9 Equilibrium Theory with Possibly Satiated Preferences*

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1 INTRODUCTION

In the applications of equilibrium theory one occasionally encounters situations where consumption sets are naturally compact. Two examples are:

(i) fix-price equilibria where consumption is restricted to a given budget set and the equilibrating variables are ration-coupons prices (as in Drèze and Muller, 1980);
(ii) models where the choice variables are probability distributions on a fixed number of indivisible objects (as in Hylland and Zeckhauser, 1979).

If consumption sets are compact then preferences are satiated. A way to proceed is to avoid this conclusion by assuming the existence of an extra commodity, a desirable *numéraire* (of course, then the consumption set ceases to be compact). This is done, for example, by Drèze and Muller, (1980). It may be worthwhile, however, to explore how far we can go in developing an equilibrium theory with satiation. This I shall try.

It is of course well known that with the usual definition a Walrasian equilibrium may fail to exist if preferences are satiated. But it is also clear from the literature that this is not the end of the story. The existence of an equilibrium (which we call Walrasian equilibrium with slack) can be restored if we give consumers appropriate extra amounts of income to spend (or not, as they see fit). Theorems 1 and 2 deal with this. A similar modification of the equilibrium notion has been proposed by Drèze and Muller, (1980). A limitation of the concept of Walrasian equilibrium with slack is that the optimality of equilibria is not guaranteed. By strengthening the notion of equilibrium Theorem 3 shows that if preferences satisfy a weak condition (representability by concave utility functions suffices) then

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among the Walrasian equilibria with slack there is at least one which is an optimum.

All the hypotheses I shall use are standard. No free-disposal is assumed. This is done not to restrict indirectly the kind of satiation phenomena to be handled.

My interest in this topic was aroused years ago by David Gale who came to me with a question similar to the lottery allocation problem of Hylland and Zeckhauser (1979). His interest in the problem, which he has pursued, has been focused on the uniqueness and incentives properties of the derived equilibrium problem, rather than on existence. Thus this never became a joint paper. But it is very appropriate that it be published in a volume in his honour.

2 SETTING AND DEFINITIONS

The description of the basic model is similar to Gale and Mas-Colell (1975).

There are M firms with nonempty *production sets* $Y_j \subset R^l$, $1 \leqslant j \leqslant M$.

There are N consumers with nonempty *consumption sets* $X_i \subset R^l$, $1 \leqslant i \leqslant N$.

The *price set* is $B = \{p \in R^l : \| p \| \leqslant 1\}$.

The *state space* is $Z = Y_1 \times \ldots \times Y_M \times X_1 \times \ldots \times X_N \times B$ with generic entry $z = (y, x, p) = (y_1, \ldots, y_M, x_1, \ldots, x_N, p)$.

Every consumer i is endowed with a *preference map* $P : Z \to X_i$.

There are N distribution functions $m_i : Y_1 \times \ldots \times Y_N \times B \to R$ satisfying $\Sigma_i\, m_i(y, p) = \Sigma_j\, p \cdot y_j$ and $m_i(y, 0) = 0$ for all y, p and i.

The *economy* is then

$$\mathcal{E} = (Z, \{P_i\}_{i=1}^N, \{m_i\}_{i=1}^N)$$

Maintained hypotheses throughout this study are:

(I) Every Y_j is *closed* and *convex*.
(II) Every X_i is *closed* and *convex*.
(III) Every P_i is *irreflexive* (i.e., $z_i \notin P_i(z)$ for every $z \in Z$), *convex-valued* (i.e., $P_i(z)$ *is convex for every* $z \in Z$) *and has an open graph.*
(IV) Every $m_i(y, p)$ is *continuous*.

The hypotheses so far do not even guarantee the existence of a *feasible state* (i.e., a $z \in Z$ with $\Sigma_j\, y_j = \Sigma_i\, x_i$), let alone any sort of equilibrium. In order to push forward the analysis I shall require that the distribution functions satisfy a survival-like hypothesis.

For any $p \in B$ let $\Pi(p) = \Sigma_j \sup p \cdot Y_j$. In particular, $\Pi(p) = +\infty$ is admissible. If there is a feasible state we obviously should have $\Pi(p) \geqslant \Sigma_i \inf p \cdot X_i$ for all p (i.e., maximum income can in principle be distributed so that every consumer can afford a consumption in his budget set). The next hypothesis requires that what can be done in principle it is in fact done by the given distribution functions (it also replaces weak by strict inequalities).

(V) For any $(y, p) \in Y_1 \times \ldots Y_M \times B$ such that $p \neq 0$ and $\Sigma_j\, p \cdot y_j = \Pi(p)$ we have $m_i(y, p) > \inf p \cdot X_i$ for all i.

Hypothesis V implies, in particular, that $\Pi(p) > \Sigma_i \inf p \cdot X_i$ for every $p \neq 0$. This will be satisfied if, for example, $0 \in X_i$ for every i and 0 belongs to the interior of $\Sigma_j\, Y_j$.

The indispensability of hypotheses I–V can be established by familiar examples. The convexity condition rules out indivisibilities.

Note that consumption, but not production, externalities are permitted.

Definition A state $z = (y, x, p) \in Z$ is a Walrasian equilibrium with slack if there is $\alpha \geqslant 0$ such that:

(a) $p \cdot y_j \geqslant p \cdot Y_j$ for all j (profit maximisation).
(b) $p \cdot x_i \leqslant m_i(y, p) + \alpha$ and '$v \in P_i(z) \Rightarrow p \cdot v > m_i(y, p) + \alpha$' (preference maximisation).
(c) $\Sigma_j\, y_j = \Sigma_i\, x_i$ (feasibility).

Remark If the income slack term α is zero then the above is the usual definition of Walrasian equilibrium. The slack, however, makes a difference in the sense that a state z may be an equilibrium with slack but not without it. This is the case for state (ω, x, p) in Example 1.

Example 1 This is sufficiently well described by Figure 9.1.

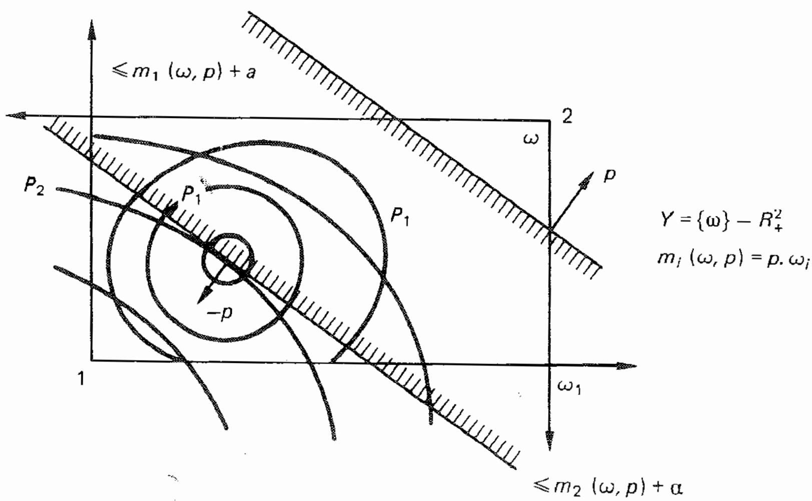

Figure 9.1

Remark I have taken the slack term to be the same for every consumer. Clearly, there is nothing essential here. I have done this just to be definite. More generally, one would have a given distribution rule accomplishing the function of transferring wealth from consumers that do not want it to consumers that can use it.

3 EXISTENCE OF EQUILIBRIA WITH SLACK

Example 2 shows that if satiation is possible then under the maintained hypothesis I–V an equilibrium may not exist.

Example 2 This is sufficiently well described by Figure 9.2. Note that in Figure 9.2 an equilibrium does not exist but that $z = (\omega, 0, \omega, p)$ is an equilibrium with slack for $\alpha = p \cdot \omega_1$.

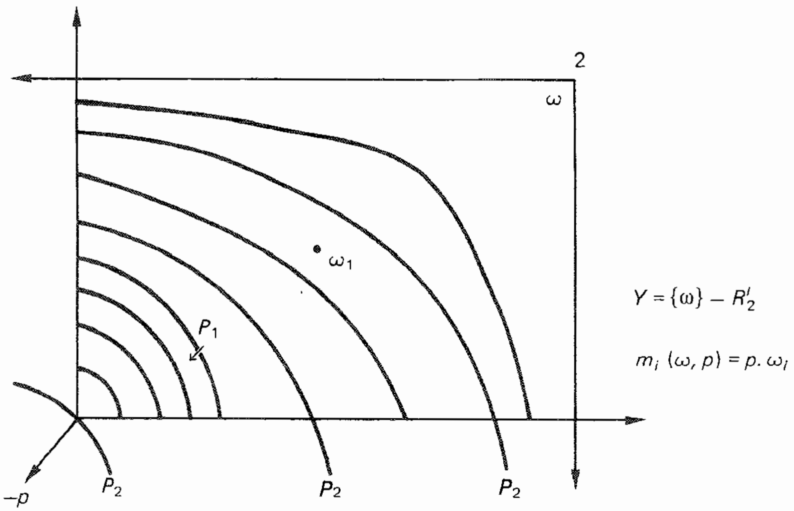

Figure 9.2

Since I do not want to rule out satiation, I will concentrate in this section on proving the existence of Walrasian equilibria with slack. As already mentioned a similar approach in the context of fix-price modelling has been followed by Drèze and Muller (1980).

For clariy this section will proceed as follows. Theorem 1 establishes existence under the maintained hypothesis I–V and the additional hypothesis that all consumption and production sets are compact. Theorem 2 then takes care of the general case. This has the advantage of concentrating all the uninteresting technicalities in the proof of Theorem 2. Also, the compactness of the consumptions is one of the main reasons why preferences may be satiated (e.g., Drèze and Muller 1980, or Hylland and Zeckhauser, 1979).

Theorem 1 An equilibrium with slack exists under the hypotheses I–V and:

(VI) All the sets Y_j and X_i are compact.

Theorem 1 will be derived by means of a fixed point theorem for open-graph, convex-valued correspondences proved in Gale and Mas-Colell (1979).

Fixed Point Theorem Let $S_1, \ldots, S_H \subset R$ be nonempty, compact, convex sets. Put $S = S_1 \ldots \times S_H$. Suppose that $F_h{:}S \to S_h$, $h = 1, \ldots, H$, are convex-valued correspondences with an open graph. Then there is $s \in S$ such that, for every h, we have either $s_h \in F_h(s)$ or $F_h(s) = \emptyset$.

The Fixed Point Theorem admits the following interpretation. We have H players and S_h is the strategy space of player h. The set $F_h(s)$ represents the preferred moves of player h. So, it is natural to require $s_h \notin F_h(s)$ for every h (irreflexivity of preferences). Then the theorem tells us that if F_h has an open-graph (continuity) and $F_h(s)$ is convex for every h, then there is $s \in S$ with $F_h(s) = \emptyset$ for all h (i.e., s constitutes an equilibrium point).

Proof of Theorem 1 In order to apply the Fixed Point Theorem, the 'players' will be: firm j with strategy set Y_j, $1 \leq j \leq M$, consumer i with strategy set X_i, $1 \leq i \leq N$, and a market agent in charge of prices with strategy set B. This makes a grand total of $M + N + 1$ players. We now define for every player a preferred move mapping from Z to his strategy set.

(a) *Producers* For every j and $z \in Z$ let $F_j(z) = \{ v \in Y_j{:}p \cdot v > p \cdot y_j\}$ (i.e., the set of productions which give higher profits at the price vector p). Obviously, the correspondence $F_j{:}Z \to Y_j$ is convex-valued, irreflexive and has an open graph.
(b) *Consumers* The interpretation of the following construction of preferred moves for the consumers is that there is an underlying priority ordering: anything in the (relaxed) budget set is preferred to anything outside of it. Once in the budget set the preference map P_i applies. Specifically,

$$F_i(z) = \begin{cases} \left\{ v \in X_i : p \cdot v < m_i(y, p) + \frac{1-\|p\|}{\|p\|} \right\} \\ \qquad \text{if } p \cdot x_i > m_i(y, p) + \frac{1-\|p\|}{\|p\|} \\ \left\{ v \in X_i : p \cdot v < m_i(y, p) + \frac{1-\|p\|}{\|p\|} \right\} \cap P_i(z) \\ \qquad \text{if } p \cdot x_i \leqslant m(y, p) + \frac{1-\|p\|}{\|p\|} \end{cases}$$

The priority ordering trick appears in Gale and Mas-Colell (1975). The budget relaxation idea (and the form $(1 - \|p\|)/\|p\|$) is due to Bergstrom (1976); see also Shafer (1976). It was invented to handle nonfree disposal. As we can see, it turns to be also very useful to deal with satiation.

The correspondence F_i: $Z \to X_i$ is obviously convex-valued and irreflexive. It is almost as obvious that it has an open graph. Note that the smaller is $\|p\|$ the more we relax the budget constraint. Remembering that X_i is compact this implies that for $\|p\|$ sufficiently small we always have $F_i(z) = P_i(z)$.

(c) *Market agent* The market agent moves prices so as to increase the value of excess demand

$$F_0(z) = \left\{ q \in B : q \cdot \left(\sum_i x_i - \sum_j y_j\right) > p \cdot \left(\sum_i x_i - \sum_j y_j\right) \right\}$$

This is a classical construction and its economic logic is clear. Every commodity which is in excess demand (respectively, supply) has its price raised (respectively, lowered). This is what we would expect if the responsibility of the market agent was to strive for the equalisation of demand and supply.

Obviously, F_0: $Z \to B$ is convex-valued, irreflexive and has an open graph.

All the conditions of the Fixed Point Theorem are met. Therefore, there is a state $z = (y, x, p) \in Z$ such that $F_j(z) = \varnothing$ for every j, $F_i(z) = \varnothing$ for every i and $F_0(z) = \varnothing$. We show now that z is an equilibrium with slack for $\alpha = (1-\|p\|)/\|p\|$ (if $\|p\| = 0$ then take α finite but very large). The three conditions of the definition must be verified:

(a) *Profit maximisation* By definition $F_j(z) = \varnothing$ means that $p \cdot y_j \geqslant p \cdot Y_j$. Hence condition (a) holds. Note that, as a consequence, $\Sigma_j\, p \cdot y_j = \Pi(p)$.

(b) *Preference maximisation* If $p = 0$ then $F_i(z) = \varnothing$ is equivalent to

$P_i(z) = \emptyset$. Hence, the relaxed budget constraint and preference maximisation condition hold vacuously. Suppose that $p \neq 0$. Then by Hypothesis V and the previous profit maximisation conclusion we have $m_i(y, p) > \min p \cdot X_i$. Therefore $F_i(z) = \emptyset$ is possible only if $p \cdot x_i \leqslant m_i(y, p) + \alpha$ and $v \in P_i(z)$ implies $p \cdot v \geqslant m_i(y, p) + \alpha$. Suppose that $v \in P_i(z)$ and $p \cdot v = m_i(y, p) + \alpha$. A standard argument will yield a contradiction. Choose $v' \in X_i$ such that $p \cdot v' < m_i(y, p)$. Because X_i is convex and $P_i(z)$ open there is $v'' \in P_i(z)$ such that $p \cdot v'' < m_i(y, p) + \alpha$. A contradiction. Therefore, $v \in P_i(z)$ implies $p \cdot v > m_i(y, p) + \alpha$.

(c) *Feasibility* Suppose, by way of contradiction, that $\Sigma_i x_i - \Sigma_j y_j \neq 0$. Then the problem $\max_{q \in B} q \cdot (\Sigma_i x_i - \Sigma_j y_j)$ has a positive value and any maximiser $\hat{q}$ is such that $\|\hat{q}\| = 1$. From $F_0(z) = \emptyset$ we know that p is one such maximiser. Hence $\|p\| = 1$, and $p \cdot (\Sigma_i x_i - \Sigma_j y_j) > 0$. But $\|p\| = 1$ implies $\alpha = 0$ and the previous conclusion on preference maximisation then yields $\Sigma_i p \cdot x_i \leqslant \Sigma_i m_i(y, p) = \Sigma_j p \cdot y_j$. This contradiction shows that $\Sigma_i x_i = \Sigma_j y_j$.

Theorem 2 strictly generalises Theorem 1 by dropping hypothesis VI. The proof involves a number of technicalities. This is why we presented the result for the compact case first and separately.

The *feasible* set of states is

$$\hat{Z} = \left\{ z \in Z: \sum_i x_i = \sum_j y_j \right\}$$

It is trivial to find examples where with $\hat{Z}$ unbounded an equilibrium, with or without slack, may not exist. Just consider a one producer, one consumer problem and choose Y, X and P so that no maximum of P is attained in Y. Hence we assume:

(VII) $\hat{Z}$ is compact.

Sufficient conditions on the primitive data Y_j, X_i for VII to hold are, for example: every X_i is bounded below, $-R^l_+ \subseteq Y_j$ for every j, and $\lambda v \in \Sigma_j Y_j$, $-\lambda v \in \Sigma_j Y_j$ for any $\lambda > 0$ implies $v = 0$ (irreversibility). See Debreu (1959, Ch.5) for the closedness and boundedness properties of feasible sets.

Theorem 2 Under hypotheses I–V and VII there is a state $z = (y, x, p) \in Z$ which is a Walrasian equilibria with slack. Moreover, for every i such that $P_i(z) \neq \emptyset$ we have $p \cdot x_i = m_i(y, p) + \alpha$.

Proof By Hypothesis VII there is $r > 0$ such that the box $(-r, r)^{l(M+N+1)}$

contains $\hat{Z}$. Put $X'_i = X_i \cap [-r, r]^l$, $Y'_j = Y_j \cap [-r, r]^l$, $Z' = Y'_1 \times \ldots \times Y'_M \times X'_1 \times \ldots \times X'_N \times B$. Of course, $\hat{Z} \subseteq Z'$.

Theorem 1 will be applied to the compact sets X'_i and Y'_j. But first I should transform the preference map and the distribution functions appropriately (see Gale and Mas-Colell, 1975 and 1979, for similar constructions). I deal with the preference maps in three steps. Fix an i.

Step 1 Let L_i be the smallest linear space containing X_i. Then for every $z \in Z$ replace $P_i(z)$ by its interior on L_i, denoted $P'_i(z)$. The map $z \mapsto P'_i(z)$ is irreflexive, convex-valued and has a graph open relative to $Z \times L_i$ (not just $Z \times X_i$).

Step 2 Replace now $P'_i(z)$ by $P''_i(z) = \{\lambda x_i + (1 - \lambda)\upsilon\colon\ 0 < \lambda \leq 1, \upsilon \in P'_i(z)\}$ see Figure 9.3. Then it is easy to see that $z \mapsto P''_i(z)$ is again irreflexive, convex-valued and has an open graph (because the graph of P'_i is open relative to $Z \times L_i$). Moreover, if $P_i(z) \neq \emptyset$ then $P'_i(z) \neq \emptyset$ and so, $x_i \in$ closure $P''_i(z)$.

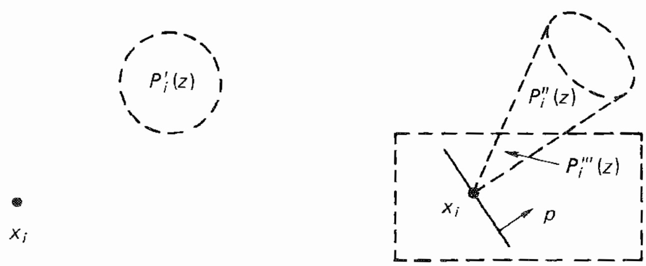

Figure 9.3

Step 3 Replace $P''_i(z)$ by $P'''_i(z) = P''_i(z) \cap (-r, r)^l$. Again $z \mapsto P'''_i$ is irreflexive, convex-valued and has an open graph. Moreover, if $z \in \hat{Z}$ and $P_i(z) \neq \emptyset$ then $x_i \in$ closure $P'''_i(z)$ (see Figure 9.3).

I now handle the distribution functions. While straightforward this is a bit technical and I skip some simple details.

Let U be a bounded, open neighbourhood of $\hat{Z}$. For every j (resp. every i) and $p \in B$ define $\Pi_j(p) = \sup p \cdot Y_j$ (resp. $g_i(p) = \inf p \cdot X_i$) and put $\Pi(p) = \Sigma_j \Pi_j(p)$ (resp. $g(p) = \Sigma_i g_i(p)$). The extended real valued function $h(p) = \Pi(p) - g(p)$ is convex and homogeneous of degree 1. Hypothesis V tells us that $h(p) > 0$ whenever $\|p\| = 1$. By a simple continuity argument we can assert that if r is large enough and we define $\Pi'_j(p) = \sup p \cdot Y'_j$, $g'_i(p) = \inf p \cdot X'$, then $h'(p) = \Sigma_j \Pi'_j(p) - \Sigma_i g'_i(p) > 0$ whenever $\|p\| = 1$ and $\Pi'_j(p) = \Pi_j(p)$, $g_i(p) = g'_i(p)$ whenever $z \in U$.

For $z \in Z'$ put then $m_i'(y, p) = g_i'(p) + (1/N)\ (\Sigma_j\ p \cdot y_j - \Sigma_i\ g_i'(p))$. By construction, the auxiliary distribution functions $m_i'(y, p)$ satisfy hypothesis V with respect to Z'. Also, if $z \in U$ and $\Sigma_j\ p \cdot y_j = \Pi'(p)$ then $\Pi'(p) = \Pi(p)$ and so, by hypothesis V, $m_i(y, p) > g_i(p) = g_i'(p)$. Finally, let $m_i''(y, p)$ be continuous distribution functions such that: $m_i''(y, p) = m_i'(y, p)$ for $z \in Z' \backslash U$, $m_i''(y, p) = m_i(y, p)$ for $z \in Z$ and $m_i''(y, p) \geqslant \min\ \{m_i(y, p), m_i'(y, p)\}$ for $z \in U$. Then the distribution functions m_i'' satisfy hypothesis V with respect to z'.

Apply now Theorem 1 to the compact economy $\mathcal{E}' = (Z', \{P_i'''\}_{i=1}^N, \{m_i''\}_{i=1}^N)$. Let $z \in Z'$ be an equilibrium with slack $\alpha \geqslant 0$. I claim that this z is as desired. Because z is feasible we have $z \in \hat{Z}$. Hence $p \cdot y_j = \Pi_j'(p) = \Pi_j(p)$ for all j and $m_i''(y, p) = m_i(y, p)$ for all i. Because $x_i \in (-r, r)^\ell$ and $\upsilon \in P_i'''$ implies $p \cdot \upsilon > m_i(y, p) + \alpha$ we have that $\upsilon \in P_i'(z)$ implies $p \cdot \upsilon > m_i(y, p) + \alpha$ (see Figure 9.3). Since $P_i(z) \subset$ closure $P_i'(z)$ we have that $\upsilon \in P_i(z)$ implies $p \cdot \upsilon \geqslant m_i(y, p) + \alpha$. However, $m_i(y, p) > \inf p \cdot X_i$ which by a familiar argument (see Proof of Theorem 1) does yield that $\upsilon \in P_i(z)$ implies $p \cdot \upsilon > m_i(y, p) + \alpha$. Therefore, z is a Walrasian equilibrium with slack.

If $P_i(z) \neq \varnothing$ then $x_i \in$ closure $P_i'''(z)$. Hence, $p \cdot x_i \leqslant m_i(y, p) + \alpha$ and $p \cdot \upsilon \geqslant m_i(y, p) + \alpha$ for all $\upsilon \in P_i'''(z)$ implies $p \cdot x_i = m_i(y, p) + \alpha$. This concludes the proof.

Example 2 has shown that a Walrasian equilibria may not exist under the hypotheses of Theorem 2. It is however, a corollary of the Theorem that under an additional nonsatiation hypothesis equilibria without slack does indeed exist. Curiously, local nonsatiation (i.e., $x_i \in$ closure $P_i(z)$) is not required. Nonsatiation (i.e., $P_i(z) \neq \varnothing$) is all that matters:

(VIII) For every $z \in Z$ and i we have $P_i(z) \neq \varnothing$ (nonsatiation).

Corollary Under hypotheses I–V, VII and VIII there is a state $z = (y, x, p) \in Z$ which is a Walrasian equilibrium with no slack. Moreover, $p \cdot x_i = m_i(y, p)$ for all i.

Proof Let z be given by Theorem 2. Then $p \cdot x_i = m_i(y, p) + \alpha$ for all i. Hence, $\Sigma_i\ p \cdot x_i = \Sigma_i\ m_i(y, p) + \alpha N = \Sigma_j\ p \cdot y_j + \alpha N$. Since $\Sigma_i\ x_i = \Sigma_j\ y_j$ we get $\alpha = 0$ and so, our proof is completed.

Remark In a context of complete, continuous, transitive preferences the Corollary implies that the presence of thick indifference surfaces does not by itself prevent the existence of equilibria (as contrasted with its optimality; see section 4). The weakest nonsatiation property yielding the existence of a Walrasian equilibrium (without slack) seems to be that at any consumption the set of at-least-as-good consumptions be unbounded.

4 WELFARE ANALYSIS

Let us now assume that the preference mappings P_i originate in individual preference relations with no externalities $\succsim_i \subseteq X_i \times X_i$ (i.e., for every $z \in Z_i$ we have $P_i(z) = \{v \in X_i : v \succ_i x_i\}$). I take every $\succsim_i$ to be transitive, complete, continuous and convex (i.e., $\{z \in X_i : z \succsim_i x_i\}$ is a convex set for every $x_i \in X_i$). Then, of course, P_i is irreflexive, convex-valued and has an open graph.

A feasible state $z = (y, x, p)$ is an *optimum* if there is no other feasible state $z' = (y', x', p')$ such that $x_i' \succsim_i x_i$ for every i and $x_i' \succ_i x_i$ for some i. The state is a *weak optimum* if there is no other feasible state $z' = (y', x', p')$ such that $x_i' \succ_i x_i$ for all i.

It follows simply from the definitions that a Walrasian equilibrium with slack is necessarily a weak optimum. Example 3, which is trivial, shows first that a Walrasian equilibrium with slack need not be an optimum and, second that no Walrasian equilibrium with slack which is an optimum may exist.

Example 3

$$l = 1,\ X_1 = X_2 = [0, 1],\ Y = R_-$$

$$\omega_1 = \omega_2 = \frac{1}{2}$$

$$u_1(x) = \begin{cases} 0 & \text{for } x \leqslant \frac{3}{4} \\ x - \frac{3}{4} & \text{for } x \geqslant \frac{3}{4} \end{cases}$$

$$u_2(x) = x$$

In this example $p = 1$, $x_1 = (1/2) - \alpha$, $x_2 = (1/2) + \alpha$ constitutes an equilibrium with slack for any $\alpha \leqslant 1/4$. Moreover, this describes the whole set of equilibria. Any of these equilibria is dominated by $x_1 = 0$, $x_2 = 1$.

Why may a Walrasian equilibrium with slack $z = (y, x, p)$ fail to be an optimum? The reason is simple enough: under the hypotheses made cost minimisation in consumption may fail (i.e., we may have $p \cdot x_i' < p \cdot x_i$, $x_i' \succsim_i x_i$, for some x_i' and i). There are at least two conditions on preferences that will guarantee cost maximisation. One, the most traditional, is local nonsatiation. The other, used by Debreu (1959, Ch. 6) and Drèze and Muller (1980), is strict convexity (note that if $\succsim_i$ is strictly convex, $x_i' \succsim_i x_i$ and $p \cdot x_i' < p \cdot x_i$, then, letting $x_i'' = (1/2)\, x_i + (1/2)\, x_i'$, we have $x_i'' \succ_i x_i$ and $p \cdot x_i'' < p \cdot x_i$. Hence preference maximisation implies cost maximisation). Local nonsatiation is against the grain of this study. Strict convexity is compatible with satiation and compact consumption sets but it is violated in some interesting applications (e.g., Hylland and Zeckhauser, 1979). So I do not want to impose this condition either. What I shall do instead is to modify the notion of equilibrium by requiring that cost maximisation be also satisfied and then show that under a weak additional condition (local nonsatiation at nonsatiation points) states satisfying the stronger definition exist.

Definition A state $z \in Z$ is a *strong Walrasian equilibrium with slack* if it is a Walrasian equilibrium with slack and, moreover, '$v \succsim_i x_i$ implies $p \cdot v \geqslant p \cdot x_i$' for every i.

Proposition Every strong Walrasian equilibrium with slack is an optimum.

Proof Standard. Let the slack term be α. Suppose that z' dominates z. Then $p \cdot x_i' \geqslant p \cdot x_i$ for every i. Also, $p \cdot x_i' > m_i(y, p) + \alpha \geqslant p \cdot x_i$ for at least one i. Hence, $p \cdot \Sigma_i x_i' > p \cdot \Sigma_i x_i = p \cdot \Sigma_j y_j$ which, by profit maximisation, implies that z' is not feasible.

Example 3 shows that with the hypotheses so far a strong Walrasian equilibria with slack need not exist. However, they do exist under a condition weaker than local nonsatiation and which in practice seems to cover most of the cases where the latter is violated:

(IX) For any i, $x_i \notin K_i = \{v\colon v \succsim_i z \text{ for all } z \in X_i\}$ and $\varepsilon > 0$ there is a x_i' such that $x_i' \succ_i x_i$ and $\|x_i - x_i'\| < \varepsilon$ (local nonsatiation except at complete satiation points).

Hypothesis ix is satisfied if, for example, $\succsim_i$ is representable by a concave utility function. Note that the preferences of the first agent in Example 3 violate it.

Theorem 3 Under hypotheses I–V, VII and IX a strong Walrasian equilibrium with slack exists.

Proof Suppose first that for every i the satiation region K_i is either empty or full-dimensional in X_i, that is, K_i^0 = Interior of K_i relative to X_i, is nonempty (if $K_i \neq \emptyset$).

Define then the price-dependent preference mappings $P_i\colon Z \to X_i$ by:

$$P_i(x_i) = \begin{cases} \{v\colon v \succ_i x_i\} & \text{if } x_i \notin K_i \\ K_i^0 \cap \{v\colon p \cdot v < p \cdot x_i\} & \text{if } x_i \in K_i \end{cases}$$

Then the maps P_i are irreflexive, convex-valued and have an open graph (this is simple to prove). By Theorem 2 there is a Walrasian equilibrium with slack z. Hypothesis IX (for $x_i \notin K_i$) and the definition of P_i (for $x_i \in K_i$) guarantee the property '$v \succsim_i x_i$ implies $p \cdot v \geq p \cdot x_i$'. So, the definition of strong equilibrium is satisfied. (The nice trick of using the P_i maps defined above I owe to K. Nehring and M. Spagat.)

If $K_i^0 = \emptyset$ for some i we must either appeal to an approximation argument or use fixed points theorems for non open-graph correspondences (but satisfying the weaker property of lower hemicontinuity – e.g., Gale and Mas-Colell, 1979). The approximation argument goes as follows. For each i with $K_i \neq \emptyset$ pick $x_i \in K_i$ and $x_{in} \to x_i$, $x_{in} \notin K_i$. Replace then $\succsim_i$ by $\succsim_{in}$ defined by letting $v' \succsim_{in} v$ if either $v' \succsim_i v$ or v, $v' \succsim_i x_{in}$. Then $K_{in} = \{v \in X_i\colon v \succsim_i x_{in}\}$ is either empty or full-dimensional. Therefore there is a modified equilibrium with slack z_n for $\mathcal{E}_n$. It is then an easy matter to show that z_n has an accumulation point z and that the latter satisfies the desired properties.

Example 4 shows that equilibria with or without slack can exist simultaneously and that the only one being an optimum may be the one with slack.

Example 4

$$l = 1,\ X_1 = X_2 = X_3 = [0, 1]$$
$$\omega_1 = \omega_2 = \omega_3 = \frac{1}{3}$$
$$u_1(t) = u_2(t) = t$$
$$u_3(t) = 0$$

In this example there is an equilibrium with slack for every $0 \leq \alpha \leq 1/6$. It is given by $p = 1$ and $x_1 = x_2 = (1/3) + \alpha$, $x_3 = (1/3) - 2\alpha$. Only for $\alpha = 1/6$ is the equilibrium strong and an optimum.

It is an easy matter to generate examples similar to Example 4 in the context of a Hylland and Zeckhauser (1979) lottery trade model. Thus, let $X_1 = X_2 = X_3 = \Delta$, the two-dimensional simplex. Suppose that $\omega_1 = \omega_2 = \omega_3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $u_1(x_1) = x_1^1 + x_1^3$, $u_2(x_2) = x_2^1 + x_2^2$, $u_3(x_3) = x_3^2 + x_3^3$. Then the allocation $((\frac{1}{2}, 0, \frac{1}{2}), (0, 1, 0), (\frac{1}{2}, 0, \frac{1}{2}))$ is an equilibrium (for $p = (\frac{1}{3}, \frac{1}{3}, \frac{1}{4})$) which is not an optimum.

References

Bergstrom, T. (1976) 'How to Discard "Free Disposability" at no Cost', *Journal of Mathematical Economics*, 3, pp. 131–4.

Debreu, G. (1959) *Theory of Value* (New York, Wiley).

Drèze, J. and H. Muller (1980) Optimality Properties of Rationing Schemes', *Journal of Economic Theory*, 23 (2) pp. 131–49.

Gale, D. and A. Mas-Colell (1975) 'An Equilibrium Existence Theorem for a General Model without Ordered Preferences', *Journal of Mathematical Economics*, 2, pp. 9–15.

Gale, D. and A. Mas-Colell (1979) 'Corrections to an Equilibrium Existence Theorem for a General Model without Ordered Preferences', *Journal of Mathematical Economics*, 6, pp. 297–8.

Hylland A. and R. Zeckhauser (1979) 'The Efficient Allocation of Individuals to Positions', *Journal of Political Economy*, 87 (2) pp. 293–313.

Shafer, W. (1976) 'Equilibrium in Economies without Ordered Preferences or Free Disposal', *Journal of Mathematical Economics*, 3, pp. 135–7.