale) Proposition 2 ensures that the existence of a Linear Cost Share Equilibrium is a consequence of the existence of a Lindahl–Foley Equilibrium (see Foley [3], Milleron [11], and Roberts [14] for the latter). The interesting point, however, is that while in the increasing returns case a Lindahl–Foley equilibrium cannot exist by definition, a Linear Cost Share Equilibrium may well exist. Heuristically, it will all depend on the relative curvatures of the indifference manifolds and the cost function (the larger the first are, the better). Rather than stating a formal theorem along this line we choose to emphasize the broader domain of existence of linear cost share equilibria in another direction.
PROPOSITION 5. If all consumers are identical a Linear Cost Share Equilibrium always exists.

Proof. Put $a_{i} = 0$, $b_{i} = 1/M$ and let $\bar{y}$ maximize $u_{i}(y, \omega_{i} - (1/M) C(y))$ for all $i$. Because $C$ is unbounded above this maximum exists. Q.E.D.

III. BALANCED LINEAR COST SHARE EQUILIBRIUM

The linear cost share equilibrium concept does not yield an endogenous theory of profit distribution. As indicated by Propositions 2-4, the concept leaves $M - 1$ degrees of freedom which correspond implicitly to profit shares. In order to arrive at an endogenous determination of the latter we need to close the degrees of freedom in a manner compatible with the basic postulate of this paper: that the individual payments for public goods be in accordance with individual benefits.

There is a natural way to introduce $M - 1$ extra constraints. Suppose that $\bar{y}$ is a linear cost share equilibrium with respect to $a_{i}, b_{i}$. Since $\sum_{i} a_{i} = 0$ it follows that $\sum_{i} a_{i} \cdot \bar{y} = 0$. Therefore $a_{i} \cdot \bar{y}$ represents a transfer from (or to) consumer $i$, the aggregate net transfer being zero. We can consider an equilibrium as a priori distributionally neutral if the net transfer of every consumer is zero, that is, $a_{i} \cdot \bar{y} = 0$. Formally,

DEFINITION 5. A linear cost share equilibrium $\bar{y}, a_{i}, b_{i}$ is balanced if $a_{i} \cdot \bar{y} = 0$ for every $i$.

Note that in the case of a single public good a linear cost share equilibrium is balanced if and only if $a_{i} = 0$, i.e., if and only if it is a Ratio Equilibrium in the sense of Kaneko [7].

In order to gain some understanding of the new equilibrium notion it is useful to go back to Proposition 4 and ask ourselves to which Lindahl–Foley equilibrium (in the convex, differentiable cost function case) does a given balanced, linear cost share equilibrium $\bar{y}, a_{i}, b_{i}$ correspond. Writing $q = \sum_{i} p_{i}$, the Lindahl–Foley personalized prices of Proposition 4 satisfy $p_{i} = a_{i} + b_{i} q$, i.e., $p_{i} \cdot \bar{y} = a_{i} \cdot \bar{y} + b_{i} q \cdot \bar{y}$. Thus, if $q \cdot \bar{y} > 0$ (otherwise profits are zero) then $\theta_{i} = b_{i} = p_{i} \cdot \bar{y} / q \cdot \bar{y}$; i.e., profits are distributed proportionally to (gross) expenditures $p_{i} \cdot \bar{y}$.

Hence, the concept of Balanced Linear Cost Share Equilibrium embodies an endogenous determination of profit (or loss) shares, i.e., a theory of the ownership of the firm. Exogenous profit shares can, under convexity, be viewed as reflecting the private ownership of a nontraded input (see McKenzie [10]); such an input should be considered here as collectively owned, its rewards being distributed according to the ratios $p_{i} \cdot \bar{y} / q \cdot \bar{y}$. (See Moulin [12] for a thought-provoking discussion of this interpretation in the one public good context.)
No justification in terms of fairness should be attempted for this distribution scheme (any such assessment should in any event cover the initial distribution of numéraire). But the scheme fits well with the benefit theory of taxation, i.e., with the idea that individual contributions to the cost reflect personal valuations (here, the marginal valuations given by \( p_i \)). The Lindahl-Foley approach translates this idea only in a partial manner: \( i \)'s net contribution is the difference between a term \( p_i \cdot \bar{y} \) of gross expenditures, that does reflect marginal valuations, and a profit term \( \theta_i (q \cdot \bar{y} - C(\bar{y})) \), unrelated to them.

The preceding analysis suggests the following characterization of the states that may arise as Balanced Linear Cost Share Equilibria.

**Proposition 6.** Suppose that \( C \) is convex and differentiable. Then \( (\bar{y}, \bar{x}) \in \mathbb{R}^{N+M}_+ \) arises from a Balanced Linear Cost Share Equilibrium if and only if there are \( p_i \in \mathbb{R}^N, i = 1, \ldots, M, \) such that

(a) for every \( i \), \( p_i \) supports consumer \( i \)'s preferred set at \((\bar{y}, \bar{x}_i)\);

(b) \( \sum_h p_h \) supports the cost function at \( \bar{y} \), i.e., \( \left( \sum_h p_h \right) \cdot (\bar{y} - y) \geq \left( C(\bar{y}) - C(y) \right) \);

(c) for every \( i \),

\[
\omega_i - \bar{x}_i = \frac{p_i \cdot \bar{y}}{\left( \sum_h p_h \right) \cdot \bar{y}} C(\bar{y}) \geq 0
\]

(by convention, \( 0/0 = 0 \)).

**Proof.** It has just been argued that, under convexity and differentiability, given a Balanced Linear Cost Share Equilibrium there are \( p_i \) \((i = 1, \ldots, M)\) with the desired properties. Conversely, let such \( p_i \) exist, and write \( q = \sum_h p_h \). Then it is easy to check that \((\bar{y}, \bar{x})\) arises from a Lindahl-Foley Equilibrium for the personalized prices \( p_i \) and for the profit shares \( \theta_i = 1/M \), if \( \bar{y} = 0 \), or \( \theta_i = p_i \cdot \bar{y}/q \cdot \bar{y} \) otherwise. By Proposition 2, \((\bar{y}, \bar{x})\) is a Linear Cost Share Equilibrium for parameters \((a_i, b_i)\) satisfying \( a_i \cdot \bar{y} = 0 \), if \( \bar{y} = 0 \), or \( a_i \cdot \bar{y} = p_i \cdot \bar{y} - \theta_i q \cdot \bar{y} = 0 \) otherwise. Thus, \((\bar{y}, \bar{x})\) is a Balanced Linear Cost Share Equilibrium. Q.E.D.

**Remark 6.** If \( C \) is not convex then the "only if" part of the above proposition remains valid (provided that \( \nabla C(\bar{y}) \) be strictly positive and replacing (b) by the corresponding first order conditions) but not the "if" part. Because the second order conditions may fail, a feasible state satisfying the conditions in the statement of Proposition 4 may not be supportable as a Balanced Linear Cost Share Equilibrium (but, again, it could; it all depends on relative curvatures).
Remark 7. If $C$ is not differentiable, then the "if" part of Proposition 6 remains valid but not the "only if" part. Note that the example after Proposition 4 is actually a Balanced Linear Cost Share Equilibrium, but if $p_i$ supports consumer $i$'s preferred set to $\langle \tilde{y}, \tilde{x}_i \rangle$, $i = 1, 2$, then $(\omega_i - \tilde{x}_i)(p_1 + p_2) \tilde{y} = \frac{5}{4}$, and this equals neither $\frac{1}{2} = C(\tilde{y}) p_1 \tilde{y}$ nor $2 - C(\tilde{y}) p_2 \tilde{y}$.

Remark 8. Chander [11] has proposed a dynamic nontâtonnement public good allocation process where, along the trajectory, costs are instantaneously distributed proportionally to marginal valuations. This can be seen as an instantaneous, nontâtonnement analog of condition (c) in the statement of Proposition 6.

What about the existence of Balanced Linear Cost Share Equilibria? The matter is now more delicate than in the preceding section. We shall prove existence for a general case with convex costs and no public bads. As with the general linear case, existence is not automatically precluded by increasing returns. The degree of increasing returns relative to the curvature of the indifference manifolds is the essential consideration. We shall not attempt however to state a formal result. For the identical consumer case, note that the linear equilibrium exhibited in Proposition 5 was balanced. If there are public bads (i.e., the utility function is not monotone) then we have no general existence theorem, nor should one be expected. This is clear from the characterization in Proposition 6 and the discussion leading to it. With public bads there is no reason why $p_i \cdot \tilde{y} / q \cdot \tilde{y} \geq 0$. Extending our definition to allow $b_i < 0$ is no remedy because the individual budget set (of consumer $i$) would then fail to be convex.

**PROPOSITION 7.** Suppose that $C$ is convex and proper (i.e., $\|y_n\| \to \infty$ implies $C(y_n) \to \infty$). Assume that every $u_i$ is weakly increasing in every coordinate (i.e., no public bads). Suppose also that the numéraire is indispensable (i.e., $u_i(y, 0) \leq u_i(y', x_i)$ for any $y, y', x_i$). Then a Balanced Linear Cost Share Equilibrium exists.

**Proof.** Without loss of generality we will only consider the case where every $u_i$ is strictly concave, differentiable, and every marginal rate of substitution $(\partial u_i / \partial y_i) / (\partial u_i / \partial x_i)$ is uniformly bounded by a constant $k$. If this is not satisfied by our original economy we can always approximate it by a well-behaved one and then make a limit argument.

Let $P = [0, k]^N$. For every $i$ and $p_i \in P$, $\pi_i \in R_+$, define $y_i(p_i, \pi_i)$ as the maximizer of $u_i(y_i, \omega_i - p_i \cdot y_i + \pi_i)$ on a box $T = [0, t]^N$. Let $s > ktMN$. Define the correspondences $\Phi_i : T \times P \times [0, s] \to P$ by $\Phi_i(y, p_i, \pi_i) = \max_{v \in P} \{v \cdot (y_i(p_i, \pi_i) - y)\}$. It is easy to verify that every $\Phi_i$ is convex-valued and upper-hemicontinuous. Also, if $p_i \in \Phi_i(y, p_i, \pi_i)$ then necessarily
y'_j(p_i, \pi_i) = y_i. Indeed, suppose that y'_j(p_i, \pi_i) < y'_i. Then p'_i = 0, which implies y'_j(p_i, \pi_i) = t > y'_i, a contradiction. Suppose that y'_j(p_i, \pi_i) > y'_i. Then p'_i = k, which implies y'_j(p_i, \pi_i) = 0 < y'_i, a contradiction.

Define $\Psi: P^M \to T$ by $\Psi(p_1, ..., p_M) = \{ \tilde{\eta} | \sum_i p_i \cdot \tilde{\eta} - C(\tilde{\eta}) \geq \sum_i p_i \cdot y - C(y) \}$ for all $y \in T$. The correspondence $\Psi$ is convex-valued and upper-hemicontinuous. Define also the continuous function $\Pi: P^M \to [0, s]$ by $\Pi(p_1, ..., p_M) = \sum_i p_i \cdot y - C(y)$ for any $y \in \Psi(p_1, ..., p_M)$. Let $A$ be the $N-1$ simplex and define a convex-valued, upper-hemicontinuous correspondence $\alpha: P^M \times T \to A$ by $\alpha(p_1, ..., p_M, y) = \{(1/\sum_i p_i \cdot y)(p_1 \cdot y, ..., p_M \cdot y)\}$. If $\sum_i p_i \cdot y = 0$ let the value be $A$.

Finally, define a convex valued, upper-hemicontinuous correspondence $F: P^M \times T \times A \to P^M \times T \times A$ by $F(p_1, ..., p_M, y, \theta) = \chi_i \Psi(y, p_i, \theta) \Pi(p_1, ..., p_M) \times \alpha(p_1, ..., p_M, y)$. By Kakutani's Theorem the correspondence has a fixed point $(\bar{p}_1, ..., \bar{p}_M, \bar{\eta}, \bar{\theta})$. It is then trivial to verify that if $t$ is large enough, the fixed point generates a Lindahl–Foley equilibrium with $(\sum_i \bar{p}_h \cdot \bar{\eta}) \theta_i = \bar{p}_i \cdot \bar{\eta}$ for all $i$. In turn this can be used, as in Proposition 6 (see Remark 7), to generate a Balanced Linear Cost Share Equilibrium. Q.E.D.

IV. More Specific Environments

IV.1. Notation

In this section the number of goods $N$ is the product of $M$ and $L$, where $L$ is interpreted as the number of distinct physical outputs. We use a double index $(h, k)$ to identify a good, instead of the previous single index, e.g., $y_{hk}$ denotes the amount of physical output $k$ assigned to consumer $h$. The cost function verifies $C(y) = C(y')$ whenever $\sum_h y_{hk} = \sum_h y'_{hk}$ for all $k$.

IV.2. Pure Private Goods

It is well known that an Arrow–Debreu equilibrium for an economy with $L$ private goods can be interpreted as a Lindahl–Foley Equilibrium for the economy where the $N = ML$ allocation vector is viewed as a vector of pure public goods (if $(p_1, ..., p_L)$ are the Arrow-Debreu prices, define the Lindahl prices $p_{ihk}$ by: $p_{ihk} = p^k$, and $p_{ihk} = 0$, for $h \neq i$). Conversely, a Lindahl–Foley equilibrium for an economy with $N = ML$ public goods, such that $u_i(y, x_i) = u_i(y', x_i)$ wherever $y_{ik} = y'_{ik}$ for all $k \in \{1, ..., L\}$, is an Arrow–Debreu equilibrium for an economy with $N$ private goods (if $q_{hk}$ are the production prices at the Lindahl–Foley equilibrium, define $p^k = \max_h q_{hk}$). Hence, Propositions 2–4 state, in this context, the equivalence between Arrow–Debreu equilibria and Linear Cost Share Equilibria (given the Arrow–Debreu parameters $p^k$, $\theta_i$, the corresponding parameters $a_i$, $b_i$ of
the Linear Cost Share Equilibrium are \( b_i = \theta_i, a_{ijk} = p^k(1 - \theta_i), a_{ikh} = -\theta_i p^k \) for \( h \neq i \). Our approach may look more complex than Arrow–Debreu’s. The added complexity is the price exacted by the absence of a profit maximization condition.

Proposition 6 in turn implies the equivalence between Balanced Linear Cost Share Equilibria and Arrow–Debreu equilibria for private ownership economies with profit shares satisfying \( \theta_i = (p \cdot \bar{y}_i / p \cdot \sum_h \bar{y}_h) \), where \( p = (p^1, \ldots, p^L) \) and \( \bar{y}_i = (\bar{y}_{i1}, \ldots, \bar{y}_{iL}) \). Hence, at the Arrow–Debreu equilibrium associated with a Balanced Linear Cost Share Equilibrium, consumer \( i \)'s consumption of numéraire is

\[
\bar{x}_i = \omega_i - p \cdot \bar{y}_i \cdot \frac{C(\bar{y})}{p \cdot \sum_h \bar{y}_h}.
\]

In other words, his net (i.e., after profits) expenditure equals an index of his consumption times the average cost of production (i.e., total cost divided by an index of total production). In this sense, consumer \( i \)'s net payment agrees with average cost pricing (if \( L = 1 \) the average cost interpretation is, of course, exact: no index is needed to aggregate), even though the allocation is optimal. This is a notable fact because our equilibrium is an optimum and we know that, for optimality, decisions must be guided by marginal, not average, cost. Optimality can be made compatible with payment according to average cost only if the payment schedule maintains the discrepancy between average and marginal costs. Our cost share scheme accomplishes precisely this.

Summarizing: Proposition 6 tells us that (when the cost function is convex and differentiable) the set of allocations supportable as a Balanced Linear Cost Share Equilibrium coincides with the set of optimal allocations compatible with average cost pricing. Provided that preferences are weakly monotone, Proposition 7 ensures that this set is nonempty.

IV.3. Externalities

The model is as in the previous section but we now allow that the private goods be externality producing, i.e., with \( y_{hk} \) as consumer \( h \)'s consumption of output \( k \), the statement “\( u_i \) is increasing (resp. decreasing) in \( y_{hk} \)” means that agent \( i \) enjoys a positive (resp. suffers a negative) externality from \( h \)'s consumption of \( k \) (or, alternatively, from the production of \( k \) if \( i \)'s utility function treats \( y_{hk} \) and \( y_{ik} \) equally whenever \( h \neq i \) and \( l \neq i \)). Thus, our Cost Share Equilibrium concepts can also be viewed as optimality-guaranteeing equilibrium concepts for economies with externalities.
To continue our discussion it will be useful if we simplify to \( L = 1 \) (i.e., \( N = M \)) and let cost and utility functions be differentiable. Moreover, the cost function is taken to be convex. Let \( (\tilde{y}, \tilde{x}) \in R^{2M} \) be an interior Linear Cost Share Equilibrium. Denote by \( p \) the (scalar) marginal cost at \( \tilde{y} \), and write \( y = (y_1, ..., y_M) \in R^M \), \( v_{ih} = (\partial u_i/\partial y_h)/(\partial u_i/\partial x_i)(\tilde{y}, \tilde{x}) \). By optimality, \( p = \sum_h v_{hi} \) \((i = 1, ..., M)\). The first order conditions for an equilibrium imply

\[
a_{ih} + b_i p = v_{ih},
\]

i.e.,

\[
\omega_i - x_i = \sum_h a_{ih} \tilde{y}_h + b_i C(\tilde{y})
\]

\[
- \sum_h v_{ih} \tilde{y}_h - b_i p \sum_h \tilde{y}_h + b_i C(\tilde{y}).
\]

It is well known that if \((\tilde{y}, \tilde{x})\) is an optimal state of the economy then it can be supported by means of a price supplemented by Pigou taxes, i.e., there is a price \( p \in R \), individual tax rates \( \tau_i \), and lump-sum subsidies (or taxes) \( T_i \) such that the firm maximizes profits and, for every \( i \), \((\tilde{y}_i, \tilde{x}_i)\) maximizes \( u_i(\tilde{y}_1, ..., \tilde{y}_{i-1}, y_i, \tilde{y}_{i+1}, ..., \tilde{y}_M, x_i) \) subject to \( x_i = \omega_i + \theta_i[p \sum_h \tilde{y}_h - C(\tilde{y})] - (p + \tau_i) y_i + T_i \).

Which are the Pigou taxes and subsidies which correspond to a (necessarily optimal) Linear Cost Share Equilibrium \((\tilde{y}, \tilde{x})\)? Clearly, for \( i \) to choose \( \tilde{y}_i \), one needs \( v_{ii} = p + \tau_i \), or, recalling that \( \sum_h v_{hi} = p \),

\[
\tau_i = - \sum_{h \neq i} v_{hi}.
\]

Moreover, if profits are distributed according to the shares \( \theta_i = b_i \), then it is easy to check that, by (2), in order to consume \( \tilde{x}_i \) of numéraire he must receive from the government the lump-sum subsidy

\[
T_i = - \sum_{h \neq i} v_{ih} \tilde{y}_h.
\]

It is clear that the government balances it budget, i.e., \( \sum_i (\tau_i \tilde{y}_i - T_i) = 0 \).

Let now \((y, x)\) correspond to a Balanced Linear Cost Share Equilibrium. Then \( \sum_h a_{ih} \tilde{y}_h = 0 \), i.e., from (1),

\[
b_i = \frac{\sum_h v_{ih} \tilde{y}_h}{p \sum_h \tilde{y}_h}.
\]
or in terms of the Pigou tax rates and lump-sum subsidies given in (3) and (4),
\[ b_i = \frac{(p + \tau_i) \bar{y}_i - T_i}{p \sum_h \bar{y}_h}. \] (6)

We conclude, as in Section III, that profits are distributed proportionally to expenditures, except that now one should also count as \( i \)'s expenditure \( i \)'s net payment to the government (of course, since the government budget balances, the sum of expenditures of all consumers equals \( p \sum_h \bar{y}_h \)).

Alternatively, \( i \)'s net expenditure is \( \omega_i - \bar{x}_i = b_i C(\bar{y}) \). From (6), (3), and (4), we can write
\[ \omega_i - \bar{x}_i = \bar{y}_i \frac{C(\bar{y})}{\sum_h \bar{y}_h} - \left( \bar{y}_i \sum_{h \neq i} v_{hi} - \sum_{h \neq i} v_{ih} \bar{y}_h \right) \frac{C(\bar{y})}{p \sum_h \bar{y}_h}. \]

Similarly to IV.2 above, this can be interpreted as compatibility with average cost pricing, corrected by externality-based transfers among consumers. If, for instance, all externalities are positive (i.e., \( v_{ih} > 0 \) for \( i \neq h \)), then \( i \) receives from other consumers an amount that depends on his consumption times the sum of the marginal benefit that he generates on others; he in turn transfers to the others an amount that depends on the sum of their consumptions weighted by the marginal benefit that each creates on \( i \).

A concluding word of caution. In an externality world there is no a priori reason for the numerator in (6) to be positive (for all consumers). Hence, as noted in Section III above, Balanced Linear Cost Share Equilibria may very well fail to exist in the case of negative externalities.


### V. Extensions

(a) An extension which is relatively simple is to allow for several private goods usable as inputs. In this case, of course, the prices and the demand–supply equations for inputs have to be appended to the equilibrium problem. Note that if this extension is done, then the private good pure exchange model with any number of goods (not just one as in Section IV.1) would be covered by our theory.

(b) Suppose that instead of one there are several cost functions \( C_h: R^N \to R_+, \ h = 1, ..., H \). Those are to be understood as technologies
embodied in $H$ institutionally separated firms. The concept of Linear Cost Share Equilibrium must then be adapted to incorporate parameters for the production and cost of every firm, i.e., $g_i: R^{NH} \to R_+: g_i(y^1, ..., y^H) = \sum_h a_{ih} y^h + \sum_h b_{ih} C_h(y^h)$ with $\sum_i a_{ih} = 0$ and $\sum_i b_{ih} = 1$ for every $h$. There are now $(M-1)H$ degrees of freedom which, as in Section III, correspond to the profit shares $\theta_{ih}$ of the Lindahl–Foley equilibrium. As in Section III, the degrees of freedom can be closed by imposing a neutrality-like condition which this time must be firm specific, i.e., $a_{ih} \cdot \bar{y}^h = 0$. As contrasted with $\sum_h a_{ih} \cdot \bar{y}^h = 0$, there does not seem to be much intuitive appeal in this balancedness axiom. In a sense, this is as it should be. While one may hope to find a natural theory for the endogenous distribution of overall social profits, it is unlikely that this could be done from first principles if the distribution has to be compatible with the vagaries of the institutional organization of production in one or several firms.

(c) For the case where the cost function can be written in the additively separable form $C(y) = \sum_j C_j(y_j)$ for some $C_j: R_+ \to R_+$, $j = 1, ..., N$, Kaneko's notion of Ratio Equilibrium (see Kaneko [7] and Ito and Kaneko [6]), is well defined and it does not coincide with our Balanced Linear Cost Share Equilibrium. In terms of the previous paragraph, Kaneko's proposal coincides with the extension of the Balanced Linear Cost Share Equilibrium concept when each public good $j$ is viewed as being produced by an institutionally distinct firm (with cost function $C_j(y_j)$).

(d) The extension that seems to us very difficult is to allow for produced intermediate goods. The reason is that our approach does depend, on the one hand, on there being a well-defined cost function but, on the other, weesch the explicit pricing of produced goods.

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