CHAPTER 12

On the uniqueness of equilibrium once again

Andreu Mas-Colell

1 Introduction

This chapter has two objectives. First, we present a survey of the state of the art in the theory of uniqueness of Walrasian equilibrium. Second, we try to push forward its frontier, especially for the case of general production economies. Roughly speaking, we take up the topic where it was left by Kehoe (1985b).

The plan of the chapter is as follows. In Sections 2 and 3, we gather a number of basic concepts and facts about the aggregate excess demand of an economy. The exposition is taken from Mas-Colell (1985, Section 5.7), and we refer there for further analysis and basic references.

In Sections 4 and 5, we study the situation where the distribution of wealth is price-independent (collinear endowments). We emphasize uniqueness conditions that involve only the demand side of the economy or, what comes to the same thing, conditions for the fulfillment of the Weak Axiom of Revealed Preference in the aggregate. It turns out to be more convenient to focus on the stronger property of monotone excess demand (precise definitions are given in Section 2) because the latter aggregates better across consumers.

In these sections, we give main billing to the Mitiushin–Polterovich theorem, a beautiful result that deserves to be much better known. The

Sections 1 to 6 constitute a revision and extension of "Large Substitution Effects and the Uniqueness of Equilibrium," presented and distributed at the March 9, 1986 NSF-NBER General Equilibrium Conference held at Berkeley, California. In particular, both Examples 1 and 2 are taken from there. In turn, the Berkeley presentation was based, in part, on material contained in a Fisher–Schultz lecture delivered at the 1985 Boston meeting of the Econometric Society. Sections 7 to 10 are new.

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theorem gives sufficient conditions on concave, C^2 utility functions for individual demand (income fixed to equal 1) to be monotone (with collinear endowments this then carries over to aggregate excess demand). It can be looked at as a concrete expression of the idea that the substitution effects dominate the income effects. What is remarkable is the simplicity of the (simplest) condition: $-[x\cdot\partial^2 u(x)x/x\cdot\partial u(x)]\leq 4$ for all x. Interestingly, an example will show that 4 is indeed the magic number for the purposes of uniqueness theory.

Our conclusion from Sections 4 and 5 is that as long as the distribution of income is price-independent, there are reasonable classes of conditions involving only the demand side of the economy and guaranteeing the uniqueness of equilibrium.

In Section 6, we look at the general case of noncollinear endowment (especially relevant in a production context). Unfortunately, the previous conclusion is then overturned. We present an extremely simple and nonpathological example with two consumers and linear utilities (in a four-commodity world) where the Weak Axiom fails to be satisfied in the aggregate. The discussion of the example tells us that there is little room for any attempt to get fruitful uniqueness conditions involving only the excess demand function: A minimally satisfactory theory will have to involve both consumption and production conditions.

In Sections 7 to 9, we pursue an approach in this direction. It consists in reducing the production economy (assumed to be of the constant return, no joint-production type) to an exchange economy for factors of production and exploit then either the general theory of the previous sections specialized to the exchange case (this leads to conditions involving factor intensities) or theories, such as gross substitution, specific to exchange economies. This last line is perhaps the most promising and leads to the main contribution of this chapter, which is Theorem 3 in Section 9. By using index theory, the theorem shows that uniqueness obtains if every utility and production function is *super Cobb–Douglas*. The concept of a super Cobb–Douglas function $h(x)$ is introduced in this chapter, and is defined by the property that at every x there is a Cobb–Douglas function h_x such that $h_x(x)=h(x)$ and $h_x(x')\leq h(x')$ for x' close to x. Of course, this includes the Cobb–Douglas case (which is dealt with in Theorem 2 of Section 8 as a step toward Theorem 3), but the extension to the super Cobb–Douglas situation adds considerable flexibility. See Section 9 for details.

Finally, in Section 10 we spell out some of the convergence implications of the uniqueness theory for dynamic economies. These constitute an important part of the initial motivation for this chapter.

2 Aggregate excess demand: Basic properties

This section describes a number of standard concepts on aggregate excess-demand functions. See Mas-Colell (1985, Section 5.7) for further details and references.

For l the number of commodities, a function $f: R^l_{++} \to R^l$ is an *excess-demand function* if it is continuous, homogeneous of degree 0, and satisfies Walras's law (i.e., $p \cdot f(p) = 0$ for all p).

Definition 1. *The excess-demand function f satisfies the* Weak Axiom (WA) *if $q \cdot f(p) \leq 0$ and $p \cdot f(q) \leq 0$ implies $f(p) = f(q)$.*

Definition 2. *The excess-demand function f is* monotone *with respect to the normalizing vector $z \geq 0$, $z \neq 0$, if $(f(p) - f(q)) \cdot (p - q) \leq 0$ whenever $p \cdot z = q \cdot z = 1$ (it is* strictly monotone *if the inequality is strict for $p \neq q$).*

An excess-demand function f that is monotone with respect to some z also satisfies the Weak Axiom (to verify this is a simple and useful exercise). The economic interpretation of the two properties is clear: the WA is the law of demand (prices and quantities move in opposite directions) for compensated price changes, whereas monotonicity is the law of demand for any kind of (normalized) price change.

If the excess-demand function was generated by a single preference-maximizing consumer, then it would satisfy the WA but not necessarily monotonicity (thus, it is immediate from the definition that strict Giffen goods cannot arise for $l = 2$). For aggregation purposes, however, monotonicity is a better behaved property. As is well known, by adding up excess-demand functions generated from utility maximization (and satisfying, therefore, the WA), one can lose any significant restriction (see Shafer and Sonnenschein, 1982). On the other hand, the sum of two excess-demand functions monotone with respect to a common normalizing factor z is monotone with respect to the same factor. It would be nice if one could get rid of the normalizing factor in the definition of monotonicity. But this cannot be done. In fact, if $f(p) \neq 0$, there is always a q such that $(f(p) - f(q)) \cdot (p - q) > 0$ (this is easy to verify graphically for $l = 2$).

Consider the two commodities case. Then (strict) monotonicity has a simple geometric interpretation. It says that the offer curve, that is, the set $f(R^l_{++})$, projects in a one-to-one manner into some one-dimensional subspace with positive normal (the projection is taken in the direction of the normal). Thus, the offer curves in Figure 1(a) and 1(b) (resp. Figure 1(c)) are (resp. is not) compatible with monotonicity with respect to some z.

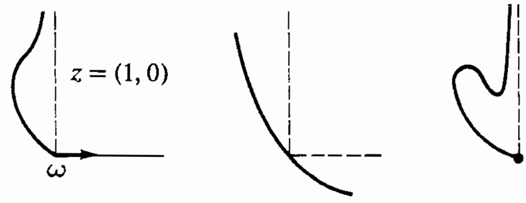

Figure 1.

Suppose now that the excess-demand function f is C^1. Then one can give natural sufficient conditions in terms of derivatives for the properties of interest. Precisely, a sufficient condition for the WA is that $v \cdot \partial f(p) v < 0$ whenever $p \cdot v = 0$, $v \neq 0$, *and* $v \cdot f(p) = 0$. A sufficient condition for monotonicity with respect to z is that $v \cdot \partial f(p) v < 0$ whenever $v \neq 0$, $z \cdot v = 0$. Abusing language somewhat, we will refer to such conditions as negative (or positive) definiteness. Although this is not an aspect to be emphasized in this chapter, we want to mention that these conditions remain valid if f is merely Lipschitzian (as would be the case if boundary consumption at the individual level were allowed) and the conditions are required whenever $\partial f(p)$ exists.

The relevance of the Weak Axiom (and, therefore, of monotonicity) to the uniqueness question is the following. If f satisfies the WA, then for any closed and convex cone $Y \subset R^l$, the set of price vectors p such that $p \cdot Y \leq 0$ and $f(p) \in Y$ is convex. This means that for the typical case of a regular economy, where the set of (normalized) equilibrium price vectors is discrete, the WA implies the uniqueness of equilibrium. Conversely, if f does not satisfy the WA (for, say, the price vectors p, q), then there is a production cone Y $(= \{x : q \cdot x \leq 0 \text{ and } p \cdot x \leq 0\})$ yielding an economy (f, Y) for which the equilibrium is not unique (this observation is due to H. Scarf).

Therefore, the WA is the minimal property on excess-demand functions that has uniqueness implications irrespective of the (constant returns) technology it is coupled with. Thus, any uniqueness condition involving only the consumption side of the economy can be presented as a sufficiency condition for the WA. Those are the conditions we will focus on in the next four sections. Because of the better aggregation properties of monotonicity, these often translate into the search of sufficiency conditions for the latter.

3 Decomposition of aggregate demand effects

We will now assume that the excess-demand function f is generated from a population of N consumers. Each consumer i is characterized by a vector of initial endowments ω_i and a demand function $h_i(p, w_i)$ defined on $R^l_{++} \times R_{++}$. Thus, $f(p) = \sum_i (h_i(p, p\cdot\omega_i) - \omega_i)$.

For every i and p, we can define

$$\theta_i(p) = \frac{1}{\sum_j p\cdot\omega_j} p\cdot\omega_i \qquad \text{(the income share)},$$

$$c_i(p) = (\partial_w h_i(p, p\cdot\omega_i))^T \quad \text{(column vector of income effects)},$$

and

$$S_i(p) = \partial_p h_i(p, p\cdot\omega_i) + c_i(p)(h_i(p, p\cdot\omega_i))^T \quad \text{(matrix of substitution effects)}.$$

Correspondingly, we define the aggregate substitution effect matrix $S(p) = \sum_i S_i(p)$, and the weighted average income effect vector $c(p) = \sum_i \theta_i c_i(p)$.

Finally, we define two covariancelike matrices measuring the association across consumers of the deviation of individual income effect from its mean, that is, $c_i(p) - c(p)$, with, respectively, the deviations of (weighted) individual demand from its mean and the deviations of (weighted) initial endowments from its mean. Precisely, the matrices are defined as

$$C_h(p) = \sum_i p\cdot\omega_i [c_i(p) - c(p)] \left[\frac{1}{p\cdot\omega_i} h_i(p, p\cdot\omega_i) - \frac{1}{p(\sum_j \omega_j)} \sum_j h_j(p, p\cdot\omega_j) \right]^T,$$

$$C_\omega(p) = \sum_i p\cdot\omega_i [c_i(p) - c(p)] \left[\frac{1}{p\cdot\omega_i} \omega_i - \frac{1}{p(\sum_j \omega_j)} \sum_j \omega_j \right]^T,$$

and they will be called, respectively, the *expenditure distribution effect matrix* and the *endowment distribution effect matrix*. Note that if there is no dispersion of income effects, that is, if $c_i(p) = c(p)$ for all i, then $C_h(p) = C_\omega(p) = 0$. Similarly, if initial endowments are collinear, that is, if $\omega_i = \theta_i(p)(\sum_j \omega_j)$, then $C_\omega(p) = 0$.

The relevance of all this is that the expression

$$\partial f(p) = \sum_i [\partial_p h_i(p, p\cdot\omega_i) + c_i(p)\omega_i^T]$$

can readily be transformed into

$$\partial f(p) = S(p) - c(p) f^T(p) + C_h(p) + C_\omega(p).$$

This decomposition into four effects will allow a systematic study of the factors affecting the fulfillment of the WA by f. Of course, decompositions of this sort are not new. I follow Mas-Colell (1985, Section 5.7), but see there for earlier references (e.g., Jerison, 1982).

We saw in the previous section that the WA will hold if, for all p, $\partial f(p)$ is negative definite on $\{v: p \cdot v = f(p) \cdot v = 0\}$. So let $p \cdot v = f(p) \cdot v = 0$. Then $v \cdot \partial f(p) v < 0$ will obtain if $v \cdot S(p) v + v \cdot C_h(p) v + v \cdot C_\omega(p) v < 0$. But $v \cdot S(p) v = \sum_i v \cdot S_i(p) v$ and $v \cdot S_i(p) v < 0$ follows from preference maximization (negative definiteness of the individual substitution matrix). Therefore, assuming the preference maximizations hypothesis, $v \cdot S(p) v < 0$ will always hold. The key effects are, therefore, the distribution effects matrices.

In the next two sections, we will assume that the distribution of income is price-independent or, equivalently, that initial endowments are collinear (those have been called *"distribution economies"* by Malinvaud, 1969). Then $C_\omega(p) = 0$ and so we only have to worry about the expenditure matrix.

In Section 4, we discuss conditions under which $v \cdot C_h(p) v > 0$. In Section 5, we discuss conditions under which $v \cdot S(p) v - v \cdot C_h(p) v < 0$.

In Section 6, we bring C_ω into the picture and consider the case where the distribution of income is not price-independent.

4 Expenditure distribution effects: Engel curves conditions

In this section, we investigate the possibility that $C_h(p)$ is positive definite.

The positive definiteness of $C_h(p)$ has a clear interpretation. The vectors $c_i(p) = \partial_{w_i} h_i(p, p \cdot \omega_i)$ and $(1/p \cdot \omega_i) h_i(p, p \cdot \omega_i)$ can be viewed, respectively, as income marginal and income average propensities to consume. Thus, $C_h(p)$ positive definite means that on average (with the weight of every consumer being its expenditure), consumers have a higher than average marginal consumption for the commodities for which they have a higher than average consumption. Marginal and average tend to go together.

One can find arguments making positive association plausible (e.g., if the average is high, then the marginal must have been high somewhere) or implausible (e.g., if the average is high, then we are relatively satiated and the marginal should be low). The truth of the matter is that introspection will not settle this question. The author of this chapter is fairly sure he consumes more books and newspapers than the mean of the population. But although for books his marginal consumption is also probably higher than the mean (hence, positive association), for newspapers it is, on account of satiation, definitely lower (hence, negative association).

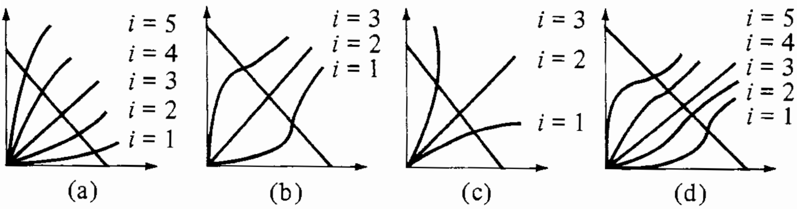

Figure 2.

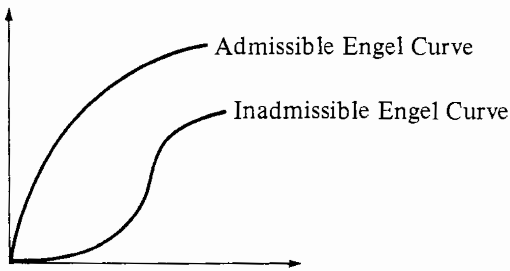

Figure 3.

Summarizing: The positive association issue is very much an empirical matter. For a recent econometric treatment, see Hildenbrand and Hildenbrand (1986).

For a fixed price vector, and assuming for simplicity that all consumers have the same income, Figure 2 represents families of population's Engel curves (for $l = 2$). Figure 2(a) and 2(c) (resp. 2(b) and 2(d)) display positive (resp. negative) association at the given price vector and income.

Suppose, to look at another limiting example, that all consumers have identical preferences. Then it follows from Hildenbrand (1983) that $C_h(p)$ will be positive definite if the distribution of income is nonincreasing on an interval $[0, b]$. (Strictly speaking, this requires a continuum of consumers. Also, the result is stronger: $\partial f(p)$ itself is negative definite.) In Freixas and Mas-Colell (1987), it is shown that if one wants the positive definiteness of $C_h(p)$ for any distribution of income on an interval $[a, b]$, then the sufficient and (in a certain sense) necessary conditions on the Engel curve associated with p is that (i) it be contained on a plane through the origin, and (ii) it display a uniform curvature condition (see Figure 3). These conditions do not guarantee the positive definiteness of $\partial f(p)$; hence, they only yield the WA.

5 Dominating substitution effects: The Mitiushin–Polterovich theorem

In this section, we still maintain the hypothesis that the distribution of income is price-independent. That is, we let $\omega_i = \theta_i \omega$, where ω is the aggregate endowment and θ_i is a constant. Then $C_\omega(p) = 0$ for all p.

Even if $C_h(p)$ fails to be positive definite, it can still happen that $S(p) + C_h(p)$, or even $\partial f(p) = S_p(p) - c(p) f^T(p) + C_h(p)$, be negative definite. This will occur if, informally, substitution effects dominate income effects.

The question is most naturally posed at the individual level: Are there conditions on the individual preferences of consumer i guaranteeing that individual excess demand be monotone with respect to ω_i? (This choice of normalizing factors makes sense since then $\omega_i = \theta_i \omega$ implies the existence of a common normalizing factor.) A remarkable and little known result of Mitiushin and Polterovich (1978) provides the answer.

Suppose that $h: R^l_{++} \to R^l_{++}$ is a demand function with income normalized to 1, that is, $p \cdot h(p) = 1$ for all p. We say that h is *strictly monotone* if $(p-q) \cdot (h(p) - h(q)) < 0$ whenever $p \neq q$.

Theorem 1. *If the C^1 demand function $h: R^l_{++} \to R^l_{++}$ (income is fixed to equal 1) is generated by the C^2, monotone, concave utility function u: $R^l_{++} \to R^l_{++}$, then a sufficient condition for the strict monotonicity of h is that*

$$\sigma(x) = -\frac{x \cdot \partial^2 u(x) x}{x \cdot \partial u(x)} < 4 \quad \textit{for all } x.$$

Proof: Mitiushin and Polterovich's proof is so simple that I would like to give it. Define $g: R^l_{++} \to R^l_{++}$ by $g(x) = [1/x \cdot \partial u(x)] \partial u(x)$, that is, $g(x)$ is collinear to $\partial u(x)$ and $x \cdot g(x) = 1$. Because $h(\cdot)$ and $g(\cdot)$ are inverses to each other, h will be strictly monotone if and only if $g(\cdot)$ is also (i.e., $(x-y) \cdot (g(x) - g(y)) < 0$ whenever $x \neq y$). A sufficient condition for the latter is that $w \cdot \partial g(x) w < 0$ for all x and $w \neq 0$.

Let $w \neq 0$ and denote $q = \partial u(x)$, $A = \partial^2 u(x)$. We can assume $q \cdot w = q \cdot x$. Differentiating, we get

$$(q \cdot x)^2 w \cdot \partial g(x) w = (q \cdot x) w \cdot A w - (q \cdot w)(x \cdot A w) - (w \cdot q)^2.$$

Since $q \cdot w > 0$, we have sign $w \cdot \partial g(x) w = \text{sign}\{w \cdot Aw - x \cdot Aw - w \cdot q\}$. But

$$w \cdot Aw - x \cdot Aw = (w - \tfrac{1}{2}x) \cdot A(w - \tfrac{1}{2}x) - \tfrac{1}{4} x \cdot Ax < -\tfrac{1}{4} x \cdot Ax < q \cdot x = q \cdot w$$

and so $w \cdot \partial g(x) w < 0$. □

If $f_i: R \to R^l$ is an excess-demand function generated by a utility function as in the theorem and the endowments ω_i, then f_i is strictly monotone with respect to the vector ω_i because $f_i(p) = h(p)$ whenever $p \cdot \omega_i = 1$. Therefore, if all utility functions satisfy the conditions of the theorem and endowments are collinear, it follows that there exists a common normalizing factor ω and so the aggregate f will be strictly monotone with respect to ω.

Note that if the utility function u is homogeneous of degree 1, then $\sigma(x) = 0$ for all x. Hence, the theorem covers the homothetic case (the case where, so to speak, the income effects are globally so well-behaved that no help is needed from substitution effects). The sufficiency condition "$\sigma(x) < 4$ for all x" is not ordinal, that is, it depends on the utility representations. The least concave representation of Debreu (1976) provides, if it is C^2, the best candidate for the sufficiency test. In fact, the following is true: If for the least concave representation we have $\sigma(x) > 4$ at all x, then h is not monotone. For an ordinal version of the condition, see Kannai (1987).

To gain some further understanding of the theorem, suppose that u is additively separable, that is, $u(x) = \sum_j u_j(x^j)$. Then the condition $\sigma(x) < 4$ for all x is equivalent to $-[x_j u_j''(x_j)/u_j'(x)] < 4$ for all x and j (see Section 7 for a discussion of the case when we have < 1 instead of < 4). If we were dealing with a von Neumann–Morgenstern utility theory over state-contingent income (so that $u_j(x_j) = \pi_j v(x_j)$, where π_j is the probability of state j and $v(\cdot)$ is the Bernouilli utility), then $-[x_j u_j''(x_j)/u_j'(x_j)]$ is the coefficient of relative risk aversion at x_j. Hence, the sufficiency condition for monotonicity is that this coefficient be less than (or equal to) 4. Apparently, this is not an implausible figure in applications (see, e.g., Friend and Blume, 1975; Grossman and Shiller, 1981; Leland, 1986; Kimball, 1988). As an incidental matter, it is worth mentioning that, as one would expect, this monotonicity result extends to the continuum-of-states case and it is thus likely that it can be used to establish uniqueness results for some general equilibrium models with financial securities.

Why the number 4? The proof makes clear where it comes from. Still one could imagine that all we are getting is a sufficiency condition and that behind it lurks a more sensible number. Surprisingly, the next example (which is the only original contribution of this section) shows that 4 is very much the magic number as far as uniqueness is concerned.

Example 1. There are two commodities and two consumers, A and B. Initial endowments are $\omega_A = \omega_B = (2, 2)$. Utility functions are

$$u_A(x_1, x_2) = x_1 + \frac{(4-\epsilon)^{\sigma_A}}{1-\sigma_A} x_2^{1-\sigma_A},$$

$$u_B(x_1, x_2) = \frac{(4-\epsilon)^{\sigma_B}}{1-\sigma_B} x_1^{1-\sigma_B} + x_2,$$

where

$$\sigma_A = \frac{\ln(1-\epsilon)}{\ln(4-\epsilon)-\ln 4}, \qquad \sigma_B = \frac{-\ln(1-\epsilon)}{\ln(4-\epsilon)-\ln(4-2\epsilon)},$$

and $\epsilon > 0$ is small.

Note that the upper bound for the "relative risk aversion" coefficients is $\max\{\sigma_A, \sigma_B\}$ and that $\sigma_A \to 4$, $\sigma_B \to 4$ as $\epsilon \to 0$ (use l'Hospital's rule to verify this). Both utility functions are linear with respect to some commodity and so they are minimally concave.

This economy has three equilibria. To make the lack of uniqueness point it is enough to exhibit two of them:

(I) $p_1 = 1,\ p_2 = 1,\ x_A = (\epsilon, 4-\epsilon),\ x_B = (4-\epsilon, \epsilon),$
(II) $p_1 = 1,\ p_2 = 1-\epsilon,\ x_A = (2\epsilon, 4),\ x_B = (4-2\epsilon, 0).$

It is easy to check that both (I) and (II) satisfy the feasibility, budget constraint, and first-order conditions.

As observed by Mitiushin and Polterovich (1978), Theorem 1 remains valid if the consumption set is not R^l_{++} but an arbitrary closed, convex cone $\Gamma \subset R^l_{++}$. To see this, note that for any p, $x = h(p)$ and $g(x)$ defined as in the proof of the theorem, the vector $g(x)-p$ supports Γ at x, that is, $(g(x)-p)\cdot(y-x) \le 0$ for any $y \in Y$. Similarly, for $y = h(q)$, we have $(g(y)-q)\cdot(x-y) \le 0$. Therefore,

$$(p-q)\cdot(h(p)-h(q)) = (p-q)\cdot(x-y) \le (g(x)-g(y))\cdot(x-y) < 0,$$

the last inequality following from the proof of the theorem. (I am grateful to V. Polterovich for clarifying this implication for me.)

Remark 1. For simplicity, we have avoided the study of correspondences. This is the only reason to require the strict concavity of utility functions. But it should be clear that with the proper definition of monotonicity ($(p-q)\cdot(x-y) \le 0$ for $x \in h(p)$, $y \in h(q)$), the previous results remain valid without strict concavity.

6 Noncollinear endowments

From the previous two sections, we could conclude that as long as the distribution of income is price-independent, we have a broad class of reasonable conditions guaranteeing the uniqueness of equilibrium.

As we will see, the situation changes drastically when initial endowments are not collinear. It is thus perhaps not surprising that the possibility of multiplicity has been taken most seriously in fields such as international trade, where differences of endowments are of the essence (see, e.g., Meade, 1952).

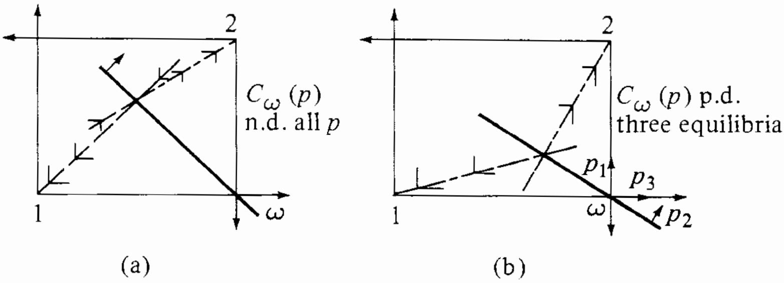

Figure 4.

Of course, there is no added complication if the covariance matrix $C_\omega(p)$ is negative definite, namely, if the marginal propensities to consume are negatively correlated with endowments. This means that people would rather consume what they don't have, which would tend to make for a lot of trade. Thus, informally, we could say that uniqueness is more likely in models that generate a large volume of trade. This is also an intuition present in international trade theory (e.g., Jones, 1970). See Figure 4 for an illustration. We will comment more on this in Section 7 when discussing production economies.

So far, so good. Lower-dimensional cases can, however, be misleading. The following two-consumers four-commodities example (taken from Mas-Colell, 1986) illustrates the difficulty we face. The utility functions in the example are as nice as they can be: linear (thus, globally, substitution effects are infinite and demand is gross substitute). Nonetheless, the weak axiom will be violated.

Example 2.

$$l=4,\ N=2;$$

$$u_1(x_1)=x_1^1, \qquad \omega_1=(0,0,a,0);$$

$$u_2(x_2)=x_2^2, \qquad \omega_2=(0,0,0,b).$$

Things could not be simpler. Endowments are only held in commodities not desired for consumption and the two agents specialize in the consumption of different commodities (in fact, there is a complete disjointness with consumer 1 living over commodities 1 and 3 and consumer 2 over 2 and 4).

Aggregate excess demand is

$$f(p)=\left(\frac{p_3}{p_1}a, \frac{p_4}{p_1}b, -a, -b\right),$$

which at $p=(1,1,1,1)$ yields the Jacobian matrix

$$\partial f(p)=\begin{bmatrix} -a & 0 & a & 0 \\ 0 & -b & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is quite clear that to exhibit a violation of the WA, it suffices to find a $v\neq 0$ such that $v\cdot f(p)=0$ and $v\cdot\partial f(p)v>0$. Put $a=60$, $b=10$, and try $v=(1/6,1,0,2)$. Then $v\cdot f(p)=0$ and $v\cdot\partial f(p)v=25/3>0$. More directly, try the price vector $p'=(25,30,24,36)$, $p''=(23,18,24,12)$. Then $f(p')=(1440/25,12,-60,-10)$ and $f(p'')=(1440/23,20/3,-60,-10)$. Hence, $p'f(p'')<0$, $p''\cdot f(p')<0$, a violation of the WA. Incidentally, it is a trivial matter to perturb the economy so as to replace the zero entries of $\partial f(p)$ by nonzero ones. To make this into a nonuniqueness example, we should specify appropriate production technologies. Because the WA is violated, we know that these exist. We point out that they can also be chosen to verify a no-joint production condition (see the technique used in Section 8).

The culprit of $v\cdot\partial f(p)v>0$ is twofold. First, $S(p)=0$, that is, there are no working substitution effects at p (this is an unavoidable consequence of assuming large substitution effects globally: consumption is pushed toward the boundary where the substitution effects are in fact nil). The matrix $C_h(p)$ is well-behaved, but $C_\omega(p)$ is quite bad. Economically, what goes on could be explained thus. If we raise p_4 by two units and p_2 by only one, then the second consumer will increase his demand of the second good (the income effect is positive). Demand and price move, therefore, in the same direction. The price change, however, is not compensated. To make it compensated in the aggregate (and this is what matters), we could try to raise the price of commodity 1 up to the point where the loss of real income by consumer 1 equals the gain by consumer 2 determined by the change of the second and fourth prices. Now, if the demand of consumer 1 is relatively large, it will only take a tiny change of the first price to accomplish this and so one may hope that when estimated quantitatively, the conclusion that prices and quantities move in the same direction is preserved. This is indeed what happens.

How particular is this example? Obviously, there is nothing pathological about it. What is essential is that when projected on the commodities never demanded, the vectors ω_1 and ω_2 be linearly independent. Suppose that these commodities are the last m. Let us assume that $c_1(p)\neq c(p_2)$. Then we can find a v_1 such that $v_1\cdot c_1(p)\neq v_1\cdot c_2(p)$ and $v_1^{l-m+1}=\cdots=v_1^l=0$. This means that, when projected on the last m coordinates,

$$(v_1\cdot c_1(p))\omega_1+(v_2\cdot c_2(p))\omega_2 \quad\text{and}\quad \omega_1+\omega_2$$

are linearly independent. Therefore, we can find a v_2 with $v_2^1 = \cdots = v_2^{l-m} = 0$ such that $v_2 \cdot (\omega_1 + \omega_2) = v_1 \cdot f(p)$ and

$$v_2 \cdot ((v_1 \cdot c_1(p))\omega_1 + (v_2 \cdot c_2(p))\omega_2) = -v_1 \cdot \partial f(p) v_1 + 1.$$

But then letting $v = v_1 + v_2$, we have $v \cdot f(p) = v_1 \cdot f(p) - v_2 \cdot (\omega_1 + \omega_2) = 0$ and $v \cdot \partial f(p) v = v_1 \cdot \partial f(p) v_1 + v_1 \cdot (c_1(p)\omega_1^T + c_2(p)\omega_2^T) v_2 = 1 > 0$. Summarizing: The WA can only hold if $c_1(p) = c_2(p)$ for all p. It is worthwhile to mention, however, that if we impose some natural restriction on technologies (such as, perhaps, no joint production), it is not at all clear that nonuniqueness examples can then be automatically generated from the violation of the WA.

For further extensions in the line of the last paragraph, see Grodal and Hildenbrand (1988) and Hildenbrand (1989).

There is some sense in which a randomly chosen economy will not have nondemanded commodities (see Kehoe, 1985a). Economies, however, do not come at random and in a production context the existence of nondemanded commodities seems a most natural phenomenon. Moreover, those are likely to be factors of production asymmetrically distributed across the population.

There is another sense in which the economy of Example 2 is special: It is the union of two disjoint economies. Again, the violation of the WA is a general feature of these economies (see Kehoe, 1986). Suppose that $f(\cdot)$ and $g(\cdot)$ are the excess demands for the two economies. Then we can choose q, q' and p, p' such that $q \cdot g(q') < 0$ and $p' \cdot f(p) < 0$. Hence, if $\alpha > 0$ is large enough, $p \cdot f(\alpha p') + \alpha q \cdot g(q') < 0$ and $\alpha p' \cdot f(p) + q' \cdot g(q) < 0$. In other words, the disjoint union of f and g violates the WA at $(p, \alpha q)$ and $(\alpha p', q')$. Of course, the fact that the economies are disjoint in consumption and endowments does not mean that they cannot be linked via the (unspecified) technology.

The conclusions we draw from the example (we emphasize again its simplicity: two consumers and linear utilities!) is that with noncollinear endowments, an attack on the uniqueness question for general production economies based entirely on demand conditions (or, in other words, via the WA) is most unpromising. Production conditions must enter the picture. The next three sections will take them into account.

7 Exchange economies and gross substitution

We now consider a very special production restriction, namely, there is no production at all. We then know that the price vector p is an equilibrium if and only if $f(p) = 0$. Can we use this information on the nature of equilibrium to sharpen the previous results?

There is a clear sense in which this can be done. The negative definiteness conditions: "$v \cdot \partial f(p) v < 0$ for $v \neq 0$, $p \cdot v = 0$" need only be required at equilibrium, that is, whenever $f(p) = 0$. This is a well-known consequence of mathematical index theory (see Dierker, 1972; Varian, 1975; Mas-Colell, 1985). In fact, the weaker property: "whenever $f(p) = 0$ a $(l-1) \times (l-1)$ principal minor of $\partial f(p)$ has sign $(-1)^{l-1}$" will do. The more interesting question, however, is: Is there some significant condition on individual agents' characteristics that would yield negative definiteness at equilibrium without necessarily giving it outside of equilibrium?

As it turns out (all this is well known), there is such a condition, namely, gross substitution, that is, the diagonal (resp. off-diagonal) entries of $\partial f(p)$ are negative (resp. positive). For the negative definiteness implication at equilibrium, see, for example, McKenzie (1960). Any example of a gross substitute excess demand violating the Weak Axiom (such as Example 2 in Section 6) is also an example of a gross substitute excess demand violating negative definiteness at some price (Example 2 of Section 6 improves on the classical example of Kehoe, 1985b, in two respects: the elasticity of substitution is infinite rather than 1 and the number of consumers is 2 rather than 4).

Undoubtedly, for highly disaggregated models, gross substitution is most implausible. But for aggregate models, while restrictive, it is not strained. For two commodities, it precisely means that substitution effects dominate income effects. More generally, gross substitution should be thought of as a stronger property than monotonicity. On the precise relationship between the two properties, we can make the following comments:

1. Both properties are preserved under aggregation with the advantage on the side of gross substitution that there is no need to keep track of normalizing factors.
2. In the two-commodity world, an excess demand function f satisfies gross substitution if and only if it is monotone for any normalizing factor $z \geq 0$, $z \neq 0$. In Figure 1(a), there is an example of an excess demand for $l = 2$, which is monotone only with respect to some z (and is not gross substitute). In Figure 1(b), there is a gross substitute example.
3. In a two-commodity world with homothetic preferences, gross substitution obtains if the elasticity of substitution is everywhere larger than 1 (see Fisher, 1972).
4. If there are more than three commodities, gross substitution does not imply monotonicity (see Example 2 in Section 6). For $l = 3$, I do not know (see Kehoe and Mas-Colell, 1984, for the Weak Axiom).

5. Suppose that f is the excess-demand function generated by some ω and the C^1 demand function $h(p,w)$. If demand is normal, that is, if $\partial_w h(p,w) \geq 0$ for all (p,w), and f is gross substitute, then f is also monotone with respect to ω (see Polterovich and Spivak, 1986).
6. Suppose that f is the excess-demand function generated by some ω and the C^2 separable utility function $u(x) = \sum_i u_i(x_i)$. If $-[x_i u_i''(x_i)/u_i'(x_i)] < 1$ for all x_i and i, then f is gross substitute (see Varian, 1985, for a proof and a history of this result; see also Slutzky, 1915, Section 10). Monotonicity with respect to ω will also obtain. This follows from the observation in number 5 or from the fact that $-[x_i u_i''(x_i)/u_i'(x_i)] < 4$ for all x_i and i suffices (see Section 5).

Is gross substitution a helpful property for the study of more general production economies? We will see in the next section that it is, but only after the production economy has been reduced to an exchange one. Any attempt at a direct application faces the difficulty that the firms' demand functions for inputs cannot be expected to be gross substitutes. Indeed, with a technology close to constant returns, an increase in the price of an input has a negative effect on the supply of output and, therefore, on the demand of all other inputs (see Rader, 1972).

8 Reducing production economies to exchange economies

In this section, we will consider production economies of a particularly simple type, namely, those characterized by constant returns no-joint production technologies.

Without loss of generality, we will assume that the l commodities are of two types: (a) factors of production (r in number) owned by consumers and not entering their utility functions, and (b) consumption, or possibly intermediate, goods (m in number). By expanding the set of commodities and technologies, it is always possible to adopt this format. Of course, $l = m + r$.

These are N consumers with utility functions $u_i: R^m_+ \to R$ and endowment vectors $\omega_i \in R^r_+$. Also, for every $j \leq m$, there is a convex production cone $Y_j \subset R^l$ with the property that at most the jth component of any $y \in Y_j$ can be positive. (Thus, we can as well identify Y_j with the production function for good j.) For every $z \in R^r_+$, let $A(z) = \{x \in R^m_+ : (x, -z) \in \sum_{j=1}^m Y_j\}$. We assume that this convex set is closed and bounded. Finally, we restrict ourselves to economies where at equilibrium every consumption good is produced.

There is a simple, and familiar (it is usually attributed to T. Rader), way to associate with the production economy defined by u_i, ω_i, Y_j a *reduced-factor exchange economy* on the space of factors of production. We simply let agent i have endowments ω_i and utility function $u_i^*(z_i) = \max\{u_i(x_i): x_i \in A(z_i)\}$. From now on, we will use an asterisk (*) to denote reference to the reduced economy. Under standard conditions on the original utility functions and technologies, the reduced-factor exchange economy generates an aggregate excess-demand function $f^*(p)$. The key fact is that the zeros of this excess demand correspond exactly to the equilibrium factor prices of the original production economy (by the nonsubstitution theorem, consumption good prices are uniquely determined from factor prices). The idea of studying the production economy by means of the reduced exchange economy goes back to at least Taylor (1938) and it has since been used many times. See, for example, Chapter 9 by J. Drèze in this volume.

Once reduced to the exchange case, we have two approaches to uniqueness: (i) the weak-axiom-based theory, and (ii) the gross-substitution-based theory. We comment on them in turn.

The weak-axiom theory still awaits careful development. What is needed is that $C_h^*(p)$ and $C_\omega^*(p)$ be, respectively, positive (and negative) semidefinite. The positive semidefiniteness of $C_h^*(p)$ will follow from considerations similar to the ones in Sections 4 and 5 once one takes into account that many properties are preserved in going from u_i to u_i^* (e.g., if u_i is homogenous of degree 1, so is u_i^*; the relationship between $\sigma^*(\cdot)$ and $\sigma(\cdot)$ – see Theorem 1 – needs to be investigated). The negative semidefiniteness of $C_\omega^*(p)$ will depend now on factor-intensity considerations (built into the definition of u_i^*). Informally, $C_\omega^*(p)$ will be negative semidefinite if, on average, consumers have a higher marginal propensity to consume commodities whose production is relatively more intensive on factors they are less endowed with than the mean of the population. In a world with two factors (say, capital and labor), two goods (say, necessities and luxuries), and two classes (say, poor and rich; the poor consumer needs relatively more necessities than the rich), this would say that the production of necessities must be more capital-intensive than the production of luxuries (for a similar insight, see Jones, 1972). Clearly, requirements of this sort may create problems in applications (thanks to T. Kehoe for this observation).

As for the gross-substitution theory, perhaps it can be said that if the number of factors is not large, then it may be the condition most often satisfied in practice. The difficulty raised in the previous section does not arise and it is, in fact, not a ridiculously implausible condition. In terms of primitives, we have, for example, the following theorem.

Theorem 2. *If every utility function u_i and every technology Y_j is of the Cobb-Douglas variety, then the utility functions u_i^* of the reduced-factor exchange economies are also Cobb-Douglas.*

Proof: Let $\beta \in R_+^m$ be the vector of Cobb-Douglas coefficients of u_i. Denote by E and F, respectively, the $m \times m$ and $r \times m$ matrices of coefficients corresponding to the technologies. If there are no intermediate goods, then $E = 0$. We can assume that $(I-E)^{-1}$ exists.

For any vector of factor prices $q \in R^r$, let $b_q \in R_+^r$ be the vector of expenditure shares solving the problem: "Max $u_i^*(z)$ s.t. $q \cdot z \leq 1$." Because of the definition of u_i^*, the solution of this problem can be attained by means of a price vector $p \in R_+^m$, a consumption that solves "Max $u_i(x_i)$ s.t. $p \cdot x_i \leq 1$" and production vectors (one for every technology) that are profit-maximizing with respect to (p, q). But then a standard computation from input-output analysis gives $b_q = F(I-E)^{-1}\beta$. Therefore, the factor demand budget shares are independent of the factor prices. Since this independence is a characterizing property of Cobb-Douglas, we have reached our conclusion. □

The implication of Theorem 2 is that $f^*(\cdot)$ will satisfy the (weak) gross-substitute property and, thus, the equilibrium will be unique (strictly speaking, this is correct only if the economy is regular). Cobb-Douglas exchange economies have been extensively studied (see Afriat, 1987; and Eaves, 1985). For the production case, it is hard to believe that the previous uniqueness result is not in the literature, but I have been unable to find a reference. The result has also been obtained (simultaneously and independently) by Jerison (1988).

In the next section, we will provide a generalization of the uniqueness implication of Theorem 2 to a more general class of preferences and technologies.

We conclude this section with a remark. Theorem 2 excludes the possibility that agents have endowments of consumption goods. This is with loss of generality because the expansion trick referred to at the beginning of the section will not preserve the Cobb-Douglas character of the technologies. However, in the terminology of the next section, it preserves the super Cobb-Douglas character and thus the uniqueness implication is covered by Theorem 3 in the next section (see example (b) in Section 9).

9 Super Cobb-Douglas economies

In this section, we generalize the uniqueness implication of Theorem 2 to the equilibria of (regular) no-joint production economies, where, in a precise sense, preferences and technologies exhibit as much substitution as in a Cobb–Douglas world.

Definition 3. *The function* $h: R^s_+ \to R$ *is* super Cobb–Douglas *if at every* $x \geq 0$, *there is a Cobb–Douglas function* $h_x: R^s_+ \to R$ *such that* $h_x(x) = h(x)$ *and* $h_x(x') \leq h(x')$ *for all* x' *in a neighborhood of* x.

A concave, super Cobb–Douglas function is necessarily homogeneous of degree 1 and C^1 on R^s_{++}. Also, the Cobb–Douglas function h_x is entirely determined by $h(x)$ and $\partial h(x)$.

If h is continuous and C^2 on R^s_{++}, the super Cobb–Douglas property can be equivalently formulated as: "For every $x \gg 0$, let h_x be the (unique) Cobb–Douglas function with $h_x(x) = h(x)$, $\partial h_x(x) = \partial h(x)$. Then $\partial^2 h(x) - \partial^2 h_x(x)$ is positive semidefinite."

It is of interest to point out that the excess-demand function generated by a super Cobb–Douglas utility function does not need to satisfy the gross-substitute property.

Examples of super Cobb–Douglas functions include the following.

(a) All the C.E.S. functions with elasticity of substitution coefficient larger or equal to one (compare with John, 1989).

(b) Any function obtained as a nesting of super Cobb–Douglas functions. Specifically, suppose that $h^j: R^q \to R_+$, $j \leq s$, and $g: R^s \to R_+$ are super Cobb–Douglas. Then $G(x) = g(h^1(x), \ldots, h^s(x))$: $R^q \to R$ is also super Cobb–Douglas. Indeed, at any x' close to x, we have

$$G(x') = g(h^1(x'), \ldots, h^s(x')) \geq g_v(h^1(x'), \ldots, h^s(x'))$$
$$\geq g_v(h^1_x(x'), \ldots, h^s_x(x')),$$

where $v = (h^1(x), \ldots, h^s(x))$ and, of course, $g_v(h^1_x(x'), \ldots, h^s_x(x'))$ is a Cobb–Douglas function. As a particular case, note that a sum of super Cobb–Douglas functions is super Cobb–Douglas.

(c) Suppose that $h^1, h^2: R^s_+ \to R$ are concave, super Cobb–Douglas functions. Then the function $h: R^s_+ \to R$ defined by

$$h(x) = \max\{h_1(x_1) + h_2(x_2): x_1 + x_2 \leq x\}$$

is also (by way of proof, note that the graph of h is the convex hull of the graphs of h^1 and h^2). Because of this, the theory of this section allows implicitly for technological substitution (among super Cobb–Douglas technologies) in the production of any good.

The proof of the following result will rely again on the concept of a reduced factor exchange economy.

Theorem 3. *Suppose that every utility function u_i and every technology Y_j is super Cobb–Douglas. Then every (regular) economy has a unique equilibrium.*

Proof: We understand that the concept of regularity includes the differentiability of the relevant functions.

The proof will be by means of index theory (see Mas-Colell, 1985, Chap. 5, for an account). Namely, we will show that if f^* is the excess demand of the reduced exchange economy and $f^*(\bar{q})=0$, then the determinant of $(-1)^{(r-1)}\partial f^*(\bar{q})$, when viewed as a map from $T_q=\{v\in R^r: q\cdot v=0\}$ to itself, is necessarily nonnegative. For this it suffices that $\partial f^*(\bar{q})$ be negative semidefinite (n.s.d.).

For every q, let $c_j(q)$ be the minimum cost at which one unit of consumption good j can be produced. By the nonsubstitution theorem, this is a well-defined concept. Put $c(q)=(c_1(q),\dots,c_m(q))$. Denote by $B(q)$ the $r\times m$ matrix of cost-minimizing total (i.e., direct and indirect) factor requirements. Of course, $\partial c(q)=B^T(q)$ (Roy's identity). Let $b_j(q)$ be the jth column of $B(q)$ and put $E(q)=[B(q),I]$, an $r\times(m+r)$ matrix.

The excess demand of the nonreduced economy is denoted $f(p,q)$.

Direct computation yields

$$\partial f^*(q)=\sum_{j=1}^{m} f_j(c(q),q)\partial_q b_j(q)+E(q)\partial f(c(q),q)E^T(q).$$

Suppose that we now consider the Cobb–Douglas economy associated with the consumptions and productions of the equilibrium $\bar{q}$ (i.e., we take locally minorizing Cobb–Douglas functions with the same values at the given consumption and production). From now on a hat (ˆ) denotes reference to the Cobb–Douglas economy.

Because of the minorizing property, we have: (a) $c_j(q)\le\hat{c}_j(q)$ for q near $\bar{q}$ and $c_j(\bar{q})=\hat{c}_j(\bar{q})$. Therefore, $\partial_q b_j(\bar{q})-\partial_q\hat{b}_j(\bar{q})$ is n.s.d. for every j. Similarly, if we denote by $S_i(p,q)$ the substitution matrix of i, we have (b) $S_i(c(\bar{q}),\bar{q})-\hat{S}_i(c(\bar{q}),\bar{q})$ is n.s.d. for every i. Denote $\gamma_j=f_j(c(\bar{q}),\bar{q})\ge 0$. Observe that because of the homogeneity of the utility functions, the income effects of the original and the Cobb–Douglas economies are the same. Of course, the same is true for the values γ_j and the matrix $E(\bar{q})$.

Substituting, we have

$$\partial f^*(\bar{q})=\partial\hat{f}^*(\bar{q})+\sum_j\gamma_j(\partial_q b_j(\bar{q})-\partial_q\hat{b}_j(\bar{q}))$$
$$+\sum_i E(\bar{q})(S_i(c(\bar{q}),\bar{q})-\hat{S}_i(c(\bar{q}),\bar{q}))E^T(\bar{q}).$$

By the observation of the last paragraph, the second and third terms of this sum are n.s.d. matrices. As for the first, note that $\hat{f}^*(\bar{q})=0$ and, by Theorem 2, $\hat{f}^*$ is generated by a Cobb–Douglas exchange economy. Therefore, $\partial\hat{f}^*(\bar{q})$ is also n.s.d. (see Section 7). We conclude that, as we wanted, $\partial f^*(\bar{q})$ is n.s.d. □

As a final remark, we point out that, as the proof makes clear, the super Cobb–Douglas character of the economy is a sufficient, but far from necessary, condition. The crucial condition is that, so to speak, the economy be Cobb–Douglas in the average (the precise average, however, depends on endogenously generated variables).

10 Some remarks on equilibrium dynamics

So far, we have concerned ourselves with static economies. However, part of the motivation for this work is the observation that there seems to be a correspondence between uniqueness conditions for static economies and conditions for the asymptotic convergence of equilibrium trajectories in stationary dynamic economies (i.e., the turnpike property). For example, in Kehoe et al. (1986), it is shown that gross substitution yields the turnpike property in exchange, stationary, overlapping generations economies. While it can be presumed that this correspondence is very general (e.g., Drandakis, 1963), the fact of the matter is that very little is known about it. It appears to be an area well deserving of further research.

In this section, we add a further piece of evidence. Consider an overlapping generations economy (with production). Generation t has an excess-demand function $f_t(p_t, p_{t+1})$ and a production set Y_t. We assume stationarity, that is, f_{t+1} and Y_{t+1} are generated from f_t, Y_t by a shift operator. Suppose that the economy admitted an equilibrium cycle in prices, that is, suppose there are $(p_1, \ldots, p_n) \in R^{ln}$ such that the price sequence $p_t = p_{\text{mod}_n t}$ supports an equilibrium allocation. We could then define a pseudo-static economy over R^{ln} by replacing the mod n shift of f_{n-1}, Y_{n-1} for f_n, Y_n (observe, incidentally, that the resulting economy cannot have a price-independent distribution of income). Because any mod n shift of $(p_1, \ldots, p_n)$ is an equilibrium of this economy, we can conclude that the pseudo-static economy can have a unique equilibrium only if the cycle is trivial, that is, $p_1 = \cdots = p_n$. Also, there can be only one stationary equilibrium price sequence. Therefore Theorem 3 (Section 9) has the following consequence.

Corollary 1. *If the generation's characteristics of the (stationary) economy are generated from super Cobb–Douglas utility and production functions, then there is at most one stationary equilibrium price sequence and there can be no nontrivial cyclical equilibrium price sequences.*

Corollary 1 is only a very partial result. It begs to be extended in several directions:

1. The corollary yields the uniqueness of the so-called monetary steady states (i.e., stationary price sequences). What about the uniqueness of the nonmonetary ones (i.e., constantly inflationary or deflationary price sequences)?
2. Does the turnpike property – that is, the asymptotic convergence to a steady state – also follow?
3. Can the result be extended to the case of infinitely lived agents and production sets?
4. In a more speculative vein: Does the super Cobb–Douglas property rule out the possibility of stationary stochastic ("sunspot") equilibria?

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