

# A Non-Cooperative Interpretation of Value and Potential<sup>1</sup>

Sergiu Hart

Tel-Aviv University and Harvard University

and

Andreu Mas-Colell

Harvard University

## 1. Introduction

One of the most successful cooperative solution concepts in game theory is that of *value*. Originally defined for  $n$ -person games with transferable utility ("TU") by Shapley [1953b], it has been extended to general games with nontransferable utility ("NTU") by Harsanyi [1959, 1963] and Shapley [1969] and applied to numerous models. These applications, in turn, have always yielded important insights (to mention just one: the "value equivalence principle" in purely competitive economies).

In Hart and Mas-Colell [1989], we have introduced the concept of the *potential* of a game, which is shown to lead precisely to the Shapley value in the TU case. We refer to that paper for the details. The definition of the potential (which is just a combination of the basic principles of "efficiency" and "marginal contributions") has proved to be a (very useful) technical tool. However, a direct game theoretical interpretation of the potential is still lacking. To have one would be highly desirable.

We propose here one such interpretation. More precisely, we will present an auxiliary two-person zero-sum game, whose minimax = maximin value is precisely the potential. What is perhaps more important is that the optimal strategy of player  $I$  turns out to be precisely to choose the Shapley' value! This yields a (noncooperative) interpretation for both the Shapley value and the potential. Moreover, the potential emerges as a kind of sequential price of the original game. A permanent theme in Harsanyi's work is the connection between non-cooperative and cooperative concepts. This paper provides, therefore, one more such link.

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There are a few other noncooperative approaches to the Shapley TU-value. Specifically, Gul [1989], O. Hart and Moore [1990] and Mas-Colell [1988]. One of the advantages of our model is that it can be directly extended to general nontransferable utility games. (See Hart and Mas-Colell [1992] for another non-cooperative approach to  $n$ -person NTU games.)

We start Section 2 with a simple example that introduces our model; the case of finite TU-games is treated there. The general NTU case is then analyzed in Section 3. In this context, the potential leads to the *egalitarian solution*, which is of interest on its own but which also constitutes the first step in the construction of the Harsanyi [1963] NTU-value (the second step is the intrinsic determination of the relative “weights” of the players’ payoffs, according to which the egalitarian solution is determined). Section 4 is devoted to the study of “large” games; more precisely, (NTU) games with a continuum of players falling into finitely many types.

## 2. Finite Games with Transferable Utility

To explain the idea behind our approach, let us start with a simple example. There are 2 players  $i = 1, 2$ ; player 1 by himself can get 10, player 2 by himself 30, and together they can get 100. There is also a “raider”  $R$ , an agent who wants to buy the game. Thus if 1 and 2 have agreed on, say, a (50, 50) split, then it appears that  $R$  will have to give 50 to each one. But suppose  $R$  wants to minimize the total amount he pays. He may then do the following (whenever the agreement (50, 50) emerges): offer<sup>2</sup> 51 to player 1 if he will break the agreement (recall that we are in a noncooperative setup where agreements are not binding). Then 1 will take 51 and “leave the game,” after which  $R$  will have to pay only 30 to player 2, for a total of 81 (which is of course less than 100). Is this the best  $R$  can do? No, since he can give 51 to player 2 to leave the game, and then 10 to player 1, in total 61. Now is (50, 50) the worst agreement from  $R$ ’s point of view? A little thought will show that the most  $R$  will have to pay is 71, which happens whenever the two players choose (40, 60). Note that (40, 60) is precisely the outcome obtained by the principle of “split the surplus equally” (1 gets 10, 2 gets 30, and the surplus 60 is evenly divided among them). It is therefore the *Shapley value* of the game (actually, any single-valued solution that is symmetric and covariant with linear transformations will yield the same).

This example generalizes. Let  $(N, v)$  be an  $n$ -person TU-game, where  $N = \{1, 2, \dots, n\}$  is the set of players and  $v : 2^N \rightarrow \mathbb{R}$  is the coalitional (or characteristic) function, with  $v(\emptyset) = 0$ .

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<sup>2</sup> Assume for simplicity that 1 is the smallest unit.

We will consider an *auxiliary two-player zero-sum game*  $\Gamma \equiv \Gamma(N, v)$ . Player *I* chooses a payoff configuration  $\mathbf{x} = (x_S)_{S \subset N}$ , where  $x_S = (x_S^i)_{i \in S} \in \mathbb{R}^S$  is a feasible payoff vector for the coalition  $S$  for each  $S \subset N$ ; i.e.,  $\sum_{i \in S} x_S^i \leq v(S)$ . Player *II* chooses an order  $\pi$  over  $N$  (i.e.,  $\{\pi(1), \pi(2), \dots, \pi(n)\} = N$ ). The payoff from *II* to *I* is then

$$\sum_{t=1}^n x_{S(t)}^{\pi(t)}, \quad (1)$$

where  $S(t) := \{\pi(t), \dots, \pi(n)\}$ . To understand this, assume for simplicity that  $\pi$  is the natural order  $1, 2, \dots, n$ . The payoff of *II* to *I* is the sum of the following amounts: First, the payoff of player 1 in the grand coalition  $N = S(1)$ , i.e.,  $x_N^1$ ; second, the payoff  $x_{N \setminus \{1\}}^2$  of player 2 in the remaining coalition  $N \setminus \{1\} = S(2)$ ; and so on, up to the last player's payoff  $x_{\{n\}}^n$ .

The interpretation is as follows: Think of *I* as the “owner” of (the players of) the game  $(N, v)$ ; then *II* is a “raider” that “buys off” the players (in  $N$ ) from *I* one by one. For each coalition  $S$ , *I* chooses a feasible payoff vector  $x_S$  (the final imputation if  $S$  “forms” — whenever  $S$  is the set of players that are still in the game), while *II* chooses the order in which he buys off the players. We assume that *II* wants to minimize the total payoff (by choosing the right order), and *I* wants to maximize it (by choosing the right payoff vectors).

The result is:

**Proposition A:** *In the two-person zero-sum game  $\Gamma(N, v)$ :*

- (i) *The unique optimal strategy of player I is to choose  $x_S$  equal to the Shapley value of the game  $(S, v)$ , for all  $S \subset N$ .*
- (ii) *The unique optimal strategy of player II is to choose all  $n!$  orders of  $N$  with equal probability.*
- (iii) *The minimax (= maximin) value equals  $P(N)$ , the potential of the game<sup>3</sup>  $(N, v)$ .*

**Proof.** Note that the payoff function of  $\Gamma$  is linear in player *I*'s strategy, and his strategy set—the set of feasible payoff configurations—is convex. Therefore player *I* has no need for mixed strategies (as we will see below, he has a unique pure optimal strategy, therefore there are no mixed optimal strategies).

For every  $S \subset N$ , let  $Q(S)$  be the minimax value of  $\Gamma(S, v)$ ; clearly<sup>4</sup>  $Q(\emptyset) = 0$  and  $Q(\{i\}) = v(i)$  for all  $i \in N$ . Assume inductively that  $Q(T) = P(T)$  for all  $T \subsetneq S$ .

<sup>3</sup> Since we are dealing with only one coalitional function  $v$ , we will write  $P(S)$  for  $P(S, v)$ ; see the “Moreover” statement in Hart and Mas-Colell [1989, Theorem A].

<sup>4</sup>  $\Gamma(\emptyset, v)$  is an “empty” game (the strategy sets are trivial); as for  $\Gamma(\{i\}, v)$ , the result is immediate.

One obtains the following recursive formulas:

$$Q(S) = \text{Max}_{x_S} \text{Min}_{p_S} \sum_{i \in S} p_S^i [x_S^i + Q(S \setminus \{i\})] \quad (2)$$

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where  $x_S$  ranges over feasible  $S$ -payoff vectors (i.e.,  $x_S = (x_S^i)_{i \in S} \in \mathbb{R}^S$  and  $\sum_{i \in S} x_S^i \leq v(S)$ ) and  $p_S$  is a probability distribution over  $S$  (i.e.,  $p_S = (p_S^i)_{i \in S} \in \mathbb{R}_+^S$  and  $\sum_{i \in S} p_S^i = 1$ ; if  $S = S(t) = \{\pi(t), \dots, \pi(n)\}$ , then  $p_S^i$  is the conditional probability that  $\pi(t) = i$  given  $\pi(1), \dots, \pi(t-1)$ ).

Now  $Q(S \setminus \{i\}) = P(S \setminus \{i\})$  by assumption; taking  $p_S = \left(\frac{1}{|S|}, \dots, \frac{1}{|S|}\right)$  in (2) yields

$$Q(S) \leq \text{Max}_{x_S} \frac{1}{|S|} \left[ \sum_{i \in S} x_S^i + \sum_{i \in S} P(S \setminus \{i\}) \right] \leq \frac{1}{|S|} \left[ v(S) + \sum_{i \in S} P(S \setminus \{i\}) \right] = P(S)$$

(recall (2.2) in Hart and Mas-Colell [1989]). Taking<sup>5</sup>  $x_S = Sh(S, v)$  in (3) gives

$$x_S^i + Q(S \setminus \{i\}) = Sh^i(S, v) + P(S \setminus \{i\}) = P(S)$$

for all  $i \in S$ , thus

$$Q(S) \geq \sum_{i \in S} p_S^i P(S) = P(S)$$

If  $x_S \neq Sh(S, v)$ , then there is  $i \in S$  such that  $x_S^i < Sh^i(S, v)$  (since  $\sum x_S^i \leq v(S) = \sum Sh^i(S, v)$ ), and then  $p_S^i = 1, p_S^j = 0$  for all  $j \neq i$  makes the payoff  $x_S^i + P(S \setminus \{i\}) < P(S)$ . If  $p_S \neq \left(\frac{1}{|S|}, \dots, \frac{1}{|S|}\right)$ , say  $p_S^1 > p_S^2$ , then choosing  $x_S = Sh(S, v) + (\epsilon, -\epsilon, 0, \dots, 0)$  for  $\epsilon > 0$  yields a payoff of  $P(S) + \epsilon(p_S^1 - p_S^2) > P(S)$ . This shows the uniqueness of the optimal strategies.  $\square$

Thus one may view the potential  $P(N)$  as a kind of *sequential price* of the game  $(N, v)$ : how much player  $II$  has to pay to “buy” all the players in  $N$  in sequence. It is interesting to note that in a *market game* (i.e., when  $(N, v)$  is totally balanced), one has  $P(N) \leq v(N)$  (see Hart and Mas-Colell [1988, Corollary 2]), thus  $II$  indeed prefers buying off the players in  $N$

<sup>5</sup> We write “ $Sh$ ” for the Shapley value payoff vector.

one by one (and paying  $P(N)$ ) rather than all at once (and paying  $v(N)$ ); see also the example at the beginning of this section.

It is possible to give an interpretation of the previous construction from a slightly different point of view. Suppose player  $I$  (the owner, or social planner) wishes to assign payoffs according to the egalitarian principle. How to do this? If the game was one of pure bargaining (for games normalized to  $v(\{i\}) = 0$  for all  $i \in N$ , this means that  $v(S) = 0$  for any  $S \subsetneq N$ ) it is clear how to proceed: simply give every player  $\frac{v(N)}{N}$ . In more strategic terms we can think of the equal split as the outcome of an attempt by the planner to “keep the game together” by maximizing the utility of the worst-off player (the cheapest one to raid). This, in turn, determines the “sequential price of the game”, as paid by an hypothetical player  $II$ , to be  $P(N) = \text{Max}_{x_N} \left\{ \text{Min}_i \{x_N^1, \dots, x_N^n\} : \sum_i x_N^i = v(N) \right\} = \frac{v(N)}{N}$ . The strategic reinterpretation gives then a hint on how to apply the egalitarian principle in the general case: let player  $I$  maximize the amount that a raider will have to pay sequentially in order to buy all the players (call this maximal amount the sequential price). It is worthwhile noting that in doing so player  $I$  can proceed recursively. If the sequential price  $P(N \setminus \{i\})$  has already been determined for all  $i$  then  $P(N)$  will obviously be given by

$$P(N) = \text{Max}_{x_N} \left\{ \text{Min}_i \{x_N^1 + P(N \setminus \{1\}), \dots, x_N^n + P(N \setminus \{n\})\} : \sum_i x_N^i \leq v(N) \right\},$$

which has as unique solution the  $\bar{x}_N$  satisfying  $\sum \bar{x}_N^i = v(N)$  and  $P(N) = \bar{x}_i + P(N \setminus \{i\})$  for all  $i$ . Hence  $P(N)$  is indeed the potential of the game  $(N, v)$ . Because of the maxmin = minmax theorem we do not actually need to think of the Raider as moving after the Owner. They can move simultaneously. By choosing the next player to be bought at random (with equal probability over the remaining  $i$ ) the Raider can guarantee that  $P(N)$  is his maximal payment.

### 3. Finite Games with Nontransferable Utility

Let us move now to the general NTU case. Let  $(N, V)$  be an NTU-game, with  $V(S) \subset \mathbb{R}^S$  being the set of feasible payoff vectors for the coalition  $S$ . We will make the standard assumptions that each one of the sets  $V(S)$  is convex, closed, comprehensive, nonempty and not the whole space  $\mathbb{R}^S$ . Moreover, we assume that for all  $S$  the set  $V(S)$  is nonlevel (i.e.,  $x, y \in \text{bd}V(S)$  = the boundary of  $V(S)$  and  $x \geq y$  imply  $x = y$ ) and has a smooth boundary (i.e., at every point on its boundary there is a unique supporting hyperplane). Define the two-player zero-sum game  $\Gamma(N, V)$  exactly as in the TU-case:  $I$  chooses a feasible payoff configuration  $\mathbf{x}$

(i.e.,  $\mathbf{x} = (x_S)_{S \subset N}$  with  $x_S \in V(S)$  for all  $S$ ) and  $II$  chooses an order  $\pi$  on  $N$ ; the payoff is then given by (1).

**Proposition B.** *In the two-person zero-sum game  $\Gamma(N, V)$ :*

- (i) *The unique optimal strategy of player I is to choose  $x_S$  equal to the egalitarian solution of the game  $(S, v)$ , for all  $S \subset N$ .*
- (ii) *The unique optimal strategy of player II is to choose the order  $\pi$  with probability  $\prod_{t=1}^n \lambda_{S(t)}^{\pi(t)}$ , where  $\lambda_S = (\lambda_S^i)_{i \in S} \in \mathbf{R}_{++}^S$  is the normal to  $bdV(S)$  at the egalitarian solution of  $(S, V)$ , normalized by  $\sum_{i \in S} \lambda_S^i = 1$ .*
- (iii) *The minimax (= maximin) value equals  $P(N)$ , the potential of<sup>6</sup>  $(N, V)$ .*

The *egalitarian value* is the first step in the construction of the *Harsanyi* [1963] NTU-value. It may be viewed as the outcome generated by the extension of the principle of “split the surplus equally” to more than 2 players. For TU-games, it coincides with the Shapley value. For general NTU-games, it has been studied and axiomatized by Kalai and Samet [1985]. In Hart and Mas-Colell [1989, Section 6], it is obtained via the potential approach, as follows:  $P(\emptyset) = 0$  and  $(P(S) - P(S \setminus \{i\}))_{i \in S} \in bdV(S)$  for all  $S$ , uniquely determine the potentials  $P(S)$ ; then  $P(S) - P(S \setminus \{i\})$  is  $Eg^i(S, V)$ , the egalitarian solution payoff of  $i$  in  $(S, V)$ .

Note that our smooth boundary assumption implies that  $\lambda_S$  in (ii) is uniquely determined (however, see the Remark following the Proof of Proposition B). The probability distribution over the  $n!$  orders of  $N$  may be described as follows: Assume that  $\pi(1), \dots, \pi(t-1)$  have been already determined and “bought off,” in the interpretation of Section 2, and let  $S := N \setminus \{\pi(1), \dots, \pi(t-1)\}$  be the set of remaining players. Let  $x_S = Eg(S, v) \in bdV(S)$  and  $\lambda_S \in \mathbf{R}_+^S$  a supporting normal there (i.e.,  $\lambda_S \cdot x_S \geq \lambda_S \cdot y$  for all  $y \in V(S)$ ). Normalize  $\lambda_S$  to be a probability vector (i.e.,  $\sum_{i \in S} \lambda_S^i = 1$ ). Then the next player  $\pi(t)$  is chosen according to the distribution  $\lambda_S$ , i.e.,  $\text{Prob}(\pi(t) = i | \pi(1), \dots, \pi(t-1)) = \lambda_S^i$  for all  $i \in S$ .

**Proof of Proposition B.** As in the proof of Proposition A, let  $Q(S)$  be the minimax value of  $\Gamma(S, V)$ ; we will show that  $Q(S) = P(S)$  by induction.

The recursive formulas (2) and (3) hold here too (with  $x_S \in V(S)$ ). Choosing  $x_S = Eg(S, V)$  in (3) implies  $Q(S) \geq P(S)$ . Next note that  $x_S^i + Q(S \setminus \{i\}) = x_S^i - (P(S) - P(S \setminus \{i\})) + P(S) = x_S^i - Eg^i(S, v) + P(S)$ ; choosing  $p_S = \lambda_S$  in (2) therefore yields  $Q(S) \leq P(S)$ . Moreover, if  $p_S \neq \lambda_S$ , then  $p_S$  is *not* a supporting normal at  $Eg(S, V)$ , thus there is  $x_S \in V(S)$  with  $p_S \cdot x_S > p_S \cdot Eg(S, V)$ , and the payoff is  $> P(S)$ . Finally, if  $x_S \neq Eg(S, V)$  then there is

<sup>6</sup> We again write  $P(N)$  for  $P(N, V)$ .

$i \in S$  such that  $x_S^i < Eg^i(S, V)$  (here we use the nonlevelness assumption); player *II* choosing  $p_S^i = 1$  (and  $p_S^j = 0$  for  $j \neq i$ ) yields a lower payoff.  $\square$

**Remark.** From the proof it follows that if we drop the assumption of the smoothness of the boundary of  $V(S)$ , then the optimal strategy of player *II* is no longer unique; any supporting  $\lambda_S$  at  $Eg(S, V)$  will work. Similarly, if we drop the nonlevelness assumption, then  $Eg(S, V)$  is no longer the unique optimal choice of player *I* for  $S$ ; any  $x_S \geq Eg(S, V)$  on the boundary of  $V(S)$  will do. However, note that the minimax value is not affected, i.e.,  $Q(S) = P(S)$  for all  $S$  holds always (i.e., *without* nonlevelness and smooth boundaries).

If the game is a TU-game, then we always have  $\lambda_S = \left(\frac{1}{|S|}, \dots, \frac{1}{|S|}\right)$  which yields the uniform distribution over orders. For an NTU-game however, the distribution will be in general *not uniform*. An interesting (and surprising) implication is that *the principle of "all orders equally likely" does not follow from considerations of players' symmetry ("anonymity"), but rather from having transferable utility!* Indeed, the egalitarian solution is, of course, "anonymous" — permuting the names of the players does not matter: however, the probability distribution over orders is in general not uniform. This may shed some further light on the differences between egalitarian solutions and other NTU concepts based on the principle of all orders being equally likely (e.g., the Shapley NTU-value).

An interesting question is whether one can obtain the *Harsanyi NTU-value* rather than the egalitarian solutions. Of course, for every given vector of weights  $w = (w^i)_{i \in N} \in \mathbf{R}_{+++}^N$ , if we define the payoff of the two-person zero-sum game by

$$\sum_{t=1}^n \frac{1}{w^{\pi(t)}} x_{S(t)}$$

instead of (1), then in Proposition B we will get the  $w$ -egalitarian solution  $Eg_w$  (and, in Proposition A for the TU-case, the  $w$ -Shapley value of Shapley [1953a]). This follows immediately by the potential approach — see Section 6 in Hart and Mas-Colell [1989]. Note that  $\lambda_S$  will now satisfy  $\sum_{i \in S} \lambda_S^i \cdot \frac{1}{w^i} y^i \leq \sum_{i \in S} \lambda_S^i \cdot \frac{1}{w^i} Eg_w^i(S, V)$  for all  $S \subset N$  and  $y \in V(S)$ . It follows that a Harsanyi NTU-value is obtained if and only if  $\lambda_N = \left(\frac{1}{n}, \dots, \frac{1}{n}\right)$  (see Theorems 6.2 and D in Hart and Mas-Colell [1989]). It would be worthwhile<sup>7</sup> to try and incorporate the choice of the weights  $w$  into the game (without, for instance, adding one more player).

<sup>7</sup> This is a suggestion of Ehud Kalai.

#### 4. Infinite Games

The construction of the previous two sections extends in a natural way to games with an infinite number of players. We will consider here the setup of Hart and Mas-Colell [1991]: There is a continuum of players of finitely many ( $n$ ) types. A coalition is represented by a profile  $x = (x^1, \dots, x^n) \in \mathbf{R}_+^n$ , with  $x^i$  being the total mass of players of type  $i$  in the coalition. The grand coalition is  $\bar{x} \gg 0$ . The coalitional function  $V$  associates to every profile  $x$  a set  $V(x) \subset \mathbf{R}_+^n$  of feasible per-capita payoff vectors (i.e.,  $a \in V(x)$  whenever the payoff vector  $a$ , where each player of type  $i$  receives  $a^i$ , is attainable by a coalition of profile  $x$ ; note that we only consider type-symmetric payoffs). The TU-case corresponds, of course, to  $V(x) = \{a \in \mathbf{R}^n : \sum_{i=1}^n x^i a^i \leq v(x)\}$ .

We will make the basic assumptions of Hart and Mas-Colell [1991], namely A.1–A.6 there.

Given such a game  $(\bar{x}, V)$ , consider the following two-person zero-sum game  $\Gamma(\bar{x}, V)$ : player  $I$  chooses, for each  $x \leq \bar{x}$ , a feasible payoff vector  $\alpha(x) \in V(x)$  (we assume  $\alpha$  to be measurable). Player  $II$  chooses a path  $\mathbf{x} : [0, 1] \rightarrow \mathbf{R}_{++}^n \cup \{0\}$ , where  $\mathbf{x}(0) = 0$ ,  $\mathbf{x}(1) = \bar{x}$  and  $\mathbf{x}$  is absolutely continuous; let  $\Pi(\bar{x})$  denote the set of all such paths (this is denoted  $\Gamma_0(\bar{x})$  in Hart and Mas-Colell [1991]). The payoff from  $II$  to  $I$  is

$$\int_0^1 \alpha(\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t) dt.$$

The interpretation is as follows: think of  $t$  (= “time”) as going from 1 to 0, then  $\mathbf{x}(t)$  is (the profile of) the set of remaining players at time  $t$ . As  $t$  moves to  $t - dt$ ,  $\mathbf{x}(t)$  changes to  $\mathbf{x}(t) - \dot{\mathbf{x}}(t) dt$ : player  $II$  buys off some additional players, namely  $\dot{\mathbf{x}}^i(t) dt$  players of type  $i$ , for each  $i = 1, \dots, n$ . He has to pay according to the current payoff vector  $\alpha(\mathbf{x}(t))$  (chosen by  $I$ ), in total  $\sum_{i=1}^n \alpha^i(\mathbf{x}(t)) \dot{\mathbf{x}}^i(t) dt = \alpha(\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t) dt$ .

In the current setting, it is shown in Hart and Mas-Colell [1991] that the *potential* is given by a variational problem, namely

$$P(x) := \inf \left\{ \int_0^1 v(\mathbf{x}(t), \dot{\mathbf{x}}(t)) dt : \mathbf{x} \in \Pi(x) \right\}$$

for every  $x \in \mathbf{R}_{++}^n$ , where  $v(x, \cdot)$  is the supporting function of the convex set  $V(x)$ , i.e.,  $v(x, p) := \sup \{p \cdot a : a \in V(x)\}$  for all  $p \in \mathbf{R}_+^n$ . A path  $\mathbf{x} \in \Pi(x)$  is called *minimal* if it is a minimizer above, that is  $P(x) = \int v(\mathbf{x}, \dot{\mathbf{x}})$ . The basic result (Theorem A) of Hart and Mas-Colell [1991a] is that minimal paths exist and that  $P$  as defined above is Lipschitz, thus differentiable almost

The construction of the previous two sections extends in a natural way to games with an infinite number of players. We will consider here the setup of Hart and Mas-Colell [1991]: There is a continuum of players of finitely many ( $n$ ) types. A coalition is represented by a profile  $x = (x^1, \dots, x^n) \in \mathbf{R}_+^n$ , with  $x^i$  being the total mass of players of type  $i$  in the coalition. The grand coalition is  $\bar{x} \gg 0$ . The coalitional function  $V$  associates to every profile  $x$  a set  $V(x) \subset \mathbf{R}_+^n$  of feasible per-capita payoff vectors (i.e.,  $a \in V(x)$  whenever the payoff vector  $a$ , where each player of type  $i$  receives  $a^i$ , is attainable by a coalition of profile  $x$ ; note that we only consider type-symmetric payoffs). The TU-case corresponds, of course, to  $V(x) = \{a \in \mathbf{R}^n : \sum_{i=1}^n x^i a^i \leq v(x)\}$ .

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Given such a game  $(\bar{x}, V)$ , consider the following two-person zero-sum game  $\Gamma(\bar{x}, V)$ : player  $I$  chooses, for each  $x \leq \bar{x}$ , a feasible payoff vector  $\alpha(x) \in V(x)$  (we assume  $\alpha$  to be measurable). Player  $II$  chooses a path  $\mathbf{x} : [0, 1] \rightarrow \mathbf{R}_{++}^n \cup \{0\}$ , where  $\mathbf{x}(0) = 0$ ,  $\mathbf{x}(1) = \bar{x}$  and  $\mathbf{x}$  is absolutely continuous; let  $\Pi(\bar{x})$  denote the set of all such paths (this is denoted  $\Gamma_0(\bar{x})$  in Hart and Mas-Colell [1991]). The payoff from  $II$  to  $I$  is

$$\int_0^1 \alpha(\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t) dt.$$

The interpretation is as follows: think of  $t$  (= “time”) as going from 1 to 0, then  $\mathbf{x}(t)$  is (the profile of) the set of remaining players at time  $t$ . As  $t$  moves to  $t - dt$ ,  $\mathbf{x}(t)$  changes to  $\mathbf{x}(t) - \dot{\mathbf{x}}(t) dt$ : player  $II$  buys off some additional players, namely  $\dot{\mathbf{x}}^i(t) dt$  players of type  $i$ , for each  $i = 1, \dots, n$ . He has to pay according to the current payoff vector  $\alpha(\mathbf{x}(t))$  (chosen by  $I$ ), in total  $\sum_{i=1}^n \alpha^i(\mathbf{x}(t)) \dot{\mathbf{x}}^i(t) dt = \alpha(\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t) dt$ .

In the current setting, it is shown in Hart and Mas-Colell [1991] that the *potential* is given by a variational problem, namely

$$P(x) := \inf \left\{ \int_0^1 v(\mathbf{x}(t), \dot{\mathbf{x}}(t)) dt : \mathbf{x} \in \Pi(x) \right\}$$

for every  $x \in \mathbf{R}_{++}^n$ , where  $v(x, \cdot)$  is the supporting function of the convex set  $V(x)$ , i.e.,  $v(x, p) := \sup \{p \cdot a : a \in V(x)\}$  for all  $p \in \mathbf{R}_+^n$ . A path  $\mathbf{x} \in \Pi(x)$  is called *minimal* if it is a minimizer above, that is  $P(x) = \int v(\mathbf{x}, \dot{\mathbf{x}})$ . The basic result (Theorem A) of Hart and Mas-Colell [1991a] is that minimal paths exist and that  $P$  as defined above is Lipschitz, thus differentiable almost everywhere, and its gradient is efficient, i.e.,  $\nabla P(x) \in bdV(x)$ , whenever  $P$  is differentiable at  $x$ .

everywhere, and its gradient is efficient, i.e.,  $\nabla P(x) \in bdV(x)$ , whenever  $P$  is differentiable at  $x$ .

We now describe the optimal strategies of the two players in  $\Gamma(\bar{x}, V)$ . For player *II*, it is just a minimal path  $\mathbf{x} \in \Pi(\bar{x})$ . For player *I* it is more complicated. First, there exist optimal strategies  $\alpha(\cdot)$  with  $\alpha(x) = \nabla P(x)$  for almost every  $x$ . And second, every optimal strategy must satisfy  $\alpha(\mathbf{x}(t)) = \nabla P(\mathbf{x}(t))$  for almost every  $t \in (0, 1]$  whenever  $\mathbf{x} \in \Pi(\bar{x})$  is a minimal path.

**Proposition C.** *In the two-person zero-sum game  $\Gamma(\bar{x}, V)$ :*

- (i) *Optimal strategies for player I are described above.*
- (ii) *The optimal strategies for player II are to choose minimal paths, i.e.,  $\mathbf{x} \in \Pi(\bar{x})$  that minimize  $\int_0^1 v(\mathbf{x}(t), \dot{\mathbf{x}}(t)) dt$  over  $\Pi(\bar{x})$ .*
- (iii) *The minimax (= maximin) value equals  $P(\bar{x})$ , the potential at  $\bar{x}$ .*

**Proof.** If player *II* chooses an optimal path  $\mathbf{x}$ , then  $\int_0^1 \alpha(\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t) dt \leq \int_0^1 v(\mathbf{x}(t), \dot{\mathbf{x}}(t)) dt = P(\bar{x})$  since  $\alpha(x) \in V(x)$  for all  $x$ . If  $\mathbf{x}$  is not an optimal path, then  $\int v(\mathbf{x}, \dot{\mathbf{x}}) > P(\bar{x})$ ; let  $\alpha$  be such that for all  $t \in (0, 1)$  we have  $\alpha(\mathbf{x}(t)) \in bdV(\mathbf{x}(t))$  and  $\alpha(\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t) = v(\mathbf{x}(t), \dot{\mathbf{x}}(t))$  (and arbitrary for other  $x$ ); then the payoff is  $\int v(\mathbf{x}, \dot{\mathbf{x}}) > P(\bar{x})$ . This proves (ii), and also that the minimax value is  $\leq P(\bar{x})$ .

Consider now player *I*. Assume first that the potential function happens to be  $C^1$ . Letting  $\alpha(x) := \nabla P(x)$  for all  $x$  yields, for any path  $\mathbf{x} \in \Pi(\bar{x})$ , a payoff of

$$\begin{aligned} \int_0^1 \alpha(\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t) dt &= \int_0^1 \nabla P(\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t) dt \\ &= \int_0^1 \frac{d}{dt} P(\mathbf{x}(t)) dt \\ &= P(\mathbf{x}(1)) - P(\mathbf{x}(0)) = P(\bar{x}). \end{aligned}$$

Thus  $\alpha = \nabla P$  guarantees  $P(\bar{x})$  to player *I*.

In general,  $P$  is not necessarily differentiable — only Lipschitz. But the essence of the above argument still goes through. We skip the details. (By approximating  $P$  by a smooth  $P_\varepsilon$  in an appropriate way we get an  $\alpha_\varepsilon(\cdot)$  which guarantees  $P(\bar{x}) - \varepsilon$ . An accumulation point of  $\{\alpha_\varepsilon : \varepsilon > 0\}$ , which can be shown to exist, then yields an  $\alpha(\cdot)$  guaranteeing  $P(\bar{x})$ .)  $\square$

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