Real Options with Unknown-Date Events*

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Abstract

The real options literature has provided new insights on how to manage irreversible capital investments whose payoffs are uncertain. Two of the most important predictions from such theory are: (i) greater risk delays a firm's investment timing, and (ii) greater risk increases the option value of waiting. This paper challenges such conclusions in a setting in which the relevant random variable is the arrival time of an unfavorable event. Another contribution of the paper is to introduce a novel framework in which a firm updates its beliefs about the profitability of an investment opportunity by simply waiting to invest. Thus, a wait-and-see approach allows the firm to capitalize on favorable market evolutions and avoid adverse ones to some extent. Our framework is simple and does not require using stochastic calculus, which allows for an economic interpretation of optimal investment policies for the cases of one-time and sequential investments.

Key words: Investment under Uncertainty, Option Value, Simple and Compound Options, Bad News Principle, Hazard Rate and Bayesian Updating.

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1 Introduction

The business world is full of situations in which the main source of uncertainty regarding the value of an investment opportunity stems from the arrival date of a crucial event. For instance, a firm contemplating when to build or expand a factory should account for a change in environmental or tax policy, if probable. As discussed by Dixit and Pindyck (1994, p. 304), the date of policy change can be considered to be unknown to the firm and "it is commonly believed that expectations of shifts of policy can have powerful effects on decisions to invest." Other representative settings are patent races, since the value of an R&D project and hence the innovation effort of a firm engaged in such a race critically depends upon the unknown discovery dates of competing firms.[1] Another example may be provided by product (or process) innovations, such as when a firm has to decide the time at which to improve the quality of an existing product under conditions of uncertainty about the future date at which a substitute product may be developed. Competition of a superior product may entail the gradual decline in the demand of the product sold by the firm, and, consequently, randomness about the maturity date of the market would be an aspect that would critically affect the firm's investment decision.[2]

The real options literature has certainly been aware of the importance of these situations in which the arrival date of a key event is uncertain. Indeed, this explains the use of Poisson processes in real option valuation. Yet, while Poisson arrivals seem the correct way to model such phenomena, this modeling approach is not particularly suited to perform many analyses beyond real option pricing. For example, the effects of mean-preserving spreads on investment timing and option values cannot be determined. For this reason, the purpose of this paper is to reexamine some of the conclusions of the theory of real options in a fairly general setting in which all uncertainty refers to the arrival date of an unfavorable event that critically affects a firm's payoff.

[1] See Weeds (2002) for a formalization of such situation in a game-theoretic real options setting.

[2] This example applied to product launch decisions is drawn from Bollen (1999).

To this end, we introduce a novel framework that is based on Bayesian updating: by simply waiting to invest, the firm can update its beliefs about the profitability of an investment opportunity, at the same time capitalizing on favorable market evolutions and avoiding adverse ones to some extent. The firm's problem is amenable to an intuitive analysis in terms of marginality conditions linked to Bernanke's (1983) "bad news principle of irreversible investment." We perform such analysis for both simple real options and compound real options,[3] with the aim of improving our conceptual understanding of investment decisions under uncertainty.

Our work is the first in disentangling the conceptual issues that are related to the bad news principle of irreversible investment, which is specially illuminating in the complicated case of compound options that may arise because of the sequentiality of investments. For instance, the compound option we analyze is solved by Majd and Pindyck (1987) by means of complex numerical methods. In this sense, our framework is simple and requires no knowledge of stochastic calculus, which significantly reduces the difficulty of the model. Therefore, it remains tractable even if new ingredients such as sequences of unknown-date events or strategic interaction among several firms are added.[4] In addition, because we model a product market that grows until a random maturity date and declines thereafter, we depart from the traditional real options modeling by providing a setting for stochastic product life cycles such as those followed by pharmaceuticals or durable goods such as televisions, calculators or DVD players.

The consideration of uncertain-date events yields that some of the fundamental results

[3] The option to start a sequential investment in which a firm has the option to abandon the project while incomplete can be viewed as a compound option. An example would be the option to construct a factory that requires some time to build. If the firm can abandon the project once it has begun investing, but before it has been completed, then the option to abandon is valuable, and should be incorporated into the value of the option to construct the factory.

[4] As examples of extensions that build on our work to study different issues, Ruiz-Aliseda (2006) analyzes a continuous-time game played by two firms that have to choose their entry and exit timing over a stochastic industry life cycle given that each of them faces investment opportunities that differ in their degree of irreversibility. In turn, Ruiz-Aliseda and Wu (2007) analyze a firm's entry and exit behavior in markets that follow random-length cycles, although they use exponentially distributed random variables, which does not allow for the comparative static analyses we perform in this paper.

of the real options approach under risk neutrality are not clear-cut. Firstly, the canonical real options model predicts that the value of an investment opportunity is non-decreasing in the variance parameter of the Geometric Brownian Motion that governs the return of the underlying asset.[5] Greater uncertainty cannot be harmful because the firm always has the option to wait for better times or even not to invest should conditions turn out to be adverse. But, at the same time, the firm can capitalize on favorable market evolution and invest right on. Therefore, there exists an asymmetry in that waiting to invest shields the firm against adverse realizations of uncertainty, but does not prevent it from taking advantage of favorable ones. This asymmetry, always present when there exist options whose payoffs are convex in the realization of the random variable, accounts for the non-negative effects of greater uncertainty on the value of investment opportunities. Notwithstanding, one of the contributions of our paper is to show that such asymmetry is not present when there exists uncertainty about the arrival date of an unfavorable event. In principle, in our setup, the event may occur before or after investing, but we show that the firm finds it optimal to undertake the investment only if uncertainty has not been resolved. Given this result and that the space of outcomes coincides with that of the control variable (namely, time), it follows that the firm would be insured against bad realizations because of waiting, and would take advantage of (very) good ones. However, it would be damaged by realizations shortly after the optimal time of investment. That is, if the event occurred right after investing, then the project may turn out to be unprofitable ex post, even though it might have seemed an excellent investment in expectation. As a result, more uncertainty may destroy option values, depending on whether the probability of occurrence of the event after the date of investment increases by a sufficiently large amount.

Secondly, real options theory usually predicts that increased risk delays investment timing, a relevant aspect for both public policy and business purposes. Such conclusion basically

[5] See, e.g., the classic paper by McDonald and Siegel (1986). To the best of our knowledge, no other real options work contradicts such empirical prediction up to date.

follows from Bernanke's (1983) bad news principle. Intuitively, the benefit of waiting arises from the avoidance of making a poor investment decision (or series of decisions in the case of compound options) when news is bad, that is, when events are unfavorable. Given that only adverse events matter and a mean-preserving spread increases their probability of occurrence, the marginal benefit of delaying investment increases with uncertainty, at least for simple real options. Since the marginal opportunity cost of waiting (namely, current profits forgone) is unaffected by the spread, the net marginal benefit of deferring investment increases with uncertainty, which in turn induces a delay in investment. We show that this need not be true in our setting. More specifically, in our model only adverse events matter too. Yet, the firm endogenously chooses whether to position itself in a situation in which a mean-preserving spread increases or decreases the conditional probability of immediate arrival of the (bad) event, which explains why the conclusions may differ.

It is widely believed that real options theory predicts that greater risk depresses investment, and at the same time increases the value of an investment opportunity. These results can already be found in McDonald and Siegel (1986, p. 714). Indeed, the prediction that more uncertainty leads to less investment appears to be supported by empirical evidence, as shown by Ferderer (1993) using aggregate data, or Leahy and Whited (1996) and Guiso and Parigi (1999) using micro data. These two last studies suggest that real options theory is the most solid theory of investment under uncertainty, which reinforces the need for a more comprehensive framework that helps to determine the factors and conditions that drive theoretical conclusions. Our work tries to be a step in this direction.

The paper is organized as follows. Section 2 introduces the basic model. Section 3 solves it and provides the marginal interpretation of the optimal investment rule linked to Bernanke's (1983) bad news principle. In addition, we identify a necessary and sufficient condition for an increase in risk to speed up investment. We also provide in this section a necessary and sufficient condition for the value of an investment opportunity to be a non-decreasing function of risk. Section 4 extends the basic model to the case of sequential investment, and interprets the optimal investment policy in the light of the bad news principle, while Section 5 concludes. A mathematical appendix with all proofs is included at the end of the paper.

2 Foundations of the basic theoretical model

Let time, denoted by t, be a continuous variable on $[0, \infty)$. We model an investment decision made by a *risk-neutral* firm which is operating in a market that is currently in growth, but that may start its decline some time in the future.[6] In particular, we assume that the firm is contemplating when to make a single discrete investment decision (such as an improvement of product quality or addition of capacity) in an industry that is known to follow a stochastic life cycle (e.g., calculators). For now, we suppose that investment takes place instantaneously, that is, there is no time to build. If the firm has not invested, then we say that it is in state $i = 0$. If it has invested, then its state is $i = 1$.

The firm's investment decision is complicated by the existence of uncertainty about the temporal evolution of the market, which in turn affects the pattern of profit evolution. Uncertainty unravels partially over time, implying that the market (and, as a result, flow profit) evolves in the following manner. In a first stage, the profit flow, which is positive at date $t = 0$, grows over time. Yet, the market reaches its ephemeral maturity at time τ, where τ is a continuous random variable with density $f(\tau)$ defined on $[0, \infty)$. (We will slightly abuse the notation and τ will also denote its realization.) Hence, in a second phase whose beginning is unknown at the initial date, instantaneous profit decreases over time and converges to 0 as $t \to \infty$, perhaps because consumers perceive that there is another product that can better serve their needs and choose to switch gradually. It is assumed that the firm knows the realization of τ only once it arrives. Formally, we assume the following.

[6] The model we put forth covers the cases in which the firm is inactive at the initial date, which may be important when studying entry decisions.

Assumption 1 *Given a realization τ of the random maturity date, flow profits made by the firm if in state $i \in \{0,1\}$ evolve continuously over time as follows:*

$$\Pi_i(t,\tau) = \begin{cases} \pi_i \exp(\alpha t) & \text{if } 0 \leq t \leq \tau \\ \pi_i \exp[\alpha(2\tau - t)] & \text{if } t > \tau \end{cases} .$$

Thus, if we supposed for example that there exists just one consumer in the market at time $t = 0$, then π_i would denote the profit per customer made by the firm if in state $i \in \{0,1\}$, whereas $\alpha > 0$ would denote the rate at which more customers join the market over time (if the market were in expansion; otherwise, it would be its decay rate).[7] Of course, we assume that $\pi_1 > \pi_0 \geq 0$. Because only the difference between both profit flows matters, we can normalize the model and let $\pi \equiv \pi_1 - \pi_0 > 0$. From now on, we simply work with the incremental profit flow π.

As for the assumptions on the arrival time of the market maturity, we have the following.

Assumption 2 *The maturity date of the market τ is a random variable with continuous density function $f(\tau)$ with support $[0,\infty)$.*

In order to set up a real options framework, we require investment to be irreversible.

Assumption 3 *Undertaking the investment requires the firm to pay an entirely sunk cost $K > 0$.*

Finally, we bound the value of the firm's investment opportunity by assuming that the expected discounted value of one dollar that is capitalized at an instantaneous rate of α is finite no matter what the length of the ascending phase of the life cycle is. Thus, if we

[7] It is straightforward to introduce different rates of growth and decay, or even more general functions for the growth and decline stages (see e.g. Ruiz-Aliseda 2006), but we choose not to do so to keep the model simple. Note also that we are implicitly assuming that operating costs are small enough, in order to avoid making disinvestment an issue and focus only on investment timing. However, zero cost is not necessary for positive profits. For example, a sufficiently low marginal cost in a standard Hotelling model with a single firm located at one extreme would suffice, since costs would always be transferred to consumers, who in turn would be willing to pay a high price.

assume that the firm discounts future payoffs at a constant risk-free interest rate $r \geq 0$, we have the following.[8]

Assumption 4 $\int_0^\infty e^{(\alpha-r)\tau} f(\tau) d\tau < \infty$.

3 Results of the basic model

3.1 Characterization and interpretation of the optimal investment rule

The firm's objective at time $t = 0$ is to choose an investment policy that maximizes the expected discounted stream of cash flows conditional upon information available at the time of investment. We proceed now to characterize such optimal investment rule for the two possible states of the system, depending on whether the maturity date of the market has arrived or not.

In the first place, it is clear that it is not optimal for the firm to disinvest at some date after having invested, given our assumptions of no scrap value and positive profit flow at any possible situation. This holds no matter if τ is known or not. In turn, Lemma 1 below describes the firm's behavior once the maturity of the market has been reached. According to this result, the firm prefers to invest immediately once τ is revealed, but only if such date is sufficiently late; otherwise, it prefers not to invest.

Lemma 1 *Immediate investment at the revealed maturity date τ is optimal for all $\tau > t^c = \max\{0, \ln[K(\alpha + r)/\pi]/\alpha\}$, whereas investment during the declining phase of the market is not profitable for all $\tau \leq t^c$.*

Hence, to characterize the optimal investment rule fully, it only remains to focus on the time t_1 at which the firm would invest if τ had not been revealed yet, and thus the

[8] In the traditional real options framework, a parallel convergence condition requires the rate of expected growth of the investment to be smaller than the risk-free rate.

profit cycle were in its ascending phase. Using Lemma 1, the value of the firm's investment opportunity at date $t = 0$ as a function of its investment time $t_1 \geq 0$ is:

$$
\begin{aligned}
V(t_1) \quad = \quad & \int_0^{t_1} f(\tau) \max(0, \int_\tau^\infty \Pi(s,\tau) e^{-rs} ds - K e^{-r\tau}) d\tau + \\
& \int_{t_1}^\infty f(\tau) (\int_{t_1}^\infty \Pi(s,\tau) e^{-rs} ds - K e^{-rt_1}) d\tau .
\end{aligned}
$$

If the firm chooses to wait until then t_1, for realizations of τ smaller than t_1, it seizes the payoff to immediate investment at τ if and only if it is positive. In contrast, if the firm ends up investing at t_1 while the profit cycle is growing, then it expects to gain a payoff that is contingent on the information gathered by the firm until time t_1 (namely, that τ must be greater than t_1). By another application of the lemma, $V(t_1)$ can be rewritten as follows:

$$
V(t_1) = \begin{cases}
\int_{t_1}^\infty f(\tau) (\int_{t_1}^\infty \Pi(s,\tau) e^{-rs} ds - K e^{-rt_1}) d\tau & \text{if } t_1 \in [0, t^c) \\
\int_{t^c}^{t_1} f(\tau) (\int_\tau^\infty \Pi(s,\tau) e^{-rs} ds - K e^{-r\tau}) d\tau + & \\
\int_{t_1}^\infty f(\tau) (\int_{t_1}^\infty \Pi(s,\tau) e^{-rs} ds - K e^{-rt_1}) d\tau & \text{else}
\end{cases} .
$$

$V(t_1)$ can be easily shown to be continuously differentiable. Therefore, unlike conventional real options analysis, in which expected payoff functions depend on Ito processes and thereby are not differentiable in the classical sense, the firm's optimization program can be solved using standard differentiation techniques:[9]

$$
\begin{aligned}
& \max_{t_1 \geq 0} V(t_1) \\
& \text{s.t.} V(t_1) > 0
\end{aligned}
$$

[9] It is worth remarking that the optimal investment time to be derived will not change as time goes by and no event occurs. In particular, let $V(t_1 \,|t)$ be the value of the firm if it chooses to invest at t_1 (conditional on the market still expanding), given its current information at $t \geq 0$. Note that if the cycle is still growing at t, then $V(t_1 \,|t) = V(t_1 \,|0) e^{rt} / \Pr(t \leq \tau)$, so the set of maximizers of $V(t_1 \,|t)$ coincides with that of $V(t_1 \,|0)$ and there is no loss of generality in letting $t = 0$. Intuitively, at date 0 the firm already takes into account the underlying Bayesian updating process of which it can benefit just by delaying investment. As a result, it anticipates having better information at the time of investment if the market keeps on growing.

Before examining the firm's optimal decision rule when the market is growing, let us introduce some notation. In particular, let $\lambda(t) = f(t)/\int_t^\infty f(\tau)d\tau$ denote the hazard rate, that is, the probability of the market immediately reaching its maturity at date t given that this event has not occurred previously. Although our results hold as long as global maxima are interior, we keep things simple and ensure that $V(t_1)$ is single-peaked by assuming that the environment is such that $\alpha + 2r \geq f'(t)/f(t)$ for all $t \geq 0$, which automatically holds if the density is decreasing.[10] We can now characterize the firm's optimal investment rule.

Proposition 1 *The firm's optimal investment policy is "invest at* $t^e = \min\{t_1 \geq 0 : W(t_1) \leq 0\}$ *if* $t^e < \tau$*, else do not invest," where* $W(t_1) = \dfrac{K(\alpha+r)}{\pi} - e^{\alpha t_1}(1 + \dfrac{\alpha}{r+\lambda(t_1)})$.

Our next result provides an economic interpretation of the firm's optimal investment policy.

Proposition 2 $t^e > 0$ *satisfies* $\pi e^{\alpha t^e} = rK + \lambda(t^e)(K - \int_{t^e}^\infty \pi e^{\alpha(2t^e - s)} e^{-r(s-t^e)} ds)$.

In words: at an interior solution, the firm decides to invest at the instant of time such that the marginal value of waiting equals the marginal cost of delaying investment. The marginal cost is the profit flow forgone by waiting dt, $\pi e^{\alpha t^e} dt$, while the marginal value is the part of sunk cost saved by delaying investment plus the marginal option value of waiting and avoiding an irreversible action. The latter value stems from the "bad news principle of irreversible investments," which can be found in Bernanke (1983). According to this principle, the firm must only care about the bad news that may arrive in the next

[10] A decreasing density is implied by a non-increasing hazard rate in the prior distribution over market size (see Barbarino and Jovanovic 2007 for empirical reasons why this may hold). At a theoretical level, a decreasing hazard rate follows if the true hazard rate is constant but unknown to the firm, which can update its beliefs in a Bayesian fashion as time goes by (see, e.g, Choi 1991, footnote 9). Familiar probability distributions with non-increasing hazard rate and support $[0,\infty)$ (other than the exponential) include the gamma, log-logistic, Weibull and F distributions for certain values of the parameters that define them. However, neither the density nor the hazard rate need to be non-increasing for the assumption to hold. For example, the density is decreasing even though the hazard rate is monotone increasing in Bass (1969) if $q \leq p$. Additional examples of common random variables with non-monotonic densities that may satisfy the requirement include the lognormal and the Gompertz (for certain parameter values), among several others.

instant of time when deciding whether or not to undertake an irreversible project.[11] Thus, the firm believes there is some positive probability that the market may suddenly start declining right after time t^e (this is the bad news). Hence, waiting allows it to avoid making a negative payoff with probability $\lambda(t^e)dt$.[12] Overall, total marginal value is equal to:

$$rKdt + \lambda(t^e)dt(K - \int_{t^e}^{\infty} \pi e^{\alpha(2t^e - s)} e^{-r(s-t^e)} ds).$$

Note from the second term that waiting allows the firm to update its beliefs about the maturity date in a Bayesian fashion. By delaying investment, the firm benefits from learning what some events cannot be (via the denominator of the hazard rate; see its definition), thus allowing for a better assessment of the probabilities of still-not-occurred-events.[13] At the same time, waiting implies that the firm faces a different ex ante probability of the market immediately reaching its maturity (via the numerator of the hazard rate).

3.2 Impact of greater uncertainty on investment timing

One of the standard predictions of real options models under risk neutrality is that higher uncertainty delays the optimal time of investment.[14] There are a few exceptions in some contexts, though, as for example in Dixit and Pindyck (1994, pp. 370-372).

We now show that the effect on investment timing of a greater spread is ambiguous when the firm's payoff crucially depends on the unknown arrival time of an unfavorable event. In particular, we give a necessary and sufficient condition for a mean-preserving

[11] Irreversibility yields no advantages but implies some costs that arise because the firm cannot recoup its investment if conditions turn out to be adverse, which creates the asymmetry that the firm cares about adverse events (which would not be regrettable were investment reversible) but not favorable ones.

[12] By Lemma 1, the payoff if the firm invests at date t_1 is negative if demand suddenly decays for all $t_1 < t^c$. In particular, it holds for $t^e < t^c$ (which is always satisfied, as shown in the proof of Proposition 1).

[13] The reason being that conditioning reduces the outcome space, which enhances the firm's information set. This contrasts with Roberts' and Weitzman's (1981) model of staged investment, in which the value of a project is also unknown to the company but the firm can reduce uncertainty by going ahead in a sequential fashion. In our stylized model, unlike theirs, information gathering does not require an earlier investment. Rather, it requires waiting for information to arrive.

[14] This prediction need not hold under risk aversion. See Gutiérrez (2007) for necessary and sufficient conditions in a Geometric Brownian Motion setting.

increase in the spread (MPIS) to shorten the optimal time of investment whenever $t^e > 0$. Since we are to compare members of a family of distribution functions based on (small) differences in the spread, with the restriction that their means be the same, we assume for the remainder of the paper that the density can be parametrized by σ, i.e., $f(\tau\,|\sigma)$. σ is a parameter of increasing risk, and we refer to it as the spread of the random variable. Hence, the hazard rate is also a function of σ, $\lambda(t\,|\sigma)$, and, for convenience, we assume that it is differentiable in both arguments, denoting partial derivatives by subscripts. Noticing that $W(t^e, \sigma) = \frac{K(\alpha + r)}{\pi} - e^{\alpha t^e}(1 + \frac{\alpha}{r + \lambda(t^e\,|\sigma)})$ is a continuously differentiable function on the neighborhood of any pair (t_0^e, σ_0) such that $W(t_0^e, \sigma_0) = 0$, we can establish the following.

Proposition 3 *Consider a family of distributions that can be parametrized by σ, a parameter such that a rise in it represents an MPIS. Then increasing σ hastens investment if and only if $\lambda_\sigma(t_0^e\,|\sigma_0) < 0$.*

Let us assume for expositional purposes that $f(\tau\,|\cdot)$ is a continuously differentiable function for all τ, so that $\lambda_\sigma(t_0^e\,|\sigma_0) < 0$ if and only $f_\sigma(t_0^e\,|\sigma_0) + \lambda(t_0^e\,|\sigma_0)\int_0^{t_0^e} f_\sigma(\tau\,|\sigma_0)d\tau < 0$. Hence, an MPIS has two distinct effects on optimal investment timing. On the one hand, it has an impact on the unconditional probability of immediate decay after investing, $f(t_0^e\,|\sigma)$. At an intuitive level, the firm tends to speed up investment if the MPIS decreases the (ex ante) probability of making losses immediately after investing. On the other hand, there is an additional impact on the probability that the firm does not end up investing because of a low realization of τ, $\int_0^{t_0^e} f(\tau\,|\sigma_0)d\tau$. Intuitively, the firm is more inclined to hasten investment if the probability of a high realization that would allow it to invest increases.

If $\lambda_\sigma(t_0^e\,|\sigma_0) < 0$,[15] the marginal option value of waiting –that is, the cost of irreversibility in an uncertain environment– would be reduced due to the decrease of the weight put on

[15] It is worth remarking that the restrictions that an MPIS imposes on the cumulative distribution function are consistent with $\lambda_{\sigma^2}(t_0^e\,|\sigma_0) < 0$. A sketch of the proof is as follows. Let $H(t) = \int_0^t \int_0^s f_\sigma(\tau\,|\sigma_0)d\tau ds$. Then an MPIS requires $H(t) \geq 0$ for all t, with $H(0) = H(\infty) = 0$. Note that $H(t)$ is twice continuously differentiable if $f_\sigma(\cdot\,|\sigma_0)$ is continuous, with $H'(t) = \int_0^t f_\sigma(\tau\,|\sigma_0)d\tau$ and $H''(t) = f_\sigma(t\,|\sigma_0)$. Since $\lambda_\sigma(t_0^e\,|\sigma_0) < 0$ if and only if $f_\sigma(t_0^e\,|\sigma_0) + \lambda(t_0^e\,|\sigma_0)\int_0^{t_0^e} f_\sigma(\tau\,|\sigma_0)d\tau < 0$, it suffices to find a non-empty region in which $H(t)$ is both decreasing and concave. The properties that $H(0) = H(\infty) = 0$ and $H(t) \geq 0$,

the losses avoided by delaying investment. Such result is in stark contrast with the standard conclusion from the real options literature, according to which an MPIS uniformly increases risk at every point in time, which in turn increases the marginal value of delaying investment and thus raises the investment threshold. In our setup with a single random variable, an MPIS does not have this property,[16] as exemplified by the following numerical example. In particular, let us assume that τ follows a gamma distribution with parameters $\gamma > 0$ and $\rho > 0$: $\tau \sim Gamma(\gamma, \rho)$. Recall that $E(\tau) = \frac{\rho}{\gamma}$ and $Var(\tau) = \frac{\rho}{\gamma^2}$, which means that we can perform an MPIS by simply multiplying both γ and ρ by any positive scalar smaller than 1. In addition, let the firm be a monopolist facing a linear inverse demand with intercept $a = 50$ and slope $b = 1$. If the firm is initially inactive and its costs are assumed to be zero, then it is well-known that $\pi = \frac{50^2}{4}$. Finally, let $K = 7000$ and $r = \alpha = 5\%$. Considering that the hazard rate of the gamma is non-increasing –and, hence, the density is decreasing– if $\rho \leq 1$, we have the following results, summarized in Table 1:

ρ	γ	$E(\tau)$	$Var(\tau)$	t^e
0.8	0.4	2	5	0.58
0.5	0.25	2	8	0.56
0.1	0.05	2	40	0.2

Table 1: Greater uncertainty speeds up investment

3.3 Impact of greater uncertainty on option values

A (perhaps) surprisingly robust prediction of traditional real options theory under risk neutrality assumptions is that an increase in uncertainty does not harm the value of a

together with the continuous differentiability of the function, yield such result at dates right after a local maximum is attained.

[16] A uniform increase in the hazard rate when augmenting σ would delay the optimal investment time. Yet, this would call for something other than an MPIS. Namely, "a distributional upgrade," as defined by Arozamena and Cantillon (2004), which is derived from a notion of first-order conditional stochastic dominance, rather than one of second-order stochastic dominance. Therefore, not only risk would be affected when varying σ.

firm's investment opportunity (see Dixit and Pindyck 1994, Chapters 5 and 6). We next study conditions under which this may not happen in our setting with unknown-date events. More precisely, we perform a comparative static analysis of the impact of an MPIS on the value of the investment opportunity.

First note that $t^e = 0$ if and only if $\pi \geq \pi^e \equiv \dfrac{K(\alpha + r)(r + f(0))}{\alpha + r + f(0)}$, since $\lim_{t\downarrow 0} \lambda(t) = \dfrac{\lim_{t\downarrow 0} f(t)}{\lim_{t\downarrow 0} \int_t^\infty f(\tau)d\tau} = \lim_{t\downarrow 0} f(t) \equiv f(0)$. Given that the density function can be parameterized by σ, the threshold that optimally triggers immediate investment at date 0 is also a function of σ. As a result, the maximal value of the investment opportunity as a function of both π and σ is as follows:

$$V(\pi, \sigma) = \begin{cases} \pi(\dfrac{1}{\alpha + r} + \dfrac{1}{\alpha - r}) \int_{t^e(\sigma)}^\infty e^{(\alpha - r)\tau} f(\tau \,|\sigma\,) d\tau - \\ (\dfrac{\pi e^{(\alpha - r)t^e(\sigma)}}{\alpha - r} + K e^{-rt^e(\sigma)}) \int_{t^e(\sigma)}^\infty f(\tau \,|\sigma\,) d\tau & \text{if } \pi \geq \pi^e(\sigma) \\ \pi(\dfrac{1}{\alpha + r} + \dfrac{1}{\alpha - r}) \int_0^\infty e^{(\alpha - r)\tau} f(\tau \,|\sigma\,) d\tau - (\dfrac{\pi}{\alpha - r} + K) & \text{else} \end{cases}$$

As readily seen from this expression, comparative statics are slightly complicated, for the value of the investment opportunity is a piecewise differentiable function of σ, and the non-differentiability point π^e depends on σ. For this reason, we make a mild assumption that can be relaxed for some specific probability distributions that do not satisfy it. In particular, we assume from now on that $f(0\,|\sigma\,)$ does not decrease when performing an MPIS,[17] which allows us to characterize some relevant properties of $\pi^e(\sigma)$.

Lemma 2 $\pi^e(\sigma)$ *is a continuous and non-decreasing function with range bounded by the interval* $[rK, (\alpha + r)K]$.

Lemma 2 states that $\pi^e(\sigma)$ is non-decreasing, so an MPIS usually makes investment at $t = 0$ more difficult as happens in traditional real options models. Yet, note that, as Proposition 3 shows, investment timing need not be delayed, because the firm may not wish

[17]For example, the lognormal distribution satisfies such assumption, since $f(0\,|\sigma\,) = 0$ for all σ.

to invest at $t = 0$ even for a low spread. Let $F(\tau\,|\sigma)$ denote the cumulative distribution function of τ, and let us impose the regularity condition that $\lim_{\tau\to\infty} e^{(\alpha-r)\tau}F_\sigma(\tau\,|\sigma) = 0$.[18] Then we can establish a necessary and sufficient condition for an MPIS not to reduce the value of the firm's investment opportunity.

Proposition 4 *Consider a family of distributions that can be parametrized by σ, a parameter such that a rise in it represents an MPIS. Then increasing σ does not reduce the value of the investment opportunity if and only if*

$$2(r + \lambda(t_0^e\,|\sigma_0)) \int_{t_0^e}^{\infty} e^{(\alpha-r)(\tau-t_0^e)} F_\sigma(\tau\,|\sigma_0) d\tau \le F_\sigma(t_0^e\,|\sigma_0). \tag{1}$$

Intuitively, the usual asymmetry due to the fact that the firm can benefit from upside risk without being affected by downside risk is no longer present in this situation. The ex post value of the investment opportunity is not convex in the realization of the random variable unless $t^e = 0$. Indeed, it is not a continuous function of τ, as illustrated by Figure 1. It is this fact that makes the results differ from conventional setups, noting that the envelope theorem implies that a change in t^e due to the MPIS has a negligible effect on the firm's maximal value. In particular, the outcome space coincides with that of t, which allows the discontinuity to appear. The firm does not invest when its worst-scenario payoff is 0 (the only way to prevent the discontinuity from arising), but rather when its expected payoff is maximal, that is, at t^e. Yet, the firm faces risk of losses if the market maturity arrives shortly after investing (since $t^e < t^c$ if $t^e > 0$), as seen in Figure 1(a). It is this risk of losses that explains why augmenting the spread may partially destroy option values.

[18] For instance, this property is automatically satisfied if $\alpha \le r$.

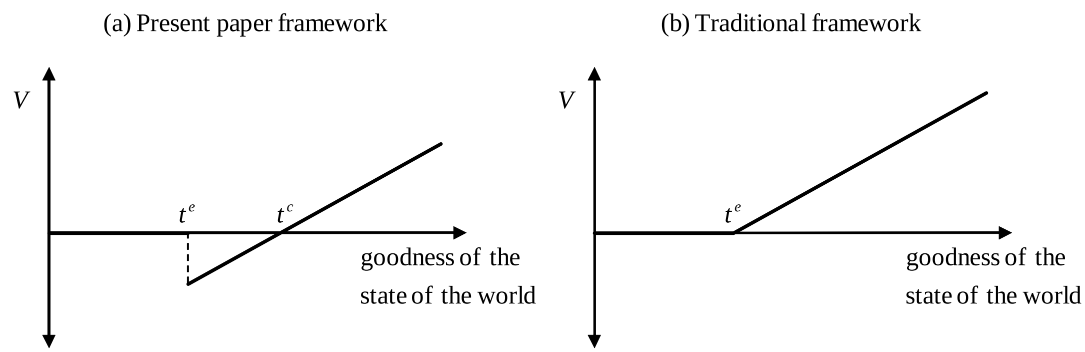

Figure 1: Ex post value of the investment opportunity

As a result, the firm is insured against bad realizations because of waiting, and definitely takes advantage of good realizations, but is damaged by realizations sufficiently close to $t^e > 0$. The overall effect of greater uncertainty is thus ambiguous. Assuming for expositional purposes that $\alpha = r$, Proposition 4 shows that if an MPIS increases the probability that the firm does not invest (i.e., $F_\sigma(t_0^e \,|\sigma_0\,) > 0$), then the value of the investment opportunity increases (since the left hand side of inequality (1) is always non-positive for $\alpha = r$,[19] while the right hand side would be positive). The point is that the firm would lose nothing if $\tau < t_0^e$ and would take advantage of a smaller probability of decay on dates following its investment. Matters may be different, though, if the probability that the firm invests increases when the spread is augmented, since $F_\sigma(t_0^e \,|\sigma_0\,)$ could be negative and very small indeed. In such a case, the probability of the market maturity date occurring at some date between t_0^e and t^c (and thus the probability of making losses) may increase so much so as to destroy part of the value of the option to invest.

Continuing with the example of Subsection 3.2, we next provide a numerical illustration that greater uncertainty harms option values. Taking into account that a gamma distribution has $f(0\,|\sigma\,) = \infty$ for all σ if its hazard rate is decreasing, we have the results depicted in Table 2:

[19] See for instance Diamond and Stiglitz (1974, p. 340).

ρ	γ	$E(\tau)$	$Var(\tau)$	V
0.8	0.4	2	5	560
0.5	0.25	2	8	627
0.1	0.05	2	40	598

Table 2: Greater uncertainty harms option values

4 Extension of the basic model to sequential investment

In this section, we modify the basic model of Section 2 to allow for sequential investment in stochastic life cycle industries. Our main goal is to understand the features of the optimal investment policy in the light of the bad news principle of irreversible investment, as was done in Proposition 2 for the case of simple real options. To allow for sequentiality, we relax the assumption that there is no time to build once the firm invests, which seems desirable too in order to come closer to the real world. In particular, we follow Majd and Pindyck (1987), and assume that the firm's investment project requires completing K steps in order to increase the flow of profits. The firm's maximum rate at which it can advance is $k > 0$ steps per unit time, and it is assumed that an investment of $I \in [0, k]$ units is irreversible and results in I steps advanced (i.e., there are constant returns to scale). Also, to facilitate tractability, we suppose that τ is exponentially distributed with parameter $\lambda > 0$.[20]

We solve the firm's sequential investment problem by using dynamic programming. To this end, let $V^+(\kappa, t)$ denote the value of an investment opportunity in a growing market given that the current date is $t \geq 0$ and the number of completed steps is $\kappa \in [0, K]$. In turn, $V^-(\kappa, t)$ denotes the value of the investment opportunity if the firm has completed κ steps and the market started declining at date t.

[20]Notwithstanding the exponential random variable assumption, we conjecture that our results that unknown-date events lead to ambiguous effects of an MPIS on option values and investment timing still hold for the case of compound real options. In fact, all our results of Section 3 would follow as k grows large enough.

If the market maturity date has not been reached yet and the project is still incomplete, then the Bellman equation that stems from the firm's optimization problem is as follows (neglecting terms of order higher than dt):

$$V^{+}(\kappa,t)=\max_{I\in[0,k]}\{-Idt+(1-rdt)[(1-\lambda dt)V^{+}(\kappa+Idt,t+dt)+\lambda dtV^{-}(\kappa+Idt,t+dt)]\}. \quad (2)$$

The Bellman equation accounts for the possibility of reaching the market maturity date over the next time interval of lenght dt. Thus, the maturity date is reached with (approximate) conditional probability λdt, and in such a case the firm seizes the value of the investment opportunity in a declining market given the number of steps completed by then and the realized maturity date of the market.

Performing a Taylor expansion on $V^{+}(\kappa+Idt,t+dt)$ and $V^{-}(\kappa+Idt,t+dt)$ respectively yields that $V^{+}(\kappa+Idt,t+dt)\approx V^{+}(\kappa,t)+V_{\kappa}^{+}(\kappa,t)Idt+V_{t}^{+}(\kappa,t)dt$ and $V^{-}(\kappa+Idt,t+dt)\approx V^{-}(\kappa,t)+V_{\kappa}^{-}(\kappa,t)Idt+V_{t}^{-}(\kappa,t)dt$, so using these approximations in (2), canceling terms, dividing through by dt and letting $dt\to 0$ yields the following Bellman equation rearranged:

$$(r+\lambda)V^{+}(\kappa,t)=\max_{I\in[0,k]}\{-I+\lambda V^{-}(\kappa,t)+V_{\kappa}^{+}(\kappa,t)I+V_{t}^{+}(\kappa,t)\}. \quad (3)$$

Because the right hand side is linear in I, the problem has a "bang-bang" structure, as was to be expected given the constant return investment technology employed by the firm. Thus, there will be a function $T(\kappa)$ such that if the firm has completed $\kappa\in[0,K)$ steps by date $t\geq T(\kappa)$ (and the market is currently growing), then the firm will invest at the maximum rate k at such date, that is, $I^{*}(\kappa,t)=k$. Otherwise, if $t<T(\kappa)$, then the firm will invest nothing, that is, $I^{*}(\kappa,t)=0$. Intuitively, $T(\kappa)$ is a decreasing function because of the monotonic growth and the memoryless property of the random variable. As a result, once the firm starts investing at time $T\equiv T(0)$, it will keep on investing at the maximum possible rate as long as the market maturity date does not arrive.

In order to work with (3), it is convenient to derive the functional form of the value of an investment opportunity once the maturity date of the market has arrived. It is standard to show that if the firm has completed κ steps at the realized maturity date τ, then the firm will either never invest or invest at the maximum rate right away until the project's completion. That is, letting $D(\kappa,\tau)$ denote the firm's payoff if it has completed κ steps and invests at the maximum rate since date τ until the project's total completion, we have that

$$\begin{aligned} D(\kappa,\tau) &= \int_{\tau+(K-\kappa)/k}^{\infty} \pi e^{(2\alpha+r)\tau} e^{-(\alpha+r)s} ds - \int_{\tau}^{\tau+(K-\kappa)/k} k e^{-r(s-\tau)} ds) \\ &= \frac{\pi e^{\alpha\tau} e^{-(\alpha+r)(K-\kappa)/k}}{\alpha+r} - \frac{k(1-e^{-r(K-\kappa)/k})}{r}, \end{aligned}$$

In short, we have shown that $V^{-}(\kappa,\tau) = \max(0, D(\kappa,\tau))$.

At this point, it is convenient to describe the qualitative features of the optimal investment path if the maturity date has not arrived. The firm simply gathers information by waiting to invest for $t < T$, but once it starts investing at the maximum rate at time T, it keeps on doing so as long as the market does not stop growing. Finally, if the maturity date is reached and the project is incomplete, the firm drops it if and only if the number of steps left is large enough. Thus, given a start date T, let κ^c be the critical number of steps above which the firm finds it optimal to continue with the project even if the market immediately begins its decay. Formally, $\kappa^c \in (0,K)$ is implicitly defined by $D(\kappa^c, T+\kappa^c/k) = 0$.

Therefore, in order to derive the value of the investment opportunity at $t = 0$ and the optimal investment policy (conditional upon the market growing), we must distinguish a number of different regions on the state space. In region I, the firm does not invest while the market is growing, and does not invest if it suddenly switches to decline. In region II of the state space, the firm exerts the maximum effort while the market is expanding, but abandons the project if the market starts decaying all of a sudden. In region III, the firm invests at the maximum rate while the market is growing, and continues investing at rate k

until the project's completion if the market suddenly begins declining. In region IV of the state space, the firm has already completed the project.

We work backwards to find out the value function in each of these regions. If the firm is in region IV (that is, $\kappa = K$) and the current date is t, then the assumption that τ is exponentially distributed yields that the value of the firm is given by

$$G(t) = \int_0^\infty \lambda e^{-\lambda\tau}(\int_0^\tau \pi e^{\alpha t} e^{(\alpha-r)s} ds + \int_\tau^\infty \pi e^{\alpha t} e^{2\alpha\tau} e^{-(\alpha+r)s} ds) d\tau = \frac{(\alpha + r + \lambda)\pi e^{\alpha t}}{(\alpha + r)(r + \lambda - \alpha)}.$$

In turn, in region III (that is, $\kappa \in [\kappa^c, K)$ and $t \in [T + \kappa^c/k, T + K/k]$), we have that $I^*(\kappa, t) = k$ and $V^-(\kappa, t) = D(\kappa, t) > 0$, so (3) implies that firm value dynamics in such region are given by the following partial differential equation:

$$(r + \lambda)V^+(\kappa, t) = V_t^+(\kappa, t) + V_\kappa^+(\kappa, t)k - k + \lambda(\frac{\pi e^{\alpha t} e^{-(\alpha+r)(K-\kappa)/k}}{\alpha + r} - \frac{k(1 - e^{-r(K-\kappa)/k})}{r}).$$

Therefore, in this region,

$$V^+(\kappa, t) = A_3(t - \kappa/k)e^{(r+\lambda)\kappa/k} - \frac{\lambda\pi e^{\alpha t} e^{-(\alpha+r)(K-\kappa)/k}}{(\alpha + r)(2\alpha - \lambda)} - \frac{k(1 - e^{-r(K-\kappa)/k})}{r},$$

where A_3 is a constant to be derived. In particular, using the value-matching condition that $V^+(K, T + K/k) = G(T + K/k)$ yields the value of A_3, and hence we have that in this region:

$$V^+(\kappa, t) = \frac{2\alpha\pi e^{\alpha T} e^{(r+\lambda)\kappa/k} e^{-(r+\lambda-\alpha)K/k}(t - \kappa/k)}{T(r + \lambda - \alpha)(2\alpha - \lambda)} - \frac{\lambda\pi e^{\alpha t} e^{-(\alpha+r)(K-\kappa)/k}}{(\alpha + r)(2\alpha - \lambda)} - \frac{k(1 - e^{-r(K-\kappa)/k})}{r}.$$

In region II, it holds that $\kappa \in [0, \kappa^c)$ and $t \in [T, T + \kappa^c/k)$, so we have that $I^*(\kappa, t) = k$ and $V^-(\kappa, t) = 0 > D(\kappa, t)$. Equation (3) leads to the following dynamics for firm value:

$$(r + \lambda)V^+(\kappa, t) = V_t^+(\kappa, t) + V_\kappa^+(\kappa, t)k - k.$$

Therefore, we have that in this region

$$V^{+}(\kappa,t) = A_2(t-\kappa/k)e^{(r+\lambda)\kappa/k} - \frac{k}{r+\lambda}$$

for some constant A_2. In order to find out the value of A_2, we must use another value-matching condition. In particular, we must have that firm value in regions II and III is the same whenever $\kappa = \kappa^c$ and $t = T + \kappa^c/k$ hold, so

$$\begin{aligned} V^{+}(\kappa,t) &= \left(\begin{array}{c} \frac{2\alpha\pi e^{\alpha(T+\kappa^c/k)}e^{-(r+\lambda-\alpha)(K-\kappa^c)/k}}{(r+\lambda-\alpha)(2\alpha-\lambda)} - \frac{\lambda\pi e^{\alpha(T+\kappa^c/k)}e^{-(\alpha+r)(K-\kappa^c)/k}}{(\alpha+r)(2\alpha-\lambda)} \\ -\frac{k(1-e^{-r(K-\kappa^c)/k})}{r} + \frac{k}{r+\lambda} \end{array} \right) \times \\ & \quad \frac{(t-\kappa/k)e^{(r+\lambda)\kappa/k}e^{-(r+\lambda)\kappa^c/k}}{T} - \frac{k}{r+\lambda}. \end{aligned}$$

Lastly, in region I, we have that $\kappa = 0$ and $t \in [0, T)$. This means that $I^*(\kappa,t) = 0$ and $V^-(\kappa,t) = 0 > D(\kappa,t)$, so expression (3) yields that the dynamics of the value of the investment opportunity are given by

$$(r+\lambda)V^{+}(\kappa,t) = V_t^{+}(\kappa,t).$$

In this region, the market is still too small and the firm simply waits to invest in order to gather information, so we have that

$$V^{+}(\kappa,t) = A_1 e^{(r+\lambda)t},$$

for some constant A_1. Using the value-matching condition that firm value in regions I and II must be the same for $\kappa = 0$ and $t = T$ allows us to definitize A_1, which yields that in region I we have that:

$$V^+(\kappa,t) = e^{-(r+\lambda)(T-t)} \left[\left(\begin{array}{c} \frac{2\alpha\pi e^{\alpha(T+\kappa^c/k)} e^{-(r+\lambda-\alpha)(K-\kappa^c)/k}}{(r+\lambda-\alpha)(2\alpha-\lambda)} - \frac{\lambda\pi e^{\alpha(T+\kappa^c/k)} e^{-(\alpha+r)(K-\kappa^c)/k}}{(\alpha+r)(2\alpha-\lambda)} - \\ \frac{k(1-e^{-r(K-\kappa^c)/k})}{r} + \frac{k}{r+\lambda} \end{array} \right) \times \right.$$
$$\left. e^{(r+\lambda)\kappa/k} e^{-(r+\lambda)\kappa^c/k} - \frac{k}{r+\lambda} \right].$$

At date $t = 0$, the firm chooses the time T at which to start investing in the project so as to maximize firm value, $V^+(0,0)$. Note that κ^c is a function of T implicitly given by

$$\frac{\pi e^{\alpha T} e^{(2\alpha+r)\kappa^c/k} e^{-(\alpha+r)K/k}}{\alpha + r} = \frac{k(1 - e^{-r(K-\kappa^c)/k})}{r}, \tag{4}$$

and with derivative with respect to T as follows:

$$\frac{d\kappa^c}{dT} = -\frac{r\alpha\pi e^{\alpha T} e^{-(\alpha+r)K/k} e^{(2\alpha+r)\kappa^c/k}}{(\alpha + r)(r + 2\alpha(1 - e^{-r(K-\kappa^c)/k}))}. \tag{5}$$

Therefore, differentiating $V_1^+(0,0)$ with respect to T, equating the derivative to 0, and performing several tedious manipulations using the fact that $\frac{d\kappa^c}{dT} = -\frac{\alpha k(1 - e^{-r(K-\kappa^c)/k})}{(r + 2\alpha(1 - e^{-r(K-\kappa^c)/k}))}$ yields the following first-order condition:

$$\frac{2\alpha\pi e^{\alpha T^*} e^{-(r+\lambda-\alpha)K/k}}{(2\alpha - \lambda)} - \frac{\lambda(r + \lambda - \alpha)\pi e^{\alpha T^*} e^{-(\alpha+r)K/k} e^{(2\alpha-\lambda)\kappa^c/k}}{(\alpha + r)(2\alpha - \lambda)} =$$
$$\frac{(r + \lambda)e^{-(r+\lambda)\kappa^c/k} k(1 - e^{-r(K-\kappa^c)/k})}{r} + k(1 - e^{-(r+\lambda)\kappa^c/k}).$$

Rewriting this expression, using (4) for $T = T^*$ and letting $\kappa^c(T^*) \equiv \kappa^*$ leads to the following expression:

$$\pi e^{\alpha(T^*+K/k)} e^{-(r+\lambda)K/k} + \int_{\kappa^*/k}^{K/k} (\pi e^{2\alpha(T^*+\tau)} e^{-\alpha(T^*+K/k)} e^{-rK/k}) f(\tau) d\tau =$$
$$k(1 - e^{-(r+\lambda)\kappa^*/k}) + k e^{-(r+\lambda)\kappa^*/k}(1 - e^{-r(K-\kappa^*)/k}). \tag{6}$$

Note that

$$k(1 - e^{-(r+\lambda)\kappa^*/k}) + ke^{-(r+\lambda)\kappa^*/k}(1 - e^{-r(K-\kappa^*)/k}) =$$
$$\int_0^{\kappa^*/k} ke^{-r\tau} f(\tau)d\tau + \int_0^{\kappa^*/k} (\frac{1}{\lambda})rke^{-r\tau} f(\tau)d\tau + e^{-\lambda\kappa^*/k} \int_{\kappa^*/k}^{K/k} rke^{-r\tau} d\tau,$$

so we can establish the following result whenever an interior solution exists:[21]

Proposition 5 *The firm's optimal investment policy calls for exerting effort at rate k at time t if either of the two following conditions hold: $t \in [T^*, \tau]$ for $\tau \in (T^*, T^* + \kappa^*/k)$; or $t \in [T^*, T^* + K/k]$ for $\tau \geq T^* + \kappa^*/k$. If none of these conditions is met, then the firm should not invest. In addition, the time T^* at which the firm finds it optimal to start investing if the market is still in growth is such that the following holds:*

$$\pi e^{\alpha(T^*+K/k)} e^{-(r+\lambda)K/k} + \int_{\kappa^*/k}^{K/k} (\pi e^{2\alpha(T^*+\tau)} e^{-\alpha(T^*+K/k)} e^{-rK/k}) f(\tau)d\tau =$$
$$\int_0^{\kappa^*/k} ke^{-r\tau} f(\tau)d\tau + \int_0^{\kappa^*/k} (\frac{1}{\lambda})rke^{-r\tau} f(\tau)d\tau + e^{-\lambda\kappa^*/k} \int_{\kappa^*/k}^{K/k} rke^{-r\tau} d\tau. \qquad (7)$$

If investment is sequential, the firm's optimal time at which to start paying k is such that the *expected* marginal cost of waiting to invest is equal to the marginal value of waiting to start investing. On the one hand, by waiting to invest, the firm forgoes earning a future profit flow if it does not abandon the project due to a premature arrival of the market maturity date. Thus, if the firm started investing, three situations could take place before the planned completion date of the project. First, with probability $e^{-\lambda K/k}$, the market might not reach its maturity between times T^* and $T^* + K/k$. In such cases, the firm would forgo earning a discounted flow profit equal to $\pi e^{\alpha(T^*+K/k)} e^{-rK/k}$. Second, if the maturity date of the market arrived at some $\tau \in [T^* + \kappa^*/k, T^* + K/k]$, then the firm would forgo earning a discounted profit flow in a declining market, since it would rather complete the project

[21] Unless π is large enough, the time $T^* > 0$ at which the firm finds it optimal to start investing is given by the solution to the system of equations formed by (4) and (6).

despite the market is decaying. In such case, a marginal delay in starting the investment would lead to forgoing a discounted flow profit equal to $\pi e^{2\alpha(T^*+\tau)}e^{-\alpha(T^*+K/k)}e^{-rK/k}$. Third and last, if the market switched to decline at some time between T^* and $T^*+\kappa^*/k$, which would happen with probability $1-e^{-\lambda\kappa^*/k}$, then the firm would forgo nothing given that it would stop the project. Hence, the left hand side of (7) quantifies the expected marginal cost of deferring the start of the investment project.

On the other hand, in the light of the bad news principle of irreversible investment, waiting to invest is valuable because the firm may avoid starting a project that should be discarded in the future if the market switches between dates T^* and $T^*+\kappa^*/k$. In such case, waiting to invest allows the firm to avoid incurring a discounted cost stream over a period of random lenght. This effect, the so-called "marginal option value of waiting to invest," is represented by the first term on the right hand side of (7). The second and third terms arise because delaying the start of the cost stream allows the firm to expect some savings in investment costs due to discounting. To compute expected savings, note that the firm continuously invests over time as long as the market maturity date does not arrive before $T^*+\kappa^*/k$. Overall, the right hand side of the above expression quantifies the expected marginal value of delaying the start of the investment project.

5 Conclusion

This paper has focused on investment contexts in which the only source of uncertainty affecting the value of a project stems from the unknown date of arrival of an unfavorable event. We have not modeled uncertainty over time by the means of a full-fledged stochastic process, and we have assumed under fairly general conditions that the date of occurrence of the event is a single random variable. In particular, it need not be exponentially distributed, which would rule out Poisson processes, since they are not well suited for some analyses.

The consideration of unknown-date events as a source of uncertainty crucially affects some of the core conclusions that characterize conventional real options theory. More specifically, we have identified a necessary and sufficient condition for the value of an investment opportunity to decrease with uncertainty. This would occur because the time space –which is the control space– and the outcome space –which is the state space– coincide. Consequently, waiting allows the firm to be insured against adverse states of the world while taking advantage of favorable states. The firm cannot avoid being damaged (and thus making losses) if the adverse event occurs shortly after investing, though. The reason is that the firm chooses to invest when expected net present value is maximal, not when downside risk vanishes, and thus greater uncertainty may increase the probability of occurrence of the adverse event right after investing by a sufficiently large amount.

In addition, we have shown that greater uncertainty may speed up investment timing in some situations. Pinpointing the conditions under which this occurs is certainly relevant for empirical work on the investment-uncertainty relationship. In our model, the firm anticipates that it will have better information at the time of investment, so at the margin, it only cares about the conditional probability of immediate arrival of the unfavorable event. Greater uncertainty may reduce this probability of receiving bad news, thus decreasing the cost of making an irreversible investment, which would hasten investment. Intuitively, the firm would face a smaller (ex ante) probability of making losses immediately after investing and/or a larger probability of being able to invest because of favorable market conditions.

Besides trying to inform empirical work, we have set up an alternative theoretical framework for continuous-time real options models of investment whenever the demand of a product follows a life cycle that is unknown to the firm.[22] The setting is simpler than that proposed by Bollen (1999), and is a potentially useful building block for complex environments such as those involving compound real options or multiple decision-makers.

[22] Such a model may be relevant based on empirical evidence. Thus, Bowman and Moskowitz (2001, p. 775) suggest that one of the mistakes made by Merck when valuing its Project Gamma was the use of the Black-Scholes formula, instead of taking into account that some biotechnology products follow a life cycle.

Appendix

Proof of Lemma 1. Since $NPV(t) = \int_t^\infty \pi e^{2\alpha\tau} e^{-(\alpha+r)s} ds - Ke^{-rt}$ is strictly quasi-convex on $(-\infty, +\infty)$,[23] it follows that, for all $\tau < t' < t''$, $NPV(t') < \max(NPV(\tau), NPV(t''))$. Taking the limit as $t'' \to \infty$, $NPV(t') < \max(NPV(\tau), 0)$. Finally, note that $NPV(\tau) > 0$ if and only if $\tau > t^c = \max\{0, \ln[K(\alpha + r)/\pi]/\alpha\}$, which completes the proof. ■

Proof of Proposition 1. We first show that $V(t_1)$ is monotone decreasing on $[t^c, \infty)$. On this set, the function becomes:

$$\begin{aligned} V(t_1) &= \int_{t^c}^{t_1} f(\tau)(\int_\tau^\infty \Pi(s,\tau)e^{-rs} ds - Ke^{-r\tau})d\tau + \\ &\quad \int_{t_1}^{\infty} f(\tau)(\int_{t_1}^\infty \Pi(s,\tau)e^{-rs} ds - Ke^{-rt_1})d\tau \\ &= \int_{t^c}^{t_1} f(\tau)(\int_\tau^\infty \pi e^{2\alpha\tau} e^{-(\alpha+r)s} ds - Ke^{-r\tau})d\tau + \\ &\quad \int_{t_1}^{\infty} f(\tau)(\int_{t_1}^\tau \pi e^{(\alpha-r)s} ds + \int_\tau^\infty \pi e^{2\alpha\tau} e^{-(\alpha+r)s} ds - Ke^{-rt_1})d\tau. \end{aligned}$$

Differentiating with respect to t_1 and performing some algebraic manipulations yields:

$$V'(t_1) = (rK - \pi e^{\alpha t_1})e^{-rt_1} \int_{t_1}^\infty f(\tau)d\tau.$$

We claim that $V'(t_1) < 0$ for $t_1 \geq t^c$. Otherwise, we would reach a contradiction:

$$0 \leq (rK - \pi e^{\alpha t_1})e^{-rt_1} \int_{t_1}^\infty f(\tau)d\tau \leq -\alpha K e^{-rt_1} \int_{t_1}^\infty f(\tau)d\tau < 0,$$

since $\dfrac{\pi e^{\alpha t_1}}{\alpha + r} \geq \dfrac{\pi e^{\alpha t^c}}{\alpha + r} \geq K$ for $t_1 \geq t^c$. Assumption 4 implies that $V(t_1)$ is bounded above, which shows that $V(t_1)$ attains a unique global maximum when $t^c = 0$, so let $t^c > 0$ and note that, to conclude the proof, we can restrict our attention to the set $[0, t^c]$. In this case,

[23] Formally, because $dNPV(t)/dt = 0$ implies $d^2NPV(t)/dt^2 > 0$.

$V(t_1)$ is as follows:

$$V(t_1) = \int_{t_1}^{\infty} f(\tau)(\int_{t_1}^{\tau} \pi e^{(\alpha-r)s} ds + \int_{\tau}^{\infty} \pi e^{2\alpha\tau} e^{-(\alpha+r)s} ds - Ke^{-rt_1}) d\tau.$$

By Assumption 4, this function is bounded above. Differentiating it with respect to t_1, solving the integrals, taking into account that $\lambda(t_1) = \frac{f(t_1)}{\int_{t_1}^{\infty} f(\tau)d\tau}$ and rearranging yields:

$$\begin{aligned} V'(t_1) &= -\frac{\pi e^{(\alpha-r)t_1} f(t_1)}{\alpha+r} - \pi e^{(\alpha-r)t_1} \int_{t_1}^{\infty} f(\tau)d\tau + Ke^{-rt_1}(f(t_1) + r\int_{t_1}^{\infty} f(\tau)d\tau) \\ &= [-\frac{\pi e^{\alpha t_1}\lambda(t_1)}{\alpha+r} - \pi e^{\alpha t_1} + K(r+\lambda(t_1))] \int_{t_1}^{\infty} e^{-rt_1} f(\tau)d\tau. \end{aligned}$$

As $\frac{\alpha+r}{\pi(r+\lambda(t_1))\int_{t_1}^{\infty} e^{-rt_1} f(\tau)d\tau} > 0$, $signV'(t_1) = signW(t_1)$, so it suffices to show that $W(\cdot)$ is a decreasing function in order to prove that a unique global maximum is attained at $t^e = \min\{t_1 \geq 0 : W(t_1) \leq 0\}$. The assumption that $\alpha + 2r \geq f'(t_1)/f(t_1)$ for all t_1 implies that

$$\begin{aligned} [\frac{f'(t_1)}{f(t_1)} - (\alpha+2r)]\lambda(t_1) \leq & \quad 0 < r(\alpha+r) \Rightarrow \\ (r+\lambda(t_1))(\alpha+r+\lambda(t_1)) > & \quad \frac{\lambda(t_1)f'(t_1)}{f(t_1)} + [\lambda(t_1)]^2 = \lambda'(t_1). \end{aligned} \tag{8}$$

As a result, it follows that $W'(t_1) = -\frac{\alpha e^{\alpha t_1}[(r+\lambda(t_1))(\alpha+r+\lambda(t_1)) - \lambda'(t_1)]}{(r+\lambda(t_1))^2} < 0$.

It is also clear that $V(t^e) > 0$, since $\lim_{t_1\to\infty} V(t_1) = 0$, and $V(t_1)$ is single-peaked. Finally, simple manipulations show that $t^e < t^c$ if $t^e > 0$, so the firm does not invest if the maturity date of the market turns out to be smaller than t^e (by Lemma 1). ■

Proof of Proposition 2. Note that $W(t^e) = 0$ if $t^e > 0$, so straightforward manipulations after multiplying such expression through by $\frac{\pi(r+\lambda(t^e))}{\alpha+r}$, and noticing that $\frac{\pi e^{\alpha t^e}}{\alpha+r} = \int_{t^e}^{\infty} \pi e^{\alpha(2t^e-s)} e^{-r(s-t^e)} ds$ imply that $\pi e^{\alpha t^e} = rK + \lambda(t^e)(K - \int_{t^e}^{\infty} \pi e^{\alpha(2t^e-s)} e^{-r(s-t^e)} ds)$. ■

Proof of Proposition 3. Differentiate $W(t^e, \sigma)$ with respect to t^e and σ and rearrange so that:

$$\begin{aligned}\frac{dt^e}{d\sigma}(\sigma_0) &= -\frac{e^{\alpha t_0^e}\left(\frac{-\alpha\lambda_\sigma(t_0^e\,|\sigma_0\,)}{(r+\lambda(t_0^e\,|\sigma_0\,))^2}\right)}{\alpha e^{\alpha t_0^e}\left(\frac{(r+\lambda(t_0^e\,|\sigma_0\,))^2+\alpha(r+\lambda(t_0^e\,|\sigma_0\,))-\lambda_t(t_0^e\,|\sigma_0\,)}{(r+\lambda(t_0^e\,|\sigma_0\,))^2}\right)} \\ &= \frac{\lambda_\sigma(t_0^e\,|\sigma_0\,)}{(r+\lambda(t_0^e\,|\sigma_0\,))^2+\alpha(r+\lambda(t_0^e\,|\sigma_0\,))-\lambda_t(t_0^e\,|\sigma_0\,)}.\end{aligned}$$

Since $(\alpha+r+\lambda(t_0^e\,|\sigma_0\,))(r+\lambda(t_0^e\,|\sigma_0\,)) > \lambda_t(t_0^e\,|\sigma_0\,)$ by expression (8), we have that $sign(\frac{dt^e}{d\sigma}(\sigma_0)) = sign(\lambda_\sigma(t_0^e\,|\sigma_0\,))$, which completes the proof. ■

Proof of Lemma 2. First note that $\pi^e(\sigma)$ is a well-defined function (and not a correspondence) by the uniqueness of t^e for a given σ. Indeed, it is continuous by the continuity of $f(\cdot\,|\sigma\,)$ for all σ. Furthermore, if an MPIS is performed so that the spread infinitesimally rises from σ_0 to σ_1, then $f(0\,|\sigma_1\,) \geq f(0\,|\sigma_0\,)$, so $\pi^e(\sigma_1) \geq \pi^e(\sigma_0)$, and thus $\pi^e(\sigma)$ is non-decreasing. Finally, note that $f(0\,|\sigma\,)$ can be neither smaller than 0 nor larger than ∞, which, together with non-decreasingness of $\pi^e(\sigma)$, implies that the range of the function must be bounded by $[rK, (\alpha+r)K]$. ■

Proof of Proposition 4. We proceed to prove the statement of the proposition for three different cases:

(i) If $\pi \geq \lim_{\sigma\to\infty} \pi^e(\sigma)$, then the firm's optimal time of investment is $t^e(\sigma) = 0$ for all σ,[24] so:

$$V(\sigma_0) = \pi(\frac{1}{\alpha+r}+\frac{1}{\alpha-r})\int_0^\infty e^{(\alpha-r)\tau} f(\tau\,|\sigma_0\,)d\tau - (\frac{\pi}{\alpha-r}+K).$$

Hence, differentiating yields:

$$V'(\sigma_0) \equiv \frac{dV(\sigma_0)}{d\sigma} = \pi(\frac{1}{\alpha+r}+\frac{1}{\alpha-r})\int_0^\infty e^{(\alpha-r)\tau} f_\sigma(\tau\,|\sigma_0\,)d\tau.$$

[24] This follows from the facts that $\pi > \pi^e(\sigma)$ for all σ if and only if the firm invests immediately for all σ.

Integrating by parts, considering that $\int_0^\infty f_\sigma(\tau\,|\sigma_0)d\tau = 0$ and $F_\sigma(\tau\,|\sigma_0) = \int_0^\tau f_\sigma(s\,|\sigma_0)ds$ when an MPIS is performed, and noticing that $\int_0^\infty e^{(\alpha-r)\tau}F_\sigma(\tau\,|\sigma)$ is well-defined we have:

$$\begin{aligned} V'(\sigma_0) &= \pi(\frac{1}{\alpha+r}+\frac{1}{\alpha-r})\left\{\begin{array}{c} \left[e^{(\alpha-r)\tau}\int_0^\tau f_\sigma(s\,|\sigma_0)ds\right]_0^\infty - \\ \int_0^\infty(\alpha-r)e^{(\alpha-r)\tau}(\int_0^\tau f_\sigma(s\,|\sigma_0)ds)d\tau \end{array}\right\} \\ &= -\left(\frac{2\alpha\pi}{\alpha+r}\right)\int_0^\infty e^{(\alpha-r)\tau}F_\sigma(\tau\,|\sigma). \end{aligned}$$

Hence, taking into account that $t^e(\sigma_0) = t_0^e = 0$ and thus $F_\sigma(t_0^e\,|\sigma_0) = 0$:

$$\begin{aligned} &V'(\sigma_0) \geq 0 \Leftrightarrow \int_0^\infty e^{(\alpha-r)\tau}F_\sigma(\tau\,|\sigma) \leq 0 \Leftrightarrow \\ &2(r+\lambda(t_0^e\,|\sigma_0))\int_{t_0^e}^\infty e^{(\alpha-r)(\tau-t_0^e)}F_\sigma(\tau\,|\sigma_0)d\tau \leq F_\sigma(t_0^e\,|\sigma_0). \end{aligned}$$

This completes the proof when $\pi \geq \lim_{\sigma\to\infty}\pi^e(\sigma)$.

(ii) If $\pi \leq \lim_{\sigma\to 0}\pi^e(\sigma)$, then we have that $t^e(\sigma_0) > 0$ for all σ_0 by Lemma 2, so:

$$\begin{aligned} V(\sigma) &= \pi(\frac{1}{\alpha+r}+\frac{1}{\alpha-r})\int_{t^e(\sigma)}^\infty e^{(\alpha-r)\tau}f(\tau\,|\sigma)d\tau - \\ &\quad (\frac{\pi e^{(\alpha-r)t^e(\sigma)}}{\alpha-r}+Ke^{-rt^e(\sigma)})\int_{t^e(\sigma)}^\infty f(\tau\,|\sigma)d\tau. \end{aligned}$$

Given that $\frac{1}{\alpha}\ln(\frac{rK}{\pi}) \leq t^e(\sigma) \leq \frac{1}{\alpha}\ln(\frac{(\alpha+r)K}{\pi})$ for all σ (this trivially follows from Proposition 1), $t^e(\sigma)$ is a continuous function by the theorem of the maximum. Indeed, we have that $t^e(\sigma)$ is differentiable on a local neighborhood of σ_0 by the implicit function theorem, and hence the envelope theorem implies that effects of changes in

σ on $V(\sigma)$ via $t^e(\sigma)$ are of second-order, so letting $t^e(\sigma_0) = t_0^e$, we have:

$$\begin{aligned} V'(\sigma_0) &= \pi(\frac{1}{\alpha+r} + \frac{1}{\alpha-r}) \int_{t_0^e}^{\infty} e^{(\alpha-r)\tau} f_\sigma(\tau\,|\sigma_0\,) d\tau - \\ & \quad (\frac{\pi e^{(\alpha-r)t_0^e}}{\alpha-r} + K e^{-rt_0^e}) \int_{t_0^e}^{\infty} f_\sigma(\tau\,|\sigma_0\,) d\tau. \end{aligned}$$

Integration by parts noting that $\int_0^\infty e^{(\alpha-r)\tau} F_\sigma(\tau\,|\sigma\,)$ is finite, and recalling that $\int_0^\infty f_\sigma(\tau\,|\sigma_0\,)d\tau = 0$ and $F_\sigma(\tau\,|\sigma_0\,) = \int_0^\tau f_\sigma(s\,|\sigma_0\,)ds$ when an MPIS is performed, yields:

$$\begin{aligned} V'(\sigma_0) &= -(\frac{\pi e^{(\alpha-r)t_0^e}}{\alpha-r} + K e^{-rt_0^e}) \int_{t_0^e}^{\infty} f_\sigma(\tau\,|\sigma_0\,) d\tau + \\ & \quad \pi(\frac{1}{\alpha+r} + \frac{1}{\alpha-r}) \left\{ \begin{array}{c} \left[e^{(\alpha-r)\tau} \int_0^\tau f_\sigma(s\,|\sigma_0\,)ds\right]_{t_0^e}^{\infty} - \\ \int_{t_0^e}^{\infty} (\alpha-r) e^{(\alpha-r)\tau} (\int_0^\tau f_\sigma(s\,|\sigma_0\,)ds) d\tau \end{array} \right\} \\ &= (\frac{\pi e^{(\alpha-r)t_0^e}}{\alpha+r} + \frac{\pi e^{(\alpha-r)t_0^e}}{\alpha-r} - \frac{\pi e^{(\alpha-r)t_0^e}}{\alpha-r} - K e^{-rt_0^e}) \int_{t_0^e}^{\infty} f_\sigma(\tau\,|\sigma_0\,) d\tau - \\ & \quad (\frac{2\alpha\pi}{\alpha+r}) \int_{t_0^e}^{\infty} e^{(\alpha-r)\tau} F_\sigma(\tau\,|\sigma_0\,) d\tau \\ &= (\frac{\pi e^{(\alpha-r)t_0^e}}{\alpha+r} - K e^{-rt_0^e}) \int_{t_0^e}^{\infty} f_\sigma(\tau\,|\sigma_0\,) d\tau - (\frac{2\alpha\pi}{\alpha+r}) \int_{t_0^e}^{\infty} e^{(\alpha-r)\tau} F_\sigma(\tau\,|\sigma_0\,) d\tau. \end{aligned}$$

Hence, we have that $V'(\sigma_0) \geq 0$ if and only if the following holds:

$$\begin{aligned} (\frac{2\alpha\pi}{\alpha+r}) \int_{t_0^e}^{\infty} e^{(\alpha-r)\tau} F_\sigma(\tau\,|\sigma_0\,) d\tau \leq -(\frac{\alpha\pi e^{(\alpha-r)t_0^e}}{(\alpha+r)(r+\lambda(t_0^e\,|\sigma_0\,))}) \int_{t_0^e}^{\infty} f_\sigma(\tau\,|\sigma_0\,) d\tau \Leftrightarrow \\ 2(r+\lambda(t_0^e\,|\sigma_0\,)) \int_{t_0^e}^{\infty} e^{(\alpha-r)(\tau-t_0^e)} F_\sigma(\tau\,|\sigma_0\,) d\tau \leq \int_0^{t_0^e} f_\sigma(\tau\,|\sigma_0\,) d\tau \Leftrightarrow \\ 2(r+\lambda(t_0^e\,|\sigma_0\,)) \int_{t_0^e}^{\infty} e^{(\alpha-r)(\tau-t_0^e)} F_\sigma(\tau\,|\sigma_0\,) d\tau \leq F_\sigma(t_0^e\,|\sigma_0\,), \end{aligned}$$

where first we have used the first-order condition of the firm's optimization program and finally we have used the fact that $F_\sigma(\infty\,|\sigma_0\,) = 0$.

(iii) If $\lim_{\sigma\to\infty} \pi^e(\sigma) > \pi \geq \lim_{\sigma\to 0} \pi^e(\sigma)$, define $\Sigma(\pi) \equiv \{\sigma : \pi^e(\sigma) = \pi\}$. Since $\pi^e(\sigma)$ is

continuous and non-decreasing by Lemma 2, then $\Sigma(\pi)$ must be convex (perhaps a singleton), so the following facts clearly hold: (1) case (i) applies for all $\sigma < \inf \Sigma(\pi)$; (2) case (ii) applies for all $\sigma \geq \inf \Sigma(\pi)$; and (3) $t^e(\sigma_0) \downarrow 0$ as $\pi \uparrow \pi^e(\sigma)$, so $t^e(\sigma)$ is a continuous function (since $\pi^e(\sigma)$ is continuous). This shows that, although $V(\sigma)$ need not be differentiable at $\inf \Sigma(\pi)$, it is clearly continuous and non-decreasing under the condition stated in the proposition.

■

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