

# Irreversible Investment in Stochastically Cyclical Markets\*

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## Abstract

This paper studies entry and exit decisions in perfectly competitive markets whose demand alternates between growth and decline phases at uncertain times. We introduce a stochastic process that captures these features of random market evolution, and we provide key mathematical results related to first passage times which make the characterization of entry and exit behavior quite simple and straightforward (even when the process is subject to an endogenously determined upper or lower barrier). We characterize entry and exit patterns in a dynamic competitive equilibrium, and we show why our results differ from those obtained if demand follows a diffusion process (e.g., a Geometric Brownian Motion). Despite the stochastic process of the underlying variable has a continuous sample path in both cases, we demonstrate in our setting that positive rates of entry and exit discontinuously fall to zero owing to informational overshooting. Another advantage of our framework is that it can explain discontinuities in stock prices even if sample paths are continuous or that it can be easily applied to (dis)investment timing games, as we illustrate.

**Keywords:** Dynamic Competitive Equilibrium, Irreversibility, Informational Overshooting, Entry and Exit in Competitive Markets, Entry and Exit in Duopoly.

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# 1 Introduction

A large number of industries exhibit a cyclical evolution of demand. Typical microeconomics textbook examples include industries producing durable goods such as motor vehicles or electrical appliances.<sup>1</sup> Other examples of cyclical industries involve very diverse sectors such as construction, household furniture, carpets and rugs, wholesale trade, legal or child care services, hotels, railroad transportation, metalworking machinery, automobile repair, etc (Berman and Pflieger 1997). The cyclical behavior of these industries typically reflects nothing but a high sensitiveness to the evolution of the state of the economy, which is well known to exhibit a cyclical but somewhat unpredictable pattern of evolution (Hamilton 1989).<sup>2</sup>

This paper studies irreversible (dis)investment decisions in industries whose demand follows random-length cycles such as the ones described above. In these industries, the dynamics of the underlying state variable (e.g., demand or profit) is not governed by the standard diffusion process traditionally employed by the literature on investment under uncertainty, namely the Geometric Brownian Motion (GBM).<sup>3</sup> To address this shortcoming of the modern approach to investment decisions under uncertainty, we focus on a stochastic process that exhibits cyclical behavior in that the underlying state variable perpetually alternates between growth and decline phases at uncertain times. The simplicity of the process allows for fully analytical results that can be easily derived both for a single firm and firms in (im)perfectly competitive environments. In addition, the process is rich enough for an option value of waiting to (dis)invest to arise. Because the length of growth and decline phases is random, a firm has an incentive to wait and continuously update its information about the duration of the current phase without making any irreversible decision, and at the same time it can capitalize on favorable market evolutions.

Our main objective is to characterize optimal/equilibrium entry and exit behavior in a variety of settings. Because of the novelty of the stochastic process in the literature on investment under uncertainty, a major contribution of the paper is to provide key mathematical results related to first passage times so as to make the analysis of entry and exit decisions quite simple and straightforward (including the cases where a firm is allowed to exit once it has entered the market). These results include expected discount factors to be used when discounting (one-shot) payoffs on the state space, or expected streams of discounted profits while transitioning from one demand/profit level to another one. With these mathematical results at hand, we do not need to resort to the complex mathematical

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<sup>1</sup>For instance, see Pindyck and Rubinfeld (2005, pp. 40-41).

<sup>2</sup>Indeed, Hamilton's (1989) paper spawned an increasingly large literature devoted to the empirical estimation of time series that are assumed to experiment unobservable changes in its growth rate at some random dates.

<sup>3</sup>Two classic papers on the modern approach to irreversible investment under uncertainty are McDonald and Siegel (1986) and Dixit (1989). Pindyck (1991) or Dixit and Pindyck (1994) provide excellent surveys of the most relevant theoretical developments. See also the recent work by Riedel and Su (2006) for an elegant general approach to irreversible investment under uncertainty.

apparatus employed by the literature on irreversible investment under uncertainty, namely dynamic programming and stochastic calculus. More specifically, we are able to characterize optimal entry and exit policies by directly applying our theorems and using ordinary calculus and algebra. Our mathematical results are powerful enough to make further research on stochastically cyclical markets quite simple and accessible. To illustrate this point, we apply them to the analysis and characterization of entry and exit policies, both for a single firm and a competitive firm, and we then compare the resulting policies. We also show how the equilibrium of an investment timing game can be easily characterized with the aid of our mathematical results without having to resort to stochastic control techniques as it is usual.

Besides the methodological contribution, another contribution of this paper is to draw implications for equilibrium entry and exit patterns in a perfectly competitive industry subject to a stochastically cyclical evolution. The stochastic process that we examine and a diffusion process have both a continuous sample path. However, unlike competitive settings in which a firm's profit flow follows a diffusion process (e.g., a GBM), entry (exit) rates are continuous even if they end up discontinuously falling from positive levels to zero whenever a growth (decline) phase ends. Thus, entry and exit waves may be observed to suddenly vanish because of "informational overshooting".<sup>4</sup> As the market goes through a growth (decline) phase whose length is random, a firm continuously updates its beliefs about the future stopping time, and entry (exit) continuously takes place; however, when the market finally stops growing (declining), a firm's expected stream of discounted profits experiences a sudden fall (rise), so entry (exit) rates drastically drop to zero.

Our paper contributes to the recent and growing literature on investment under uncertainty that builds on regime-switching models, as pioneered by Hassett and Metcalf (1999). The closest paper to ours is Drifill, Raybaudi and Sola (2003), which analyzes how the value of a single firm's investment opportunity is affected by the existence of regime shifts. Their stochastic process is more general in that they assume that the underlying state variable alternates between two different GBMs at random times, but this setting is not analytically tractable and they have to resort to simulations with which it is not possible to derive general results. Besides our specific focus on long-term random fluctuations as opposed to short-term volatility, our paper differs from theirs in other respects that are novel to the literature. From a technical standpoint, we characterize some general properties of a regime-switching stochastic process such as the (conditionally) expected stream of discounted profits harvested while the process transitions from one state to another. In addition, we are the first to examine perfectly competitive industries in a regime-switching framework, whose analysis turns out to be quite simple given our results for computing expected discounted profit streams with an upper and a lower barrier. Indeed, we show that the results (and reasoning) in Leahy (1993) directly extend to stochastically cyclical

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<sup>4</sup>See Zeira (1999) for an illustration of this concept in a setting without (dis)investment.

markets without any loss of substance. In particular, a competitive firm that acted myopically by ignoring all future entry (exit) would behave correctly by entering (exiting) at the optimal entry (exit) threshold of a single firm. Our empirical predictions regarding entry and exit paths significantly differ from his, though, since the paths we derive are continuous, even if positive entry and exit rates discontinuously fall to zero because of informational overshooting, an aspect that is also missing in his framework. Hence, our predictions are different from those derived using a diffusion process such as a GBM.

The remainder of the paper is organized as follows. Section 2 describes the stochastic process that constitutes the starting point of our analysis and sets the stage for Section 3, which provides several mathematical results (such as expected discount factors on the state space and present value calculations) that make all our subsequent results quite straightforward. Section 4 characterizes the entry and exit dynamics of a competitive industry in which demand follows random-length cycles, while Section 5 applies our framework to timing games. Section 6 offers concluding remarks. Proofs of the results not proven in the text can be found in two appendices.

## 2 The model

In this section we construct a stochastic process with continuous sample paths that represents the random evolution of a certain variable  $\Pi$ .<sup>5</sup> For the sake of concreteness,  $\Pi(t)$  denotes instantaneous profit at time  $t$ , although it could certainly represent any other variable such as demand of a product (as it is assumed in Section 4). Let the dynamics of flow profits be such that  $d\Pi = \alpha(t)\Pi dt$ , where  $\{\alpha(t), t \geq 0\}$  is a Markov chain with states  $\{\alpha_1, \alpha_2\} \in \mathfrak{R}_+ \times \mathfrak{R}_-$ . It is assumed for convenience that the chain starts at state  $\alpha_1$  (i.e.,  $\alpha(0) = \alpha_1$ ), while the transition probabilities of this process are as follows. On the one hand, if the chain is in state  $\alpha_1$  at time  $t \geq 0$ , then the probability that it moves to state  $\alpha_2$  between times  $t$  and  $t + dt$  is

$$\Pr(\alpha(t + dt) = \alpha_2 | \alpha(t) = \alpha_1) = \lambda_1 dt + o(dt).$$

On the other hand, if the chain is in state  $\alpha_2$  at time  $t > 0$ , then the probability that it moves to state  $\alpha_1$  between times  $t$  and  $t + dt$  is

$$\Pr(\alpha(t + dt) = \alpha_1 | \alpha(t) = \alpha_2) = \lambda_2 dt + o(dt).$$

Letting  $\tilde{\tau}_i$  denote the inter-arrival time of event  $i \in \{1, 2, \dots\}$  (where an event is a change in the state of the chain), we have that  $\{\tilde{\tau}_i\}_{i=1}^\infty$  is a sequence of exponential random variables such that the inter-arrival times with odd (even) subscripts are exponentially distributed

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<sup>5</sup>This process is the continuous-time limit of that put forward by Bagwell and Staiger (1997) in their analysis of collusive pricing over the business cycle.

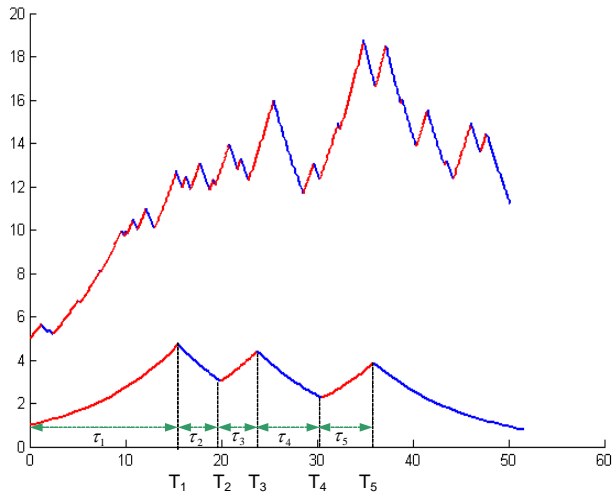


Figure 1: Two sample paths

with rate  $\lambda_1 > 0$  ( $\lambda_2 > 0$ ). We define  $T_i = T_{i-1} + \tau_i$  for all  $i \in \{1, 2, \dots\}$ , where the initial date is  $T_0 = 0$ , and we refer to each  $T_i$  as a (realized) switching date.

Figure 1 shows two sample paths of the process we have described (for different parameter values). As seen in the figure,  $\Pi(t)$  grows exponentially at rate  $\alpha_1 > 0$  during the random length period  $(T_{i-1}, T_i)$  ( $i = 1, 3, \dots$ ), and decreases exponentially at rate  $\alpha_2 < 0$  during  $(T_{i-1}, T_i)$  ( $i = 2, 4, \dots$ ). The sample path is continuous because  $\lim_{t \uparrow T_i} \Pi(t) = \lim_{t \downarrow T_i} \Pi(t)$ , although the path will exhibit a kink at any realized switching date  $T_i$  ( $i = 1, 2, \dots$ ), so it will be almost everywhere differentiable. Lastly, it holds that  $\Pi(t) > 0$  for any  $t \geq 0$  and any set of realizations of the random variables involved because we assume that  $\Pi(0) > 0$ .

We say that the process (or, more concretely, market) is in a growth phase when it is characterized by a positive instantaneous growth rate; otherwise, we say that the process is in a decline phase. Because of the i.i.d. exponential random variables, the current state of the market is clearly summarized by the current level of the flow profit and the type of phase through which the market is currently going.

This paper is concerned with (dis)investment decisions given the random temporal evolution of a variable that determines the profitability of a (dis)investment opportunity. We first focus on the cases in which there exists many atomistic decision-makers, and we relate the results we derive from this setting to those obtained when there exists a single decision-maker. We also consider entry timing games in a later section. Given these settings, the paper studies a firm's decisions to enter and exit a market given that the flow of profits reaped if active in the market follows the stochastic process previously described. Specifically, at each date, a firm does not know when the next upturns or downturns will happen, although it knows the current level of the profit flow and whether the cycle is growing or declining. A firm also observes realized switching dates immediately after they arrive. Any

firm is assumed to be risk-neutral and to use a constant discount rate  $r > 0$ . If a firm enters the market, then it incurs a positive fixed cost  $K$  and in return starts operating immediately (i.e., there is no time-to-build). Similarly, if a firm exits the market, then it is assumed to recover a non-negative value  $S$ , which can be the salvage or redeployment value of the asset.

In what follows, we study market entry and exit decisions for the case of atomistic firms, and we show how they are related to those made by a single firm that can optimally choose its entry and exit policies. Before, though, we derive several mathematical results that make these analyses quite simple.

### 3 Mathematical preliminaries

In this section we derive some relevant mathematical results regarding the stochastic process defined in Section 2 (proofs can be found in Appendix A). They are useful properties for working on the state space of the process and they will be used in Sections 4 and 5. However, the results are general enough to be directly applicable to (dis)investment problems we do not deal with in this paper (such as combined entry and exit for the case of a single firm).

Our first result deals with the expected stream of discounted profits if a firm is active in the market forever given the current state  $\pi_0$ .<sup>6</sup> We also allow the profit process to (temporarily) stay at level  $\bar{\pi} \in (0, \infty]$  if it ever reaches such level in a growth phase, at least until the process starts declining. Further, we allow the profit process to (temporarily) stay at level  $\underline{\pi} \in [0, \bar{\pi})$  if it ever reaches such level in a decline phase, at least until the process starts growing.<sup>7</sup> To this end, let  $\bar{\mathcal{E}}(\pi_0 | \bar{\pi}, \underline{\pi})$  denote the expected stream of discounted profits if the process is in a growth phase at  $\pi_0$ , given the upper bound  $\bar{\pi}$  and the lower bound  $\underline{\pi}$ .<sup>8</sup> Similarly, let  $\underline{\mathcal{E}}(\pi_0 | \bar{\pi}, \underline{\pi})$  denote the expected stream of discounted profits if the process is in a decline phase at  $\pi_0$ , given the upper bound  $\bar{\pi}$  and the lower bound  $\underline{\pi}$ . In order to compute these expected values of discounted streams of flow profits, it is useful to introduce some notation. In particular, let

$$\rho_1 \equiv \frac{\lambda_1 + \lambda_2 + r - \alpha_2}{(r + \lambda_1 - \alpha_1)(r + \lambda_2 - \alpha_2) - \lambda_1 \lambda_2}$$

and

$$\rho_2 \equiv \frac{\lambda_1 + \lambda_2 + r - \alpha_1}{(r + \lambda_1 - \alpha_1)(r + \lambda_2 - \alpha_2) - \lambda_1 \lambda_2}$$

denote the perpetuity factors to be used when the profit process is in a growth or a decline

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<sup>6</sup>Throughout, we denote the current level of the profit flow by  $\pi_0$ , whereas we denote growth and decline phases by an upper bar and a lower bar, respectively.

<sup>7</sup>Although not conventional, the notation  $\bar{\pi} = \infty$  ( $\underline{\pi} = 0$ ) is to be understood as the process not having an upper (lower) bound, since the process can never hit level  $\bar{\pi} = \infty$  ( $\underline{\pi} = 0$ ).

<sup>8</sup>Expectations are conditional upon the current state being  $\pi_0$  and upon whether the process is growing or not.

phase. Also, assume that

$$(r + \lambda_1 - \alpha_1)(r + \lambda_2 - \alpha_2) > \lambda_1 \lambda_2 \quad (1)$$

so that we have an economically meaningful setting (otherwise, streams of discounted profits fail to be integrable). A necessary condition for (1) to hold is that  $r + \lambda_1 > \alpha_1$ . We now state our first result regarding expected streams of discounted profits.

**Theorem 1** *Suppose that the firm is operating at  $\pi_0$  and is active forever. Suppose also that a profit process in a growth phase stays at level  $\bar{\pi} \in (0, \infty]$  if it ever reaches such level, and that it leaves  $\bar{\pi}$  only once the process starts declining. In addition, suppose that a profit process in a decline phase stays at level  $\underline{\pi} \in [0, \bar{\pi})$  if it ever reaches such level, and that it leaves  $\underline{\pi}$  only once the process starts growing. Then the expected stream of discounted profits if the process is in a growth phase is*

$$\bar{\mathcal{E}}(\pi_0 | \bar{\pi}, \underline{\pi}) = \xi(\bar{\pi}, \underline{\pi}) \pi_0^{\beta_1} + \delta_1 \zeta(\bar{\pi}, \underline{\pi}) \pi_0^{\beta_2} + \rho_1 \pi_0,$$

while the expected stream of discounted profits if the process is in a decline phase is

$$\underline{\mathcal{E}}(\pi_0 | \bar{\pi}, \underline{\pi}) = \delta_2 \xi(\bar{\pi}, \underline{\pi}) \pi_0^{\beta_1} + \zeta(\bar{\pi}, \underline{\pi}) \pi_0^{\beta_2} + \rho_2 \pi_0,$$

where  $\beta_1 \equiv \frac{\alpha_1(r + \lambda_2) + \alpha_2(r + \lambda_1) - \sqrt{\Delta}}{2\alpha_1\alpha_2} > 1$ ,  $\beta_2 \equiv \frac{\alpha_1(r + \lambda_2) + \alpha_2(r + \lambda_1) + \sqrt{\Delta}}{2\alpha_1\alpha_2} < 0$ ,  
 $\Delta \equiv [\alpha_1(\lambda_2 + r) - \alpha_2(\lambda_1 + r)]^2 + 4\alpha_1\alpha_2\lambda_1\lambda_2 > 0$ ,  $\delta_1 \equiv \frac{\lambda_2 + r - \alpha_2\beta_2}{\lambda_2} \in (0, 1)$ ,  $\delta_2 \equiv \frac{r + \lambda_1 - \alpha_1\beta_1}{\lambda_1} \in (0, 1)$ ,  $\xi(\bar{\pi}, \underline{\pi}) \equiv \frac{\delta_1 \bar{\pi}^{\beta_2} \rho_2 \underline{\pi} - \rho_1 \bar{\pi} \underline{\pi}^{\beta_2}}{\beta_1(\underline{\pi}^{\beta_2} \bar{\pi}^{\beta_1} - \delta_1 \delta_2 \underline{\pi}^{\beta_1} \bar{\pi}^{\beta_2})}$  and  $\zeta(\bar{\pi}, \underline{\pi}) \equiv \frac{\delta_2 \underline{\pi}^{\beta_1} \rho_1 \bar{\pi} - \rho_2 \underline{\pi} \bar{\pi}^{\beta_1}}{\beta_2(\bar{\pi}^{\beta_2} \underline{\pi}^{\beta_1} - \delta_1 \delta_2 \bar{\pi}^{\beta_1} \underline{\pi}^{\beta_2})}$ .

As we shall show below, Theorem 1 is useful when analyzing perfectly competitive industries in which free entry and exit lead to profits being bounded above and below (the bounds being determined endogenously). In competitive industries in which entry or exit do not take place or in industries in which there exists a single firm with the opportunity to (dis)invest, it is necessary to analyze the stochastic process subject to just one or no barrier at all. Because  $\beta_1 > 1$  and  $\beta_2 < 0$ , it holds that  $\xi(\bar{\pi}, 0) = -\frac{\rho_1 \bar{\pi}^{1-\beta_1}}{\beta_1}$  and  $\zeta(\infty, \underline{\pi}) = -\frac{\rho_2 \underline{\pi}^{1-\beta_2}}{\beta_2}$ , so we have the following results based on Theorem 1.

**Corollary 1** *It holds that  $\bar{\mathcal{E}}(\pi_0 | \bar{\pi}, 0) = \rho_1 \pi_0 - \frac{\rho_1 \bar{\pi}}{\beta_1} \left(\frac{\pi_0}{\bar{\pi}}\right)^{\beta_1}$  and  $\underline{\mathcal{E}}(\pi_0 | \bar{\pi}, 0) = \rho_2 \pi_0 - \frac{\delta_2 \rho_1 \bar{\pi}}{\beta_1} \left(\frac{\pi_0}{\bar{\pi}}\right)^{\beta_1}$ , while  $\bar{\mathcal{E}}(\pi_0 | \infty, \underline{\pi}) = \rho_1 \pi_0 - \frac{\delta_1 \rho_2 \underline{\pi}}{\beta_2} \left(\frac{\pi_0}{\underline{\pi}}\right)^{\beta_2}$  and  $\underline{\mathcal{E}}(\pi_0 | \infty, \underline{\pi}) = \rho_2 \pi_0 - \frac{\rho_2 \underline{\pi}}{\beta_2} \left(\frac{\pi_0}{\underline{\pi}}\right)^{\beta_2}$ . In addition,  $\bar{\mathcal{E}}(\pi_0 | \infty, 0) = \rho_1 \pi_0$  and  $\underline{\mathcal{E}}(\pi_0 | \infty, 0) = \rho_2 \pi_0$ .*

Henceforth, we focus on results that are useful for environments that are not perfectly competitive (such as those faced by an oligopolistic firm), so we let  $\underline{\pi} = 0$  and  $\bar{\pi} = \infty$  unless otherwise stated. In order to properly discount one-shot payoffs—such as investment costs or scrap values—attained when the process reaches a certain state, it is also necessary to derive the (conditionally) expected discounted value of a claim to a dollar at the first date at which the profit process hits a certain state  $\pi^*$  from above or below, starting from  $\pi_0$ . Such value is commonly referred to as "the expected discount factor," and we stick to this terminology in the remainder of the paper. The expected discount factor to be used when the dollar is achieved the first time the process reaches a certain state from below takes the following forms.

**Theorem 2** (i) *Suppose that the profit process is in a growth phase and that the current state is  $\pi_0$ . Then the expected discounted value of a claim to a dollar when the process first hits  $\pi^* > \pi_0$  from below is*

$$\overline{\varphi}_1(\pi_0, \pi^*) = \left(\frac{\pi_0}{\pi^*}\right)^{\beta_1}.$$

(ii) *Suppose that the profit process is in a decline phase and that the current state is  $\pi_0$ . Then the expected discounted value of a claim to a dollar when the process first hits  $\pi^* \geq \pi_0$  from below is*

$$\underline{\varphi}_1(\pi_0, \pi^*) = \delta_2 \left(\frac{\pi_0}{\pi^*}\right)^{\beta_1}.$$

Figure 2 provides a visual illustration of the problem. The profit process starts growing from  $\pi_0$  and after several cycles first hits  $\pi^*$  from below at the (random) first passage time  $T^*$ . Theorem 2 shows that the expected discounted value of a claim to a dollar attained at the random time  $T^*$  is given by  $\overline{\varphi}_1(\pi_0, \pi^*)$ .

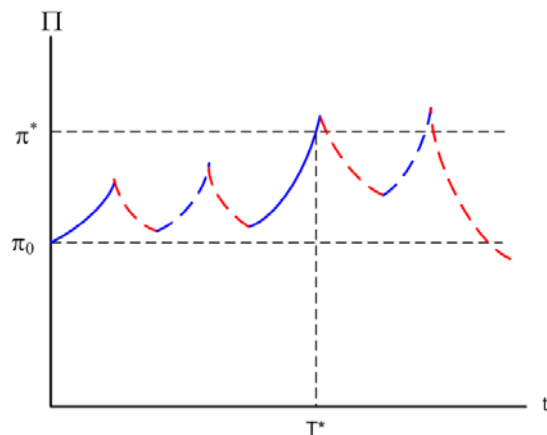


Figure 2: Expected discounted factor conditional upon the process growing at  $\pi_0$

Theorem 2 deals with expected discount factors that are useful when discounting one-shot payoffs that are attained when the process first hits a certain state *from below*. In

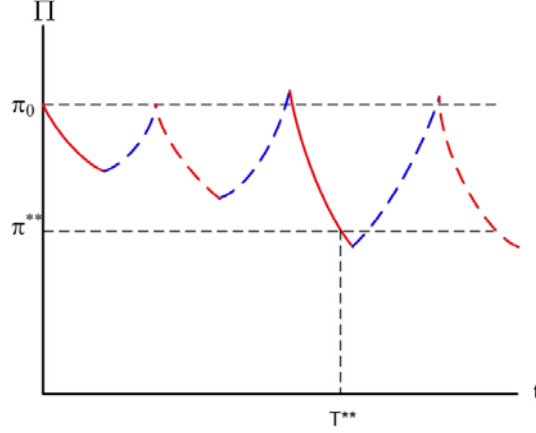


Figure 3: Expected discounted factor conditional upon the process declining at  $\pi_0$

turn, Theorem 3 deals with the discounting of one-shot payoffs achieved when the process first hits a certain level *from above*.

**Theorem 3** (i) *Suppose that the profit process is in a decline phase and that the current state is  $\pi_0$ . Then the expected discounted value of a claim to a dollar when the process first hits  $\pi^* < \pi_0$  from above is*

$$\underline{\varphi}_2(\pi_0, \pi^*) = \left(\frac{\pi_0}{\pi^*}\right)^{\beta_2}.$$

(ii) *Suppose that the profit process is in a growth phase and that the current state is  $\pi_0$ . Then the expected discounted value of a claim to a dollar when the process first hits  $\pi^* \leq \pi_0$  from above is*

$$\overline{\varphi}_2(\pi_0, \pi^*) = \delta_1 \left(\frac{\pi_0}{\pi^*}\right)^{\beta_2}.$$

This problem is represented in Figure 3, which depicts a situation in which the profit process is in a decline phase and the current state is  $\pi_0$ . It can be observed that, after several cycles, the process first hits  $\pi^{**}$  from above at the (random) first-passage time  $T^{**}$ . Theorem 3 shows that the expected discounted value of a claim to a dollar attained at time  $T^{**}$  is given by  $\underline{\varphi}_2(\pi_0, \pi^{**})$ .

We can draw a useful corollary from the previous two theorems.

**Corollary 2** (i) *Suppose that the profit process is in a growth phase and that the current state is  $\pi_0$ . Then the expected discounted value of a claim to a dollar when the process first hits  $\pi_0$  from above is  $\overline{\varphi}_2(\pi_0, \pi_0) = \delta_1 \in (0, 1)$ .*

(ii) *Suppose that the profit process is in a decline phase and that the current state is  $\pi_0$ . Then the expected discounted value of a claim to a dollar when the process first hits  $\pi_0$  from below is  $\underline{\varphi}_1(\pi_0, \pi_0) = \delta_2 \in (0, 1)$ .*

To conclude with our results in this section, notice that the expected stream of discounted profits derived in Theorem 1 is based on the hypothesis that an active firm never

becomes inactive, which is relevant for completely irreversible investment decisions. Sometimes, as when a firm foresees exiting at some random time in the future, it is also necessary to compute the (conditionally) expected stream of discounted profits harvested until a certain state is first hit by a process subject to no barriers. The following result deals with the expected discounted profit stream while transitioning from the current state  $\pi_0$  to another one denoted by  $\pi^*$ .

**Theorem 4** (i) *Suppose that the profit process is in a decline phase and that the current state is  $\pi_0$ . Then the expected stream of discounted profits while the process transitions from  $\pi_0$  until it first hits  $\pi^* < \pi_0$  from above is*

$$\underline{\mathcal{E}}(\pi_0, \pi^*) = \rho_2 \pi_0 \left[ 1 - \left( \frac{\pi_0}{\pi^*} \right)^{\beta_2 - 1} \right].$$

(ii) *Suppose that the profit process is in a growth phase and that the current state is  $\pi_0$ . Then the expected stream of discounted profits while the process transitions from  $\pi_0$  until it first hits  $\pi^* \leq \pi_0$  from above is*

$$\bar{\mathcal{E}}(\pi_0, \pi^*) = \rho_1 \pi_0 - \rho_2 \delta_1 \pi_0 \left( \frac{\pi_0}{\pi^*} \right)^{\beta_2 - 1}.$$

Figure 3 illustrates the situation faced by the firm when computing the expected stream of discounted profits while the process transitions from  $\pi_0$  until it first hits  $\pi^{**} \leq \pi_0$  from above, which happens at the first-passage time  $T^{**}$ . The formula for computing such expected payoff is given by  $\underline{\mathcal{E}}(\pi_0, \pi^{**})$ .

It is worth noting that the last statement in Corollary 1 also follows from Theorem 4, since we have that  $\underline{\mathcal{E}}(\pi_0, \pi^*) = \underline{\mathcal{E}}(\pi_0 | \infty, 0)$  and  $\bar{\mathcal{E}}(\pi_0, \pi^*) = \bar{\mathcal{E}}(\pi_0 | \infty, 0)$  for  $\pi^* = 0$ . This theorem is also helpful in drawing a simple but relevant result that can be quite useful when dealing with disinvestment decisions in a setting with few firms. Letting  $\gamma_1 \equiv \rho_1 - \rho_2 \delta_1$ , which (using the fact that  $\rho_1 = \frac{\rho_2(r + \lambda_2 - \alpha_2) - 1}{\lambda_2}$ ) can be rewritten as

$$\gamma_1 = \frac{\rho_2[\lambda_2(1 - \delta_1) + r - \alpha_2] - 1}{\lambda_2}, \quad (2)$$

we have that Theorem 4 leads to the following result.

**Corollary 3** *Suppose that the profit process is in a growth phase and that the current state is  $\pi_0$ . Then the expected stream of discounted profits while the process transitions from  $\pi_0$  until it first hits  $\pi_0$  from above is  $\bar{\mathcal{E}}(\pi_0, \pi_0) = \gamma_1 \pi_0$ .*

We present a final result that is related to Theorem 4 and is useful when dealing with investment timing games such as the one we consider in Section 5.

**Theorem 5** (i) *Suppose that the profit process is in a growth phase and that the current state is  $\pi_0$ . Then the expected stream of discounted profits while the process transitions from*

$\pi_0$  until it first hits  $\pi^* > \pi_0$  from below is

$$\bar{\mathcal{E}}'(\pi_0, \pi^*) = \rho_1 \pi_0 \left[ 1 - \left( \frac{\pi_0}{\pi^*} \right)^{\beta_1 - 1} \right].$$

(ii) Suppose that the profit process is in a decline phase and that the current state is  $\pi_0$ . Then the expected stream of discounted profits while the process transitions from  $\pi_0$  until it first hits  $\pi^* > \pi_0$  from below is

$$\underline{\mathcal{E}}'(\pi_0, \pi^*) = \rho_2 \pi_0 - \rho_1 \delta_2 \pi_0 \left( \frac{\pi_0}{\pi^*} \right)^{\beta_1 - 1}.$$

## 4 Entry and exit in competitive industries

The purpose of this section is to characterize the main properties of entry and exit behavior in a perfectly competitive industry in which demand perpetually alternates between growth and decline phases at uncertain times. In particular, we extend the results in Leahy (1993) to stochastically cyclical markets, and we show that the entry (exit) threshold of a competitive firm exactly coincides with that of a single firm that can optimally choose its optimal entry (exit) threshold. In so doing, we illustrate how the results developed in Section 3 make it straightforward to derive the entry/exit thresholds both for a single firm and one that faces perfect competition. Last but not least, this section highlights the deep and significant differences that arise at the predictive level between our framework and others that are widely used. More precisely, we show that positive entry and exit rates discontinuously fall to zero despite the stochastic process of the underlying state variable always has a continuous sample path. This does not happen when the stochastic process adopts the form of a Geometric Brownian Motion (Leahy 1993) because barriers are reflecting (see Dixit and Pindyck 1994, p. 83) rather than partially absorbing, as happens in our framework.

More specifically, this section deals with a perfectly competitive industry in which there are many firms that sell a homogeneous good and act as price-takers. We assume that any firm can operate a single unit of capacity whose investment cost is  $K > 0$  and whose redeployment value equals  $S \geq 0$ . Operating costs are set equal to zero, and the demand elasticity is large enough so that any active firm produces at full capacity. We suppose that the (inverse) market demand equals  $\pi = \Pi D(Q)$ , where  $\pi$  and  $Q$  respectively denote the market price and the number of active firms. Furthermore,  $D(\cdot)$  is a known decreasing function, whereas  $\Pi$  is assumed to follow the stochastic process introduced in Section 2.

The stationarity of the stochastic process that determines the (short-run) market price at any time for a given  $Q$  implies that there will be a threshold  $\bar{\pi} \leq \infty$  that will trigger entry by inactive firms as long as price is high enough (i.e.,  $\Pi D(Q) \geq \bar{\pi}$ ). Similarly, there will be a threshold  $\underline{\pi} \geq 0$  that will trigger exit by active firms as long as price is low enough (i.e.,  $\Pi D(Q) \leq \underline{\pi}$ ). (The extent of entry and exit for a given number of (in)active firms will be

given by the specific properties of  $D(\cdot)$  so as to keep the price at a constant level.) Noting that, given our assumptions, a firm's profit coincides with the short-run market price, we have that the profit variable  $\pi$  inherits exactly the same stochastic behavior of variable  $\Pi$  except for two aspects. On the one hand, if  $\Pi$  is going through a growth phase, then entry by other firms implies that  $\pi$  stays constant at level  $\bar{\pi} \in (0, \infty]$  if the price ever reaches such state, and it leaves  $\bar{\pi}$  only once the process for  $\Pi$  starts declining. On the other hand, if  $\Pi$  is going through a decline phase, then exit by other firms implies that  $\pi$  stays constant at level  $\underline{\pi} \in [0, \infty)$  if the price ever reaches such state, and it leaves  $\underline{\pi}$  only once the process for  $\Pi$  starts growing. As a result, the variable  $\pi$  follows the same process as that of  $\Pi$  except that it can never go higher than  $\bar{\pi}$  and lower than  $\underline{\pi}$ .<sup>9</sup> If all firms have rational expectations about  $\Pi$  and other firms' entry/exit rules, then these properties of the stochastic variable  $\pi$  are known, and in addition, all firms know that enough entry (exit) takes place so as to keep  $\pi$  at the upper (lower) barrier while  $\Pi$  is growing (declining).

Theorem 1 is particularly powerful in this case in which we have to find out the values of  $\bar{\pi}$  and  $\underline{\pi}$ . Recall that  $\bar{\mathcal{E}}(\bar{\pi} | \bar{\pi}, \underline{\pi})$  represents the stream of discounted profits expected by an active firm if  $\pi$  is currently in growth at level  $\bar{\pi}$ , taking into account that  $\pi$  can never go higher than  $\bar{\pi}$  and lower than  $\underline{\pi}$ . Similarly,  $\underline{\mathcal{E}}(\underline{\pi} | \bar{\pi}, \underline{\pi})$  represents the stream of discounted profits expected by an active firm if  $\pi$  is currently in decline at level  $\underline{\pi}$ , taking into account that  $\pi$  can never go higher than  $\bar{\pi}$  and lower than  $\underline{\pi}$ . Then it is clear that free entry and exit imply that the values of  $\bar{\pi}$  and  $\underline{\pi}$  in equilibrium must be such that

$$\bar{\mathcal{E}}(\bar{\pi} | \bar{\pi}, \underline{\pi}) = K \tag{3}$$

and

$$\underline{\mathcal{E}}(\underline{\pi} | \bar{\pi}, \underline{\pi}) = S. \tag{4}$$

#### 4.1 Entry in competitive industries under uncertainty

When  $S = 0$  and  $K < \infty$ , it is clear that  $\underline{\pi} = 0$ , so Corollary 1 applied on (3) implies that

$$\rho_1 \bar{\pi} - \frac{\rho_1 \bar{\pi}}{\beta_1} \left( \frac{\bar{\pi}}{\bar{\pi}} \right)^{\beta_1} = K,$$

that is,

$$\bar{\pi} = \pi_E^* \equiv \frac{\beta_1 K}{\rho_1 (\beta_1 - 1)}.$$

As the following result shows, the competitive entry threshold coincides with that of a single firm that can optimally choose the profit level at which to enter.

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<sup>9</sup>The preceding discussion relies on the implicit assumption that entry never takes place while  $\Pi$  is in a decline phase, whereas exit never takes place while  $\Pi$  is in a growth phase. This intuitive result can be shown to hold in any dynamic equilibrium of a competitive industry. (See for instance Lemma 1 in Appendix B for the single firm case.)

**Proposition 1** *Suppose that  $S = 0$  and  $K < \infty$ . Then a competitive firm that is inactive enters whenever the market first reaches state  $\pi_E^* = \frac{\beta_1 K}{\rho_1(\beta_1 - 1)}$  from below. This entry threshold is identical to the one that is chosen optimally by a single firm that ignores the effect that other firms exert on the price process.*

**Proof.** It only remains to show that a single firm that acts as if  $\bar{\pi} = \infty$  would choose to enter the first time the stochastic process governing profits hit state  $\pi_E^*$  from below. (The stochastic process of Section 2 is Markovian and homogeneous, so a single firm's optimal investment rule for each phase of a cycle is stationary.) As Lemma 1 in Appendix B shows, a single firm would choose to enter only if the market is growing. Therefore, Lemma 1 implies that it suffices to pay attention to phases in which the market is in growth when solving for a single firm's optimal investment threshold. Thus, suppose that the market is currently growing and denote the current state of the market by  $\pi_0$ . Because the stochastic process is Markovian, the firm simply chooses a threshold  $\pi_E$  such that it enters the market the first time the process hits such threshold from below (given that  $\bar{\pi} = \infty$ ). Hence, the firm solves the following optimization problem:

$$\max_{\pi_E} \bar{V}_E(\pi_E|\pi_0) = [\bar{\mathcal{E}}(\pi_E, 0) - K] \bar{\varphi}_1(\pi_0, \pi_E) = (\rho_1 \pi_E - K) \left( \frac{\pi_0}{\pi_E} \right)^{\beta_1}, \quad (5)$$

where the last equality makes use of Theorems 2 and 4. Thus, a single firm that believes that  $\bar{\pi} = \infty$  achieves an expected net payoff of  $\bar{\mathcal{E}}(\pi_E, 0) - K$  the first time the market reaches level  $\pi_E$  starting from state  $\pi_0$ . For this reason,  $\bar{\varphi}_1(\pi_0, \pi_E)$  is the expected discount factor that must be used when discounting this payoff on the state space.

Because  $\bar{V}_E(\pi_E|\pi_0)$  is strictly quasi-concave and

$$\frac{\partial \bar{V}_E(\pi_E|\pi_0)}{\partial \pi_E} = \left[ \frac{\rho_1(1 - \beta_1)\pi_E + \beta_1 K}{\pi_E} \right] \left( \frac{\pi_0}{\pi_E} \right)^{\beta_1},$$

easy manipulations lead to the optimal investment threshold:

$$\pi_E^* = \frac{\beta_1 K}{\rho_1(\beta_1 - 1)}.$$

■

The result that the competitive entry threshold is exactly equal to that of a single firm was first derived by Leahy (1993) for diffusion processes (such as the GBM). Proposition 1 shows that this result extends to stochastically cyclical markets, albeit in quite a straightforward manner (i.e., as a direct application of Corollary 1). As discussed by Leahy (1993), competition reduces the value of the option to build one unit of capacity by reducing the value of installed capacity. These two effects cancel out each other in equilibrium, which explains why the competitive entry threshold coincides with that of a single firm that ignores the effect that other firms exert on the stochastic process governing profits.

That the competitive entry threshold coincides with that of a single firm for a stochastic process that need not follow a diffusion was already conjectured by Leahy (1993). However, the predictions regarding entry rates that can be drawn from a stochastically cyclical process are very different from those derived from an Ito process. Despite both types of stochastic processes exhibit continuous sample paths, in a stochastically cyclical market it holds that the value of a competitive firm if it becomes active experiences jumps upwards (downwards) whenever a growth (decline) phase starts. There is "informational overshooting" in that a firm continuously updates its beliefs about the future stopping time as the market goes through a growth (decline) phase whose length is random, but it discontinuously changes its belief when the end of the phase suddenly arrives. Hence, entry rates must discretely drop to zero if  $\pi = \bar{\pi}$  and all of a sudden a decline phase that makes the market less profitable in expected terms starts. This discontinuity result that does not obtain for diffusion processes can be stated as follows.

**Proposition 2** *Suppose that  $S = 0$  and  $K < \infty$ . Then positive entry rates must discontinuously fall to zero owing to the start of a decline phase.*

**Proof.** It suffices to show that  $\bar{\mathcal{E}}(\pi_0 | \bar{\pi}, 0) > \underline{\mathcal{E}}(\pi_0 | \bar{\pi}, 0)$  for all  $\pi_0$  and  $\bar{\pi}$  such that  $\pi_0 \leq \bar{\pi}$ . Using the facts that  $\delta_2 = (r + \lambda_1 - \alpha_1 \beta_1) / \lambda_1 \in (0, 1)$  and  $1 + \lambda_1 \rho_2 = (\lambda_1 + r - \alpha_1) \rho_1$  (by expression (30) in Appendix B), we have that

$$\begin{aligned} \bar{\mathcal{E}}(\pi_0 | \bar{\pi}, 0) - \underline{\mathcal{E}}(\pi_0 | \bar{\pi}, 0) &= (\rho_1 - \rho_2)\pi_0 - \frac{(1 - \delta_2)\rho_1\pi_0}{\beta_1} \left(\frac{\pi_0}{\bar{\pi}}\right)^{\beta_1 - 1} \\ &\geq \left(\rho_1 - \rho_2 - \frac{(1 - \delta_2)\rho_1}{\beta_1}\right)\pi_0 \\ &= \left(\lambda_1\rho_1 - \lambda_1\rho_2 - \alpha_1\rho_1 + \frac{r\rho_1}{\beta_1}\right)\frac{\pi_0}{\lambda_1} \\ &= \left(\frac{\beta_1}{(\beta_1 - 1)\rho_1} - r\right)\frac{(\beta_1 - 1)\rho_1\pi_0}{\lambda_1\beta_1}, \end{aligned}$$

so the result follows because  $\frac{\beta_1}{(\beta_1 - 1)\rho_1} > r$  (as shown in the proof of Lemma 1 in Appendix B). ■

## 4.2 Exit in competitive industries under uncertainty

When  $S > 0$  and  $K = \infty$ , we have that  $\bar{\pi} = \infty$ , so Corollary 1 applied on (4) implies that

$$\rho_2\bar{\pi} - \frac{\rho_2\bar{\pi}}{\beta_2} \left(\frac{\bar{\pi}}{\bar{\pi}}\right)^{\beta_2} = S,$$

so

$$\bar{\pi} = \pi_X^* \equiv \frac{\beta_2 S}{\rho_2(\beta_2 - 1)}.$$

Again, this is an identical result to that obtained when price follows a diffusion process: the competitive exit threshold coincides with that of a single firm that can optimally choose the profit level at which to exit (as in Leahy 1993).

**Proposition 3** *Suppose that  $S > 0$  and  $K = \infty$ . Then a competitive firm that is active exits whenever the market first reaches state  $\pi_X^* = \frac{\beta_2 S}{\rho_2(\beta_2 - 1)}$  from above. This exit threshold is identical to the one that is chosen optimally by a single firm that ignores the effect that other firms exert on the price process.*

**Proof.** We simply have to show that a single firm that acts as if  $\underline{\pi} = 0$  would choose to exit the first time the stochastic process governing profits hit state  $\pi_X^*$  from above. Not surprisingly, and contrary to the entry problem, it can be shown that exit takes place only if the market is declining,<sup>10</sup> so it suffices to pay attention to phases in which the market is in decline when solving for a single firm's optimal disinvestment threshold. The homogeneity of the (Markov) stochastic process implies that the firm simply chooses a threshold  $\pi_X$  such that the firm exits the market the first time the process hits such threshold from above. Formally, the firm solves the following problem:

$$\begin{aligned} \max_{\pi_X} \underline{V}_X(\pi_X | \pi_0) &= \underline{\mathcal{E}}(\pi_0, \pi_X) + S \underline{\varphi}_2(\pi_0, \pi_X) \\ &= \rho_2 \pi_0 \left[ 1 - \left( \frac{\pi_0}{\pi_X} \right)^{\beta_2 - 1} \right] + S \left( \frac{\pi_0}{\pi_X} \right)^{\beta_2}. \end{aligned}$$

Thus, starting from a level  $\pi_0$  at which the market is declining, the firm collects a discounted stream of profits until state  $\pi_X < \pi_0$  is first reached (from above). The value of such discounted profit stream while transitioning from  $\pi_0$  to  $\pi_X$  is

$$\underline{\mathcal{E}}(\pi_0, \pi_X) = \rho_2 \pi_0 \left[ 1 - (\pi_0 / \pi_X)^{\beta_2 - 1} \right],$$

as shown by Theorem 4. However, there is an additional source of value, since the firm seizes the outside option, whose value is  $S$ , when state  $\pi_X$  is first reached. The proper expected discount factor, given that the market is in a decline phase, is

$$\underline{\varphi}_2(\pi_0, \pi_X) = (\pi_0 / \pi_X)^{\beta_2},$$

as shown by Theorem 3.

Noticing that  $\underline{V}_X(\pi_X | \pi_0)$  is strictly quasi-concave and

$$\frac{\partial \underline{V}_X(\pi_X | \pi_0)}{\partial \pi_X} = \left[ \frac{\rho_2(\beta_2 - 1)\pi_X - \beta_2 S}{\pi_X} \right] \left( \frac{\pi_0}{\pi_X} \right)^{\beta_2},$$

---

<sup>10</sup>The argument is identical in spirit to the one used for entry by a sole firm, and it makes use of the fact that  $r > \frac{\beta_2}{(\beta_2 - 1)\rho_2}$ , which is shown in the proof of Proposition 4.

it is easy to prove that the optimal disinvestment threshold for a single firm is as follows:

$$\pi_X^* \equiv \frac{\beta_2 S}{\rho_2(\beta_2 - 1)}.$$

■

As in the entry case, the prediction regarding the pattern of competitive exit is in stark contrast to the one derived based on diffusion processes. Because informational overshooting implies that the value of an active firm experiences jumps upwards (downwards) whenever a growth (decline) phase starts, exit rates must discretely drop to zero if  $\pi = \underline{\pi}$  and a growth phase that makes the market more profitable in expected terms starts all of a sudden. The following proposition summarizes this result.

**Proposition 4** *Suppose that  $S > 0$  and  $K = \infty$ . Then positive exit rates must discontinuously fall to zero owing to the start of a growth phase.*

**Proof.** It suffices to show that  $\bar{\mathcal{E}}(\pi_0 | \infty, \underline{\pi}) > \underline{\mathcal{E}}(\pi_0 | \infty, \underline{\pi})$  for all  $\pi_0$  and  $\underline{\pi}$  such that  $\pi_0 \geq \underline{\pi}$ . The proof of this result parallels that of  $\bar{\mathcal{E}}(\pi_0 | \bar{\pi}, 0) > \underline{\mathcal{E}}(\pi_0 | \bar{\pi}, 0)$  in Proposition 2, except that it uses the facts that  $\delta_1 = (r + \lambda_2 - \alpha_2 \beta_2) / \lambda_2 \in (0, 1)$ ,  $1 + \lambda_2 \rho_1 = (\lambda_2 + r - \alpha_2) \rho_2$  and  $r > \frac{\beta_2}{(\beta_2 - 1) \rho_2}$ . To prove that  $r > \frac{\beta_2}{(\beta_2 - 1) \rho_2}$ , notice that

$$(r + \lambda_1 - \alpha_1)(r + \lambda_2 - \alpha_2) - \lambda_1 \lambda_2 = \alpha_1 \alpha_2 (1 - \beta_1)(1 - \beta_2),$$

so we have that

$$\rho_2 = \frac{1}{r} \left[ \frac{r(r + \lambda_1 + \lambda_2) - r \alpha_1}{\alpha_1 \alpha_2 (1 - \beta_1)(1 - \beta_2)} \right] = \frac{1}{r} \left[ \frac{\beta_1 \beta_2 - \frac{r}{\alpha_2}}{(1 - \beta_1)(1 - \beta_2)} \right],$$

where the last equality follows because it holds that  $\alpha_1 \alpha_2 \beta_1 \beta_2 = r(r + \lambda_1 + \lambda_2)$ . As a result, using the fact that  $\alpha_2 \beta_2 - r = \lambda_2(1 - \delta_1)$  (see definition of  $\delta_1$  in Theorem 3) yields that

$$\frac{(\beta_2 - 1) \rho_2 r - \beta_2}{(\beta_2 - 1) \rho_2} = \frac{\beta_1 \beta_2 - \frac{r}{\alpha_2} - \beta_2(\beta_1 - 1)}{\rho_2(\beta_1 - 1)(\beta_2 - 1)} = \frac{\lambda_2(1 - \delta_1)}{\alpha_2 \rho_2(\beta_1 - 1)(\beta_2 - 1)},$$

whence it is clear that  $\frac{\rho_2 r (\beta_2 - 1) - \beta_2}{\rho_2 (\beta_2 - 1)} = r - \frac{\beta_2}{(\beta_2 - 1) \rho_2} > 0$ , since  $\delta_1 \in (0, 1)$ ,  $\beta_1 > 1$  and  $\beta_2 < 0$ . ■

### 4.3 Entry and exit in competitive industries under uncertainty

When it holds that  $0 < S < K < \infty$ ,<sup>11</sup> it is no longer possible to provide a closed-form solution to  $\bar{\pi}$  and  $\underline{\pi}$  by solving the system of equations (3) and (4), but it is easy to show

<sup>11</sup>It is assumed that  $S < K$  to prevent "money pumps."

that  $\pi^* \equiv \frac{\bar{\pi}}{\underline{\pi}}$  solves the following equation:

$$\pi^* \left( \frac{\frac{\delta_1 \rho_2 (\pi^*)^{\beta_2 - 1} - \rho_1}{\beta_1 (1 - \delta_1 \delta_2 (\pi^*)^{\beta_2 - \beta_1})} + \frac{\delta_1 \delta_2 \rho_1 (\pi^*)^{\beta_2 - \beta_1} - \delta_1 \rho_2 (\pi^*)^{\beta_2 - 1}}{\beta_2 (1 - \delta_1 \delta_2 (\pi^*)^{\beta_2 - \beta_1})} + \rho_1}{\frac{\delta_1 \delta_2 \rho_2 (\pi^*)^{\beta_2 - \beta_1} - \delta_2 \rho_1 (\pi^*)^{1 - \beta_1}}{\beta_1 (1 - \delta_1 \delta_2 (\pi^*)^{\beta_2 - \beta_1})} + \frac{\delta_2 \rho_1 (\pi^*)^{1 - \beta_1} - \rho_2}{\beta_2 (1 - \delta_1 \delta_2 (\pi^*)^{\beta_2 - \beta_1})} + \rho_2} \right) = \frac{K}{S}.$$

Numerical solutions can be easily obtained, although the entry and exit thresholds can be analytically shown to be identical to those selected by a single firm that can choose at what states to enter and exit. In addition, upward (downward) jumps in firm value also happen whenever a growth (decline) phase starts. As a result, entry and exit rates that are positive exhibit sudden drops to zero whenever a growth and decline phase ends, so entry and exit waves may be observed to suddenly vanish, even if the resulting entry and exit paths are continuous. This contrasts with settings based on diffusion processes in which entry (exit) is intermittent because the upper (lower) barrier is reflecting.<sup>12</sup>

Figure 4 depicts the sample path for short-run price  $\pi$  (see top graph) that is induced by a particular sample path for the demand shock  $\Pi$ , and also depicts the sample path for the number of active firms (see bottom graph).<sup>13</sup> As shown in the figure, positive entry rates discontinuously fall to zero as soon as the market starts declining, whereas positive exit rates discontinuously fall to zero as soon as the market starts growing.

## 5 Entry in duopolistic industries under uncertainty

We now illustrate how the results developed in Section 3 make it very simple to analyze investment timing games under uncertainty. Although the game we consider is quite standard and hence we shall be succinct, it is worth mentioning that our results can still be applied when dealing with more complex settings involving entry and exit timing over a stochastic industry life cycle (see e.g. Ruiz-Aliseda 2008). Thus, we consider a new market in which two ex ante identical firms can each decide at which state to invest in a single unit of capacity at a sunk cost  $K > 0$ . If operating costs are zero, then a firm's profit flow if there are  $i \in \{1, 2\}$  active firms in the market is  $\Pi(i) = \pi D(i)$ , where  $\pi$  follows the stochastic process introduced in Section 2 (subject to no barriers), and  $D(\cdot)$  is a downward sloping demand function that is elastic enough to ensure full capacity utilization when two firms are active.

To find the Markov perfect equilibrium outcome of the game, we start by analyzing those

<sup>12</sup>The difference is that in our case the upper (lower) barrier is absorbing for a random length of time, which implies that there will be a continuous flow of entry (exit) until the demand shock starts declining (growing).

<sup>13</sup>The assumptions made to perform the simulation that we exhibit are as follows:  $\pi = \Pi Q^{-1}$ ,  $\Pi(0) = Q(0) = 70$ ,  $\alpha_1 = 0.1$ ,  $\alpha_2 = -0.1$ ,  $\lambda_1 = \lambda_2 = r = 0.1$ ,  $K = 20$ , and  $S = 10$ . These assumptions imply that inactive firms enter whenever the price is at level  $\bar{\pi} = 2.2$ , while active firms exit whenever firm is at level  $\underline{\pi} = 0.86$ .

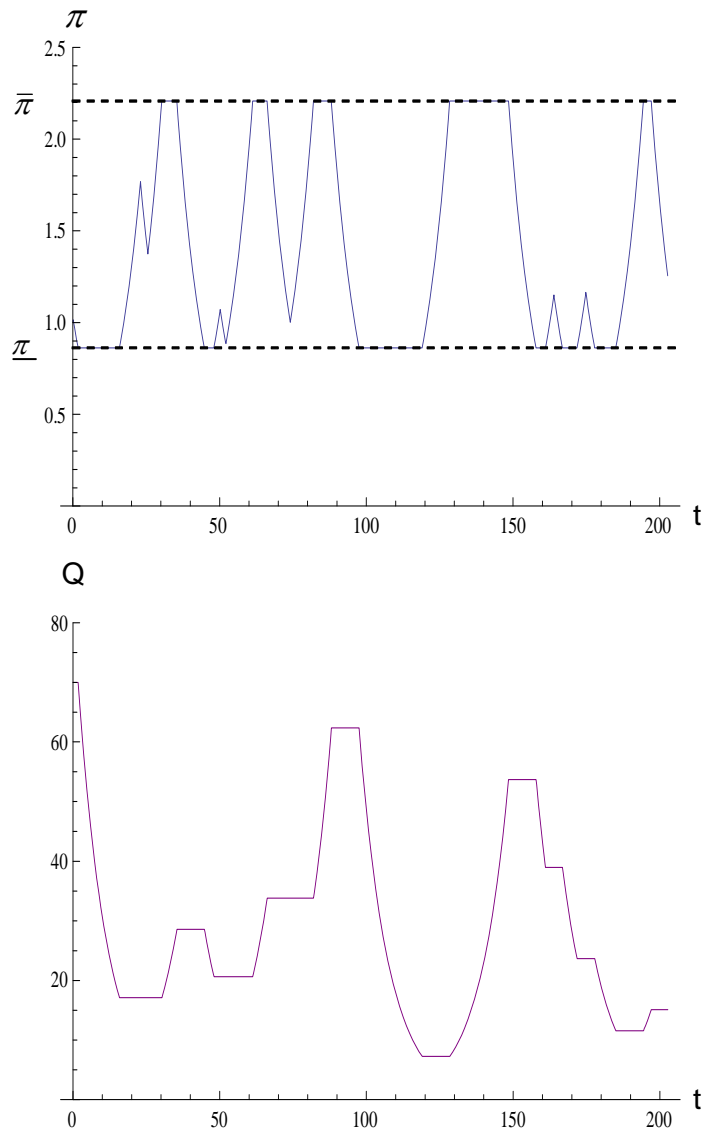


Figure 4: Sample paths for short-run price, net entry and number of active firms

subgames in which one firm has already invested, taking into account that we always focus on states at which the market is growing because no firm will ever invest during a market decline phase (see Lemma 1 in Appendix 6 for a proof). If a firm is already active, then its competitor moving second will find it optimal to invest at state  $\pi_f^* = \arg \max_{\pi_f} V^f(\pi_f | \pi_0)$ , where  $V^f(\pi_f | \pi_0) = (\rho_1 \pi_f D(2) - K) (\pi_0 / \pi_f)^{\beta_1}$  is the payoff to investing second at state  $\pi_f$  given that the current state is  $\pi_0$ . Using the strict quasi-concavity of  $V^f(\pi_f | \pi_0)$ , it is easy to show that  $\pi_f^* = \frac{\beta_1 K}{\rho_1 (\beta_1 - 1) D(2)}$ , so a firm that ends up investing in the second place will enter at state  $\pi_f^*$  in order to attain an expected payoff equal to  $V^f(\pi_f^* | \pi_0)$ .

We now turn to the payoff to a firm if it invests in the first place, taking into account at which state the competitor will find it optimal to invest given that it moves second. Using Theorems 2, 4 and 5, we have that the payoff to moving first at state  $\pi_l$  is

$$V^l(\pi_l | \pi_0) = \left[ \rho_1 \pi_l D(1) \left( 1 - \left( \frac{\pi_l}{\pi_f^*} \right)^{\beta_1 - 1} \right) - K \right] \left( \frac{\pi_0}{\pi_l} \right)^{\beta_1} + \rho_1 \pi_f^* D(2) \left( \frac{\pi_0}{\pi_f^*} \right)^{\beta_1}.$$

Note that

$$\frac{\partial V^l(\pi_l | \pi_0)}{\partial \pi_l} = \left( \rho_1 D(1) (1 - \beta_1) + \frac{\beta_1 K}{\pi_l} \right) \left( \frac{\pi_0}{\pi_l} \right)^{\beta_1},$$

so  $\pi_l^* = \frac{\beta_1 K}{\rho_1 (\beta_1 - 1) D(1)} < \pi_f^*$  is the global maximizer of  $V^l(\pi_l | \pi_0)$  (since  $\frac{\partial^2 V^l(\pi_l^* | \pi_0)}{\partial \pi_l^2} = -\beta_1 K (\pi_l^*)^{-2} (\pi_0 / \pi_l^*)^{\beta_1} < 0$ ).

Because  $V^l(\pi_l^* | \pi_0) > V^l(\pi_f^* | \pi_0) = V^f(\pi_f^* | \pi_0)$ , the facts that  $V^l(\pi_l | 0) = 0$  and  $V^f(\pi_f^* | \pi_0) > V^f(\infty | \pi_0) = 0$  imply that we can find  $\pi_0$  small enough so that  $V^l(\pi_0 | \pi_0) < V^f(\pi_f^* | \pi_0)$  (because of the continuity of  $V^l(\pi_l | \pi_0)$ ). So let  $\pi_0$  be small enough to rule out immediate investment as an equilibrium outcome, and note that the properties of  $V^l(\pi_l | \pi_0)$  imply that there exists a unique state  $\pi_P^* \in (\pi_0, \pi_f^*)$  for which  $V^l(\pi_P^* | \pi_0) = V^f(\pi_f^* | \pi_0)$ . Based on Fudenberg and Tirole's (1985) rent equalization principle, we have that the first-mover in equilibrium must invest at that state, with the competitor investing at  $\pi_f^*$ . With symmetric firms, expected payoffs are identical even if firms invest at different states. The first mover enjoys the advantage of earning monopoly profits for some additional period of time, whereas the second mover enjoys the full option value of waiting to invest, unlike the preempting firm. In a (Markov) perfect equilibrium, both of these advantages must exactly offset each other so that a firm expects earning the same payoff regardless of whether it moves first or second.

## 6 Conclusion

This paper has modeled the stochastic evolution of markets that exhibit a somewhat unpredictable cyclical behavior in that flow profit alternates between growth and decline phases

at random times. We have derived several results that make the analysis of entry and exit decisions in these environments quite straightforward for the cases of monopolistic, oligopolistic, and perfectly competitive firms. In addition, we have characterized the entry and exit dynamics displayed by a competitive equilibrium in this type of markets. In contrast with empirical predictions based on diffusion processes, we have shown that, even though the sample path of profits is continuous, positive entry (exit) rates in a competitive industry drastically fall from positive levels to zero owing to the end of a growth (decline) phase owing to informational overshooting, an aspect missing in settings using diffusion processes.

There are at least a couple of aspects that are worthwhile emphasizing about our framework and that may prove to be useful for future work on the properties of irreversible (dis)investment in stochastically cyclical markets. On the one hand, we would like to remark that our setting does not require making use of the heavy mathematical apparatus traditionally employed by the modern literature on irreversible investment under uncertainty. For instance, direct applications of our theorems allow us to avoid using stochastic calculus and even dynamic programming techniques when solving (dis)investment models that involve lumpiness. As we have shown, (dis)investment timing problems can be directly formulated and solved using algebra and ordinary calculus. Indeed, it is relatively simple to solve the problem faced by a single firm that must choose when to enter and when to exit, as in Dixit (1989).

Another aspect worthwhile pointing out is that the model can be enriched in order to improve its explanatory power, possibly at the expense of analytical tractability. Thus, when a switching date is realized, the sample path of the process that governs profit evolution is assumed to change to a different but known growth rate. Assuming that the growth rate is random may lead to possibly different (dis)investment dynamics given a market evolution that need not be stochastically cyclical despite the regime shifts. This may prove fruitful for the construction of random processes that better represent the stochastic dynamics of a variety of economic and financial variables. Both theoretical and empirical work may benefit from pursuing this promising research avenue.

## Appendix A

**Proof of Theorem 1.** Given our (memoryless) assumptions on the independent random variables involved,  $\bar{\mathcal{E}}(\pi_0 | \bar{\pi}, \underline{\pi})$  and  $\underline{\mathcal{E}}(\pi_0 | \bar{\pi}, \underline{\pi})$  can be related as follows:

$$\begin{aligned} \bar{\mathcal{E}}(\pi_0 | \bar{\pi}, \underline{\pi}) &= \int_{\frac{1}{\alpha_1} \ln(\bar{\pi}/\pi_0)}^{\infty} \lambda_1 e^{-\lambda_1 \tau_1} \left( \int_0^{\frac{1}{\alpha_1} \ln(\bar{\pi}/\pi_0)} \pi_0 e^{\alpha_1 s} e^{-rs} ds + \bar{\mathcal{E}}(\bar{\pi} | \bar{\pi}, \underline{\pi}) e^{-\frac{r}{\alpha_1} \ln(\bar{\pi}/\pi_0)} \right) d\tau_1 + \\ &\int_0^{\frac{1}{\alpha_1} \ln(\bar{\pi}/\pi_0)} \lambda_1 e^{-\lambda_1 \tau_1} \left( \int_0^{\tau_1} \pi_0 e^{\alpha_1 s} e^{-rs} ds + \underline{\mathcal{E}}(\pi_0 e^{\alpha_1 \tau_1} | \bar{\pi}, \underline{\pi}) e^{-r\tau_1} \right) d\tau_1 \end{aligned} \quad (6)$$

and

$$\begin{aligned} \underline{\mathcal{E}}(\pi_0 | \bar{\pi}, \underline{\pi}) &= \int_{\frac{1}{\alpha_2} \ln(\underline{\pi}/\pi_0)}^{\infty} \lambda_2 e^{-\lambda_2 \tau_2} \left( \int_0^{\frac{1}{\alpha_2} \ln(\underline{\pi}/\pi_0)} \pi_0 e^{\alpha_2 s} e^{-rs} ds + \underline{\mathcal{E}}(\underline{\pi} | \bar{\pi}, \underline{\pi}) e^{-\frac{r}{\alpha_2} \ln(\underline{\pi}/\pi_0)} \right) d\tau_2 + \\ &\int_0^{\frac{1}{\alpha_2} \ln(\underline{\pi}/\pi_0)} \lambda_2 e^{-\lambda_2 \tau_2} \left( \int_0^{\tau_2} \pi_0 e^{\alpha_2 s} e^{-rs} ds + \bar{\mathcal{E}}(\pi_0 e^{\alpha_2 \tau_2} | \bar{\pi}, \underline{\pi}) e^{-r\tau_2} \right) d\tau_2. \end{aligned} \quad (7)$$

To see how  $\bar{\mathcal{E}}(\pi_0 | \bar{\pi}, \underline{\pi})$  arises, note that there is some chance that the process will hit the barrier  $\bar{\pi}$  during the current growth phase, at  $\tau_1 = \frac{\ln(\bar{\pi}/\pi_0)}{\alpha_1}$ . In such a case, the firm harvests a stream of profits that must be discounted back to the current date, together with an asset worth  $\bar{\mathcal{E}}(\bar{\pi} | \bar{\pi}, \underline{\pi}) e^{-\frac{r}{\alpha_1} \ln(\bar{\pi}/\pi_0)}$ . However, if the process starts declining at some level lower than  $\bar{\pi}$ , then the firm gets a stream of discounted profits until the switching date  $\tau_1$ , together with an asset worth  $\underline{\mathcal{E}}(\pi_0 e^{\alpha_1 \tau_1} | \bar{\pi}, \underline{\pi}) e^{-r\tau_1}$ . Regarding  $\underline{\mathcal{E}}(\pi_0 | \bar{\pi}, \underline{\pi})$ , there is some chance that the process will hit the barrier  $\underline{\pi}$  during the current decline phase (at  $\tau_2 = \frac{\ln(\underline{\pi}/\pi_0)}{\alpha_2}$ ), so in such a case the firm harvests a stream of profits that must be discounted back to the current date, together with an asset worth  $\underline{\mathcal{E}}(\underline{\pi} | \bar{\pi}, \underline{\pi}) e^{-\frac{r}{\alpha_2} \ln(\underline{\pi}/\pi_0)}$ . However, if the process starts growing at some level higher than  $\underline{\pi}$ , then the firm gets a stream of discounted profits until the switching date  $\tau_2$ , together with an asset worth  $\bar{\mathcal{E}}(\pi_0 e^{\alpha_2 \tau_2} | \bar{\pi}, \underline{\pi}) e^{-r\tau_2}$ .

To solve the system of functional equations that consists of (6) and (7), guess that  $\bar{\mathcal{E}}(\pi_0 | \bar{\pi}, \underline{\pi}) = C\pi_0^{\mu_1} + F\pi_0^{\mu_2} + M\pi_0$  and  $\underline{\mathcal{E}}(\pi_0 | \bar{\pi}, \underline{\pi}) = H\pi_0^{\mu_1} + D\pi_0^{\mu_2} + N\pi_0$  (where  $\mu_1$  and  $\mu_2$  are constants to be found out, whereas  $C, F, M, H, D$  and  $N$  do not depend on  $\pi_0$  although any may depend on  $\bar{\pi}$  or  $\underline{\pi}$ ).

Plugging the assumed functional forms of  $\underline{\mathcal{E}}(\pi_0 | \bar{\pi}, \underline{\pi})$  and  $\bar{\mathcal{E}}(\bar{\pi} | \bar{\pi}, \underline{\pi})$  into (6) and per-

forming some manipulations yields:

$$\begin{aligned}
\bar{\mathcal{E}}(\pi_0 | \bar{\pi}, \underline{\pi}) &= \int_{\frac{1}{\alpha_1} \ln(\bar{\pi}/\pi_0)}^{\infty} \lambda_1 e^{-\lambda_1 \tau_1} \left( \int_0^{\frac{1}{\alpha_1} \ln(\bar{\pi}/\pi_0)} \pi_0 e^{\alpha_1 s} e^{-rs} ds + \bar{\mathcal{E}}(\bar{\pi} | \bar{\pi}, \underline{\pi}) e^{-\frac{r}{\alpha_1} \ln(\bar{\pi}/\pi_0)} \right) d\tau_1 + \\
&\int_0^{\frac{1}{\alpha_1} \ln(\bar{\pi}/\pi_0)} \lambda_1 e^{-\lambda_1 \tau_1} \left( \int_0^{\tau_1} \pi_0 e^{\alpha_1 s} e^{-rs} ds + \underline{\mathcal{E}}(\pi_0 e^{\alpha_1 \tau_1} | \bar{\pi}, \underline{\pi}) e^{-r\tau_1} \right) d\tau_1 \\
&= \frac{\pi_0 \left( \left( \frac{\pi_0}{\bar{\pi}} \right)^{\frac{\lambda_1}{\alpha_1}} - \left( \frac{\pi_0}{\bar{\pi}} \right)^{\frac{r+\lambda_1-\alpha_1}{\alpha_1}} \right)}{r-\alpha_1} + \bar{\mathcal{E}}(\bar{\pi} | \bar{\pi}, \underline{\pi}) \left( \frac{\pi_0}{\bar{\pi}} \right)^{\frac{r+\lambda_1}{\alpha_1}} + \\
&\frac{\pi_0 \left( 1 - \left( \frac{\pi_0}{\bar{\pi}} \right)^{\frac{\lambda_1}{\alpha_1}} \right)}{r-\alpha_1} - \frac{\lambda_1 \pi_0 \left( 1 - \left( \frac{\pi_0}{\bar{\pi}} \right)^{\frac{r+\lambda_1-\alpha_1}{\alpha_1}} \right)}{(r-\alpha_1)(r+\lambda_1-\alpha_1)} + \frac{\lambda_1 H \pi_0^{\mu_1} \left( 1 - \left( \frac{\pi_0}{\bar{\pi}} \right)^{\frac{r+\lambda_1-\alpha_1\mu_1}{\alpha_1}} \right)}{r+\lambda_1-\alpha_1\mu_1} + \\
&\frac{\lambda_1 D \pi_0^{\mu_2} \left( 1 - \left( \frac{\pi_0}{\bar{\pi}} \right)^{\frac{r+\lambda_1-\alpha_1\mu_2}{\alpha_1}} \right)}{r+\lambda_1-\alpha_1\mu_2} + \frac{\lambda_1 N \pi_0 \left( 1 - \left( \frac{\pi_0}{\bar{\pi}} \right)^{\frac{r+\lambda_1-\alpha_1}{\alpha_1}} \right)}{r+\lambda_1-\alpha_1} \\
&= \frac{\pi_0 \left( 1 - \left( \frac{\pi_0}{\bar{\pi}} \right)^{\frac{r+\lambda_1-\alpha_1}{\alpha_1}} \right)}{r-\alpha_1} + (C \bar{\pi}^{\mu_1} + F \bar{\pi}^{\mu_2} + M \bar{\pi}) \left( \frac{\pi_0}{\bar{\pi}} \right)^{\frac{r+\lambda_1}{\alpha_1}} - \\
&\frac{\lambda_1 \pi_0 \left( 1 - \left( \frac{\pi_0}{\bar{\pi}} \right)^{\frac{r+\lambda_1-\alpha_1}{\alpha_1}} \right)}{(r-\alpha_1)(r+\lambda_1-\alpha_1)} + \frac{\lambda_1 H \pi_0^{\mu_1} \left( 1 - \left( \frac{\pi_0}{\bar{\pi}} \right)^{\frac{r+\lambda_1-\alpha_1\mu_1}{\alpha_1}} \right)}{r+\lambda_1-\alpha_1\mu_1} + \\
&\frac{\lambda_1 D \pi_0^{\mu_2} \left( 1 - \left( \frac{\pi_0}{\bar{\pi}} \right)^{\frac{r+\lambda_1-\alpha_1\mu_2}{\alpha_1}} \right)}{r+\lambda_1-\alpha_1\mu_2} + \frac{\lambda_1 N \pi_0 \left( 1 - \left( \frac{\pi_0}{\bar{\pi}} \right)^{\frac{r+\lambda_1-\alpha_1}{\alpha_1}} \right)}{r+\lambda_1-\alpha_1} \\
&= \frac{\lambda_1 H \pi_0^{\mu_1}}{r+\lambda_1-\alpha_1\mu_1} + \left( C - \frac{\lambda_1 H}{r+\lambda_1-\alpha_1\mu_1} \right) \pi_0^{\mu_1} \left( \frac{\pi_0}{\bar{\pi}} \right)^{\frac{r+\lambda_1-\alpha_1\mu_1}{\alpha_1}} + \\
&\frac{\lambda_1 D \pi_0^{\mu_2}}{r+\lambda_1-\alpha_1\mu_2} + \left( F - \frac{\lambda_1 D}{r+\lambda_1-\alpha_1\mu_2} \right) \pi_0^{\mu_2} \left( \frac{\pi_0}{\bar{\pi}} \right)^{\frac{r+\lambda_1-\alpha_1\mu_2}{\alpha_1}} + \\
&\frac{\pi_0 (1 + \lambda_1 N)}{r+\lambda_1-\alpha_1} + \left( M - \frac{1 + \lambda_1 N}{r+\lambda_1-\alpha_1} \right) \pi_0 \left( \frac{\pi_0}{\bar{\pi}} \right)^{\frac{r+\lambda_1-\alpha_1}{\alpha_1}}.
\end{aligned}$$

Because we supposed that  $\bar{\mathcal{E}}(\pi_0 | \bar{\pi}, \underline{\pi}) = C \pi_0^{\mu_1} + F \pi_0^{\mu_2} + M \pi_0$ , we must have that the following is satisfied:

$$C = \frac{\lambda_1 H}{r + \lambda_1 - \alpha_1 \mu_1}, \quad (8)$$

$$F = \frac{\lambda_1 D}{r + \lambda_1 - \alpha_1 \mu_2}, \quad (9)$$

and

$$M = \frac{1 + \lambda_1 N}{r + \lambda_1 - \alpha_1}. \quad (10)$$

On the other hand, plugging the assumed functional forms of  $\bar{\mathcal{E}}(\pi_0 | \bar{\pi}, \underline{\pi})$  and  $\underline{\mathcal{E}}(\underline{\pi} | \bar{\pi}, \underline{\pi})$

into (7) and performing some manipulations yields:

$$\begin{aligned}
\underline{\mathcal{E}}(\pi_0 | \bar{\pi}, \underline{\pi}) &= \int_{\frac{1}{\alpha_2} \ln(\underline{\pi}/\pi_0)}^{\infty} \lambda_2 e^{-\lambda_2 \tau_2} \left( \int_0^{\frac{1}{\alpha_2} \ln(\underline{\pi}/\pi_0)} \pi_0 e^{\alpha_2 s} e^{-rs} ds + \underline{\mathcal{E}}(\underline{\pi} | \bar{\pi}, \underline{\pi}) e^{-\frac{r}{\alpha_2} \ln(\underline{\pi}/\pi_0)} \right) d\tau_2 + \\
&\int_0^{\frac{1}{\alpha_2} \ln(\underline{\pi}/\pi_0)} \lambda_2 e^{-\lambda_2 \tau_2} \left( \int_0^{\tau_2} \pi_0 e^{\alpha_2 s} e^{-rs} ds + \bar{\mathcal{E}}(\pi_0 e^{\alpha_2 \tau_2} | \bar{\pi}, \underline{\pi}) e^{-r\tau_2} \right) d\tau_2 \\
&= \frac{\pi_0 \left( \left( \frac{\pi_0}{\underline{\pi}} \right)^{\frac{\lambda_2}{\alpha_2}} - \left( \frac{\pi_0}{\underline{\pi}} \right)^{\frac{r+\lambda_2-\alpha_2}{\alpha_2}} \right)}{r-\alpha_2} + \underline{\mathcal{E}}(\underline{\pi} | \bar{\pi}, \underline{\pi}) \left( \frac{\pi_0}{\underline{\pi}} \right)^{\frac{r+\lambda_2}{\alpha_2}} + \\
&\frac{\pi_0 \left( 1 - \left( \frac{\pi_0}{\underline{\pi}} \right)^{\frac{\lambda_2}{\alpha_2}} \right)}{r-\alpha_2} - \frac{\lambda_2 \pi_0 \left( 1 - \left( \frac{\pi_0}{\underline{\pi}} \right)^{\frac{r+\lambda_2-\alpha_2}{\alpha_2}} \right)}{(r-\alpha_2)(r+\lambda_2-\alpha_2)} + \frac{\lambda_2 C \pi_0^{\mu_1} \left( 1 - \left( \frac{\pi_0}{\underline{\pi}} \right)^{\frac{r+\lambda_2-\alpha_2\mu_1}{\alpha_2}} \right)}{r+\lambda_2-\alpha_2\mu_1} + \\
&\frac{\lambda_2 F \pi_0^{\mu_2} \left( 1 - \left( \frac{\pi_0}{\underline{\pi}} \right)^{\frac{r+\lambda_2-\alpha_2\mu_2}{\alpha_2}} \right)}{r+\lambda_2-\alpha_2\mu_2} + \frac{\lambda_2 M \pi_0 \left( 1 - \left( \frac{\pi_0}{\underline{\pi}} \right)^{\frac{r+\lambda_2-\alpha_2}{\alpha_2}} \right)}{r+\lambda_2-\alpha_2} \\
&= \frac{\pi_0 \left( 1 - \left( \frac{\pi_0}{\underline{\pi}} \right)^{\frac{r+\lambda_2-\alpha_2}{\alpha_2}} \right)}{r-\alpha_2} + (H \underline{\pi}^{\mu_1} + D \underline{\pi}^{\mu_2} + N \underline{\pi}) \left( \frac{\pi_0}{\underline{\pi}} \right)^{\frac{r+\lambda_2}{\alpha_2}} - \\
&\frac{\lambda_2 \pi_0 \left( 1 - \left( \frac{\pi_0}{\underline{\pi}} \right)^{\frac{r+\lambda_2-\alpha_2}{\alpha_2}} \right)}{(r-\alpha_2)(r+\lambda_2-\alpha_2)} + \frac{\lambda_2 C \pi_0^{\mu_1} \left( 1 - \left( \frac{\pi_0}{\underline{\pi}} \right)^{\frac{r+\lambda_2-\alpha_2\mu_1}{\alpha_2}} \right)}{r+\lambda_2-\alpha_2\mu_1} + \\
&\frac{\lambda_2 F \pi_0^{\mu_2} \left( 1 - \left( \frac{\pi_0}{\underline{\pi}} \right)^{\frac{r+\lambda_2-\alpha_2\mu_2}{\alpha_2}} \right)}{r+\lambda_2-\alpha_2\mu_2} + \frac{\lambda_2 M \pi_0 \left( 1 - \left( \frac{\pi_0}{\underline{\pi}} \right)^{\frac{r+\lambda_2-\alpha_2}{\alpha_2}} \right)}{r+\lambda_2-\alpha_2} \\
&= \frac{\lambda_2 C \pi_0^{\mu_1}}{r+\lambda_2-\alpha_2\mu_1} + \left( H - \frac{\lambda_2 C}{r+\lambda_2-\alpha_2\mu_1} \right) \pi_0^{\mu_1} \left( \frac{\pi_0}{\underline{\pi}} \right)^{\frac{r+\lambda_2-\alpha_2\mu_1}{\alpha_2}} + \\
&\frac{\lambda_2 F \pi_0^{\mu_2}}{r+\lambda_2-\alpha_2\mu_2} + \left( D - \frac{\lambda_2 F}{r+\lambda_2-\alpha_2\mu_2} \right) \pi_0^{\mu_2} \left( \frac{\pi_0}{\underline{\pi}} \right)^{\frac{r+\lambda_2-\alpha_2\mu_2}{\alpha_2}} + \\
&\frac{\pi_0 (1 + \lambda_2 M)}{r+\lambda_2-\alpha_2} + \left( N - \frac{1 + \lambda_2 M}{r+\lambda_2-\alpha_2} \right) \pi_0 \left( \frac{\pi_0}{\underline{\pi}} \right)^{\frac{r+\lambda_2-\alpha_2}{\alpha_2}}.
\end{aligned}$$

Because we supposed that  $\underline{\mathcal{E}}(\pi_0 | \bar{\pi}, \underline{\pi}) = H \pi_0^{\mu_1} + D \pi_0^{\mu_2} + N \pi_0$ , we must have that the following is satisfied:

$$H = \frac{\lambda_2 C}{r + \lambda_2 - \alpha_2 \mu_1}, \quad (11)$$

$$D = \frac{\lambda_2 F}{r + \lambda_2 - \alpha_2 \mu_2}, \quad (12)$$

and

$$N = \frac{1 + \lambda_2 M}{r + \lambda_2 - \alpha_2}. \quad (13)$$

Using (10) and (13) yields that  $M = \rho_1$  and  $N = \rho_2$ . It is easy to show that we cannot

have  $C = 0$  (or  $H = 0$ ), so (8) and (11) imply that  $\mu_1$  solves the following equation:

$$(r + \lambda_1 - \alpha_1\mu_1)(r + \lambda_2 - \alpha_2\mu_1) = \lambda_1\lambda_2.$$

There exist two values of  $\mu_1$  that solve this quadratic equation. Since  $(r + \lambda_1 - \alpha_1)(r + \lambda_2 - \alpha_2) > \lambda_1\lambda_2$ , one of the roots,  $\beta_1$  say, can be easily shown to be greater than 1, whereas the other one,  $\beta_2$  say, is negative. Note also that we cannot have that  $F = 0$  (or  $D = 0$ ), so (9) and (12) imply that  $\mu_2$  solves the following equation:

$$(r + \lambda_1 - \alpha_1\mu_2)(r + \lambda_2 - \alpha_2\mu_2) = \lambda_1\lambda_2.$$

In short, noting that  $\Delta \equiv [\alpha_1(\lambda_2 + r) - \alpha_2(\lambda_1 + r)]^2 + 4\alpha_1\alpha_2\lambda_1\lambda_2 > 0$ , we have that

$$\mu_1 = \beta_1 = \frac{\alpha_1(r + \lambda_2) + \alpha_2(r + \lambda_1) - \sqrt{\Delta}}{2\alpha_1\alpha_2} > 1$$

and

$$\mu_2 = \beta_2 = \frac{\alpha_1(r + \lambda_1) + \alpha_2(r + \lambda_1) + \sqrt{\Delta}}{2\alpha_1\alpha_2} < 0$$

(or equivalently,  $\mu_1 = \beta_2$  and  $\mu_2 = \beta_1$ ).<sup>14</sup>

It only remains to determine the values of  $C$  and  $D$  (and therefore those of  $H$  and  $F$ ). To arrive at them, take into account that

$$\bar{\mathcal{E}}(\bar{\pi} | \bar{\pi}, \underline{\pi}) = \int_0^\infty \lambda_1 e^{-\lambda_1 \tau_1} \left( \int_0^{\tau_1} \bar{\pi} e^{-rs} ds + \underline{\mathcal{E}}(\bar{\pi} | \bar{\pi}, \underline{\pi}) e^{-r\tau_1} \right) d\tau_1,$$

so that using the hypotheses that  $\bar{\mathcal{E}}(\bar{\pi} | \bar{\pi}, \underline{\pi}) = C\bar{\pi}^{\beta_1} + F\bar{\pi}^{\beta_2} + \rho_1\bar{\pi}$  and  $\underline{\mathcal{E}}(\bar{\pi} | \bar{\pi}, \underline{\pi}) = H\bar{\pi}^{\beta_1} + D\bar{\pi}^{\beta_2} + \rho_2\bar{\pi}$  yields after some algebra that

$$C\bar{\pi}^{\beta_1} + F\bar{\pi}^{\beta_2} + \rho_1\bar{\pi} = \frac{\lambda_1 H \bar{\pi}^{\beta_1}}{r + \lambda_1} + \frac{\lambda_1 D \bar{\pi}^{\beta_2}}{r + \lambda_1} + \frac{(1 + \lambda_1 \rho_2) \bar{\pi}}{r + \lambda_1}.$$

Hence, using (9) (for  $\mu_2 = \beta_2$ ) and (11) (for  $\mu_1 = \beta_1$ ) yields that

$$\frac{[(r + \lambda_1)(r + \lambda_2 - \alpha_2\beta_1) - \lambda_1\lambda_2]C\bar{\pi}^{\beta_1}}{(r + \lambda_1)(r + \lambda_2 - \alpha_2\beta_1)} + \frac{\lambda_1\alpha_1\beta_2 D\bar{\pi}^{\beta_2}}{(r + \lambda_1 - \alpha_1\beta_2)(r + \lambda_1)} = \left( \frac{(1 + \lambda_1\rho_2)}{r + \lambda_1} - \rho_1 \right) \bar{\pi}.$$

The facts that  $(r + \lambda_1 - \alpha_1\beta_1)(r + \lambda_2 - \alpha_2\beta_1) = \lambda_1\lambda_2$  and  $\rho_1 = \frac{1 + \lambda_1\rho_2}{\lambda_1 + r - \alpha_1}$  allow us to rewrite the above expression after some simple manipulations:

$$\beta_1 C \bar{\pi}^{\beta_1} + \frac{\lambda_1 \beta_2 D \bar{\pi}^{\beta_2}}{r + \lambda_1 - \alpha_1 \beta_2} = -\rho_1 \bar{\pi}.$$

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<sup>14</sup>It is straightforward to show that either  $\mu_1 = \mu_2 = \beta_1$  or  $\mu_1 = \mu_2 = \beta_2$  cannot be solutions.

Note that the fact that  $(r + \lambda_1 - \alpha_1\beta_2)(r + \lambda_2 - \alpha_2\beta_2) = \lambda_1\lambda_2$  leads to

$$\frac{\lambda_1}{r + \lambda_1 - \alpha_1\beta_1} = \frac{r + \lambda_2 - \alpha_2\beta_2}{\lambda_2} \equiv \delta_1,$$

so it holds that

$$\beta_1 C \bar{\pi}^{\beta_1} + \delta_1 \beta_2 D \bar{\pi}^{\beta_2} = -\rho_1 \bar{\pi}. \quad (14)$$

Equation (14) is part of a system of two equations whence we can get the values of  $C$  and  $D$ . To obtain the other equation, observe that it holds that

$$\underline{\mathcal{E}}(\underline{\pi} | \bar{\pi}, \underline{\pi}) = \int_0^\infty \lambda_2 e^{-\lambda_2 \tau_2} \left( \int_0^{\tau_2} \underline{\pi} e^{-rs} ds + \bar{\mathcal{E}}(\underline{\pi} | \bar{\pi}, \underline{\pi}) e^{-r\tau_2} \right) d\tau_2,$$

so using the hypotheses that  $\underline{\mathcal{E}}(\underline{\pi} | \bar{\pi}, \underline{\pi}) = H \underline{\pi}^{\beta_1} + D \underline{\pi}^{\beta_2} + \rho_2 \underline{\pi}$  and  $\bar{\mathcal{E}}(\underline{\pi} | \bar{\pi}, \underline{\pi}) = C \bar{\pi}^{\beta_1} + F \bar{\pi}^{\beta_2} + \rho_1 \bar{\pi}$  yields after some algebra that

$$H \underline{\pi}^{\beta_1} + D \underline{\pi}^{\beta_2} + \rho_2 \underline{\pi} = \frac{\lambda_2 C \bar{\pi}^{\beta_1}}{r + \lambda_2} + \frac{\lambda_2 F \bar{\pi}^{\beta_2}}{r + \lambda_2} + \frac{(1 + \lambda_2 \rho_1) \bar{\pi}}{r + \lambda_2}.$$

Hence, using (9) (for  $\mu_2 = \beta_2$ ) and (11) (for  $\mu_1 = \beta_1$ ) yields that

$$\frac{\lambda_2 \alpha_2 \beta_2 C \bar{\pi}^{\beta_1}}{(r + \lambda_2)(r + \lambda_2 - \alpha_2 \beta_2)} + \frac{[(r + \lambda_2)(r + \lambda_1 - \alpha_1 \beta_1) - \lambda_1 \lambda_2] D \bar{\pi}^{\beta_2}}{(r + \lambda_2)(r + \lambda_1 - \alpha_1 \beta_1)} = \left( \frac{(1 + \lambda_2 \rho_1)}{r + \lambda_2} - \rho_2 \right) \bar{\pi}.$$

Using the facts that  $(r + \lambda_1 - \alpha_1 \beta_2)(r + \lambda_2 - \alpha_2 \beta_2) = \lambda_1 \lambda_2$  and  $\rho_2 = \frac{1 + \lambda_2 \rho_1}{\lambda_2 + r - \alpha_2}$ , we can rewrite the above expression after some simple manipulations as:

$$\frac{\lambda_2 \beta_1 C \bar{\pi}^{\beta_1}}{r + \lambda_2 - \alpha_2 \beta_1} + \beta_2 D \bar{\pi}^{\beta_2} = -\rho_2 \bar{\pi}.$$

Note that the fact that  $(r + \lambda_1 - \alpha_1 \beta_1)(r + \lambda_2 - \alpha_2 \beta_1) = \lambda_1 \lambda_2$  leads to

$$\frac{\lambda_2}{r + \lambda_2 - \alpha_2 \beta_2} = \frac{r + \lambda_1 - \alpha_1 \beta_1}{\lambda_1} \equiv \delta_2,$$

so

$$\delta_2 \beta_2 C \bar{\pi}^{\beta_1} + \beta_2 D \bar{\pi}^{\beta_2} = -\rho_2 \bar{\pi}. \quad (15)$$

Solving the system of equations that consists of (14) and (15) yields that

$$C = \frac{\delta_1 \bar{\pi}^{\beta_2} \rho_2 \bar{\pi} - \rho_1 \bar{\pi} \bar{\pi}^{\beta_2}}{\beta_1 (\bar{\pi}^{\beta_2} \bar{\pi}^{\beta_1} - \delta_1 \delta_2 \bar{\pi}^{\beta_1} \bar{\pi}^{\beta_2})} \equiv \xi(\bar{\pi}, \underline{\pi}).$$

and

$$D = \frac{\delta_2 \bar{\pi}^{\beta_1} \rho_1 \bar{\pi} - \rho_2 \bar{\pi} \bar{\pi}^{\beta_1}}{\beta_2 (\bar{\pi}^{\beta_2} \bar{\pi}^{\beta_1} - \delta_1 \delta_2 \bar{\pi}^{\beta_1} \bar{\pi}^{\beta_2})} \equiv \zeta(\bar{\pi}, \underline{\pi}).$$

Therefore, it follows that

$$H = \frac{\lambda_2 C}{r + \lambda_2 - \alpha_2 \beta_2} = \frac{(r + \lambda_1 - \alpha_1 \beta_1) C}{\lambda_1} = \delta_2 \xi(\bar{\pi}, \underline{\pi})$$

and

$$F = \frac{\lambda_1 D}{r + \lambda_1 - \alpha_1 \beta_1} = \delta_1 \zeta(\bar{\pi}, \underline{\pi}).$$

It only remains to show that  $\delta_1 \in (0, 1)$  and  $\delta_2 \in (0, 1)$ . Note that

$$\delta_1 = \frac{r + \lambda_2 - \alpha_2 \beta_2}{\lambda_2} = \frac{\alpha_1(r + \lambda_2) - \alpha_2(r + \lambda_1) - \sqrt{\Delta}}{2\alpha_1\lambda_2} \in (0, 1),$$

whereas

$$\delta_2 = \frac{r + \lambda_1 - \alpha_1 \beta_1}{\lambda_1} = -\frac{\alpha_1(r + \lambda_2) - \alpha_2(r + \lambda_1) - \sqrt{\Delta}}{2\alpha_2\lambda_1} \in (0, 1).$$

■

**Proof of Theorem 2.** Let  $\bar{\varphi}_1(\pi_0, \pi^*)$  denote the expected discounted value of a claim to a dollar when the process first hits  $\pi^*$  from below, conditional on the process being in a growth phase and on its state being  $\pi_0 < \pi^*$ . Also, let  $\underline{\varphi}_1(\pi_0, \pi^*)$  denote the expected discounted value of a claim to a dollar when the process first hits  $\pi^*$  from below, conditional on the process being in a decline phase and on the current state being  $\pi_0$ . Given our (memoryless) assumptions on the independent random variables involved, we can relate  $\bar{\varphi}_1(\pi_0, \pi^*)$  and  $\underline{\varphi}_1(\pi_0, \pi^*)$  as follows:

$$\begin{aligned} \bar{\varphi}_1(\pi_0, \pi^*) &= \int_0^{\frac{1}{\alpha_1} \ln(\pi^*/\pi_0)} \lambda_1 e^{-\lambda_1 \tau_1} \underline{\varphi}_1(\pi_0 e^{\alpha_1 \tau_1}, \pi^*) e^{-r \tau_1} d\tau_1 + \\ &\int_{\frac{1}{\alpha_1} \ln(\pi^*/\pi_0)}^{\infty} \lambda_1 e^{-\lambda_1 \tau_1} e^{-r \frac{1}{\alpha_1} \ln(\pi^*/\pi_0)} d\tau_1 \end{aligned} \quad (16)$$

and

$$\underline{\varphi}_1(\pi_0, \pi^*) = \int_0^{\infty} \lambda_2 e^{-\lambda_2 \tau_2} \bar{\varphi}_1(\pi_0 e^{\alpha_2 \tau_2}, \pi^*) e^{-r \tau_2} d\tau_2. \quad (17)$$

To understand the relation between  $\bar{\varphi}_1(\pi_0, \pi^*)$  and  $\underline{\varphi}_1(\pi_0, \pi^*)$ , note that if the realized length of the growth phase  $\tau_1$  is smaller than  $\frac{1}{\alpha_1} \ln(\frac{\pi^*}{\pi_0})$  (the time it would take to hit state  $\pi^*$  from below during the current growth phase), then the firm would acquire an asset whose discounted value would be  $\underline{\varphi}_1(\pi_0 e^{\alpha_1 \tau_1}, \pi^*) e^{-r \tau_1}$ ; if  $\tau_1$  were greater than  $\frac{1}{\alpha_1} \ln(\frac{\pi^*}{\pi_0})$ , then the firm would acquire the discounted value of a dollar. As for  $\underline{\varphi}_1(\pi_0, \pi^*)$ , note that the process will start growing at some random future date  $\tau_2$ , and hence the firm will acquire at such date an asset whose discounted value is  $\bar{\varphi}_1(\pi_0 e^{\alpha_2 \tau_2}, \pi^*) e^{-r \tau_2}$ .

To solve the system of functional equations that consists of (16) and (17), guess that  $\bar{\varphi}_1(\pi_0, \pi^*) = Y \left(\frac{\pi_0}{\pi^*}\right)^{\theta_1}$  (where  $Y$  and  $\theta_1$  are constants to be found out), and use such

functional form for  $\bar{\varphi}_1(\pi_0, \pi^*)$  in (17) so as to get:

$$\underline{\varphi}_1(\pi_0, \pi^*) = \frac{Y\lambda_2}{r + \lambda_2 - \alpha_2\theta_1} \left(\frac{\pi_0}{\pi^*}\right)^{\theta_1}. \quad (18)$$

Therefore, we have

$$\underline{\varphi}_1(\pi_0 e^{\alpha_1\tau_1}, \pi^*) = \frac{Y\lambda_2 e^{\alpha_1\theta_1\tau_1}}{r + \lambda_2 - \alpha_2\theta_1} \left(\frac{\pi_0}{\pi^*}\right)^{\theta_1},$$

so (16) becomes:

$$\begin{aligned} \bar{\varphi}_1(\pi_0, \pi^*) &= \frac{Y\lambda_1\lambda_2}{r + \lambda_2 - \alpha_2\theta_1} \left(\frac{\pi_0}{\pi^*}\right)^{\theta_1} \int_0^{\frac{1}{\alpha_1} \ln(\pi^*/\pi_0)} e^{-(r+\lambda_1-\alpha_1\theta_1)\tau_1} d\tau_1 + \left(\frac{\pi_0}{\pi^*}\right)^{\frac{r+\lambda_1}{\alpha_1}} \\ &= \frac{Y\lambda_1\lambda_2}{(r + \lambda_2 - \alpha_2\theta_1)(r + \lambda_1 - \alpha_1\theta_1)} \left(\frac{\pi_0}{\pi^*}\right)^{\theta_1} \left[1 - \left(\frac{\pi_0}{\pi^*}\right)^{\frac{r+\lambda_1-\alpha_1\theta_1}{\alpha_1}}\right] + \left(\frac{\pi_0}{\pi^*}\right)^{\frac{r+\lambda_1}{\alpha_1}} \end{aligned}$$

We assumed that  $\bar{\varphi}_1(\pi_0, \pi^*) = Y \left(\frac{\pi_0}{\pi^*}\right)^{\theta_1}$ , so the following must hold:

$$Y \left(\frac{\pi_0}{\pi^*}\right)^{\theta_1} - \left(\frac{\pi_0}{\pi^*}\right)^{\frac{\lambda_1+r}{\alpha_1}} = \frac{Y\lambda_1\lambda_2}{(r + \lambda_2 - \alpha_2\theta_1)(r + \lambda_1 - \alpha_1\theta_1)} \left[\left(\frac{\pi_0}{\pi^*}\right)^{\theta_1} - \left(\frac{\pi_0}{\pi^*}\right)^{\frac{r+\lambda_1}{\alpha_1}}\right].$$

Therefore,  $Y = 1$ , whereas  $\theta_1$  must satisfy the following equation:

$$(r + \lambda_2 - \alpha_2\theta_1)(r + \lambda_1 - \alpha_1\theta_1) = \lambda_1\lambda_2. \quad (19)$$

There exist two values of  $\theta_1$  that solve this quadratic equation, which we labeled before as  $\beta_1$  and  $\beta_2$ . Clearly, only  $\beta_1 > 1$  can be an admissible solution, so  $\theta_1 = \beta_1$ . Also, note that expression (18) with  $Y = 1$  and  $\theta_1 = \beta_1$  implies that

$$\underline{\varphi}_1(\pi_0, \pi^*) = \frac{\lambda_2}{r + \lambda_2 - \alpha_2\beta_1} \left(\frac{\pi_0}{\pi^*}\right)^{\beta_1} = \left(\frac{r + \lambda_1 - \alpha_1\beta_1}{\lambda_1}\right) \left(\frac{\pi_0}{\pi^*}\right)^{\beta_1},$$

since  $(r + \lambda_2 - \alpha_2\beta_1)(r + \lambda_1 - \alpha_1\beta_1) = \lambda_1\lambda_2$ . ■

**Proof of Theorem 3.** Let  $\bar{\varphi}_2(\pi_0, \pi^*)$  denote the expected discounted value of a claim to a dollar when the process first hits  $\pi^*$  from above, conditional on the process being in a growth phase and on the current state being  $\pi_0 \geq \pi^*$ . In addition, let  $\underline{\varphi}_2(\pi_0, \pi^*)$  denote the expected discounted value of a claim to a dollar when the process first hits  $\pi^*$  from above, conditional on the process being in a decline phase and on the current state being  $\pi_0 > \pi^*$ . Then we have the following:

$$\bar{\varphi}_2(\pi_0, \pi^*) = \int_0^\infty \lambda_1 e^{-\lambda_1\tau_1} \underline{\varphi}_2(\pi_0 e^{\alpha_1\tau_1}, \pi^*) e^{-r\tau_1} d\tau_1$$

and

$$\begin{aligned}\varphi_2(\pi_0, \pi^*) &= \int_0^{\frac{1}{\alpha_2} \ln(\pi^*/\pi_0)} \lambda_2 e^{-\lambda_2 \tau_2} \bar{\varphi}_2(\pi_0 e^{\alpha_2 \tau_2}, \pi^*) e^{-r \tau_2} d\tau_2 + \\ &\int_{\frac{1}{\alpha_2} \ln(\pi^*/\pi_0)}^{\infty} \lambda_2 e^{-\lambda_2 \tau_2} e^{-r \frac{1}{\alpha_2} \ln(\pi^*/\pi_0)} d\tau_2.\end{aligned}$$

To understand the relation between  $\bar{\varphi}_2(\pi_0, \pi^*)$  and  $\varphi_2(\pi_0, \pi^*)$ , note that if the process is going through a growth phase at  $\pi_0$ , then state  $\pi^* \leq \pi_0$  will never be hit in such a phase. Hence, when a switching date is realized at  $\tau_1$ , the firm will get  $\varphi_2(\pi_0 e^{\alpha_1 \tau_1}, \pi^*) e^{-r \tau_1}$ . However, if the process is going through a declining phase at  $\pi_0$ , the process will grow at some random future date  $\tau_2$ , so two situations must be distinguished. On the one hand, if the process starts declining before reaching  $\pi^*$  at the random date  $\tau_2$ , then the firm will attain  $\bar{\varphi}_2(\pi_0 e^{\alpha_2 \tau_2}, \pi^*) e^{-r \tau_2}$ . On the other hand, if the declining process goes all the way down to  $\pi^*$ , then the firm will attain the discounted value of the dollar as soon as state  $\pi^*$  is reached.

Suppose that  $\bar{\varphi}_2(\pi_0, \pi^*) = A \left(\frac{\pi_0}{\pi^*}\right)^{\theta_2}$  (where  $A$ , and  $\theta_2$  are constants to be found out). In this case, we have

$$\begin{aligned}\varphi_2(\pi_0, \pi^*) &= \left(\frac{\pi_0}{\pi^*}\right)^{\frac{r+\lambda_2}{\alpha_2}} + \int_0^{\frac{1}{\alpha_2} \ln(\pi^*/\pi_0)} \lambda_2 e^{-\lambda_2 \tau_2} \left[ A \left(\frac{\pi_0 e^{\alpha_2 \tau_2}}{\pi^*}\right)^{\theta_2} \right] e^{-r \tau_2} d\tau_2 \\ &= \frac{\lambda_2 A}{\lambda_2 + r - \alpha_2 \theta_2} \left(\frac{\pi_0}{\pi^*}\right)^{\theta_2} - \frac{\lambda_2 A}{\lambda_2 + r - \alpha_2 \theta_2} \left(\frac{\pi_0}{\pi^*}\right)^{\frac{r+\lambda_2}{\alpha_2}} + \left(\frac{\pi_0}{\pi^*}\right)^{\frac{r+\lambda_2}{\alpha_2}} \\ &= \frac{\lambda_2 A}{\lambda_2 + r - \alpha_2 \theta_2} \left(\frac{\pi_0}{\pi^*}\right)^{\theta_2} + \left(\frac{\lambda_2(1-A) + r - \alpha_2 \theta_2}{\lambda_2 + r - \alpha_2 \theta_2}\right) \left(\frac{\pi_0}{\pi^*}\right)^{\frac{\lambda_2+r}{\alpha_2}}.\end{aligned}\quad (20)$$

In addition,

$$\begin{aligned}\bar{\varphi}_2(\pi_0, \pi^*) &= \int_0^{\infty} \lambda_1 e^{-(r+\lambda_1)\tau_1} \left[ \frac{\lambda_2 A}{\lambda_2 + r - \alpha_2 \theta_2} \left(\frac{\pi_0 e^{\alpha_1 \tau_1}}{\pi^*}\right)^{\theta_2} + \left(\frac{\lambda_2(1-A) + r - \alpha_2 \theta_2}{\lambda_2 + r - \alpha_2 \theta_2}\right) \left(\frac{\pi_0 e^{\alpha_1 \tau_1}}{\pi^*}\right)^{\frac{\lambda_2+r}{\alpha_2}} \right] d\tau_1 \\ &= \frac{\lambda_1 \lambda_2 A}{(r + \lambda_2 - \alpha_2 \theta_2)(r + \lambda_1 - \alpha_1 \theta_2)} \left(\frac{\pi_0}{\pi^*}\right)^{\theta_2} + \\ &\quad \left[ \frac{\lambda_2(1-A) + r - \alpha_2 \theta_2}{r + \lambda_2 - \alpha_2 \theta_2} \right] \left(\frac{\pi_0}{\pi^*}\right)^{\frac{\lambda_2+r}{\alpha_2}} \left( \frac{\lambda_1}{r + \lambda_1 - \frac{\alpha_1}{\alpha_2}(\lambda_2 + r)} \right)\end{aligned}$$

Because we supposed that  $\bar{\varphi}_2(\pi_0, \pi^*) = A \left(\frac{\pi_0}{\pi^*}\right)^{\theta_2}$ , we must have that the following two equations hold:

$$\frac{\lambda_1 \lambda_2 A}{(r + \lambda_2 - \alpha_2 \theta_2)(r + \lambda_1 - \alpha_1 \theta_2)} = A, \quad (21)$$

and

$$\frac{\lambda_2(1-A) + r - \alpha_2 \theta_2}{\lambda_2 + r - \alpha_2 \theta_2} = 0. \quad (22)$$

Equation (21) implies that  $\theta_2$  solves

$$(\lambda_2 + r - \alpha_2\theta_2)(\lambda_1 + r - \alpha_1\theta_2) = \lambda_1\lambda_2.$$

We have shown before that the roots of this quadratic equation are  $\beta_1 > 1$  and  $\beta_2 < 0$ . Only the negative root can be admissible now, so  $\theta_2 = \beta_2$ . In addition, equation (22) with  $\theta_2 = \beta_2$  implies that

$$A = \frac{r + \lambda_2 - \alpha_2\beta_2}{\lambda_2}. \quad (23)$$

Therefore,  $\bar{\varphi}_2(\pi_0, \pi^*) = \delta_1 \left(\frac{\pi_0}{\pi^*}\right)^{\beta_2}$ . Finally, making use of (20) and (22) with  $\theta_2 = \beta_2$ , together with expression (23), yields that

$$\underline{\varphi}_2(\pi_0, \pi^*) = \left(\frac{\pi_0}{\pi^*}\right)^{\beta_2}.$$

■

**Proof of Theorem 4.** Let  $\bar{\mathcal{E}}(\pi_0, \pi^*)$  denote the expected value of the stream of discounted profits collected while the process transitions from  $\pi_0$  until it first hits  $\pi^*$  from above, conditional on the process being in a growth phase and on the current state being  $\pi_0 \geq \pi^*$ . Also, let  $\underline{\mathcal{E}}(\pi_0, \pi^*)$  be the expected stream of discounted profits collected while the process transitions from  $\pi_0$  until it first hits  $\pi^*$  from above, conditional on the process being in a decline phase and on the current state being  $\pi_0 > \pi^*$ . We can relate  $\bar{\mathcal{E}}(\pi_0, \pi^*)$  and  $\underline{\mathcal{E}}(\pi_0, \pi^*)$  as follows:

$$\bar{\mathcal{E}}(\pi_0, \pi^*) = \int_0^\infty \lambda_1 e^{-\lambda_1 \tau_1} \left[ \int_0^{\tau_1} \pi_0 e^{\alpha_1 s} e^{-rs} ds + \underline{\mathcal{E}}(\pi_0 e^{\alpha_1 \tau_1}, \pi^*) e^{-r\tau_1} \right] d\tau_1 \quad (24)$$

and

$$\begin{aligned} \underline{\mathcal{E}}(\pi_0, \pi^*) = & \int_0^{\frac{1}{\alpha_2} \ln(\pi^*/\pi_0)} \lambda_2 e^{-\lambda_2 \tau_2} \left[ \int_0^{\tau_2} \pi_0 e^{\alpha_2 s} e^{-rs} ds + \bar{\mathcal{E}}(\pi_0 e^{\alpha_2 \tau_2}, \pi^*) e^{-r\tau_2} \right] d\tau_2 + \\ & \int_{\frac{1}{\alpha_2} \ln(\pi^*/\pi_0)}^\infty \lambda_2 e^{-\lambda_2 \tau_2} \left( \int_0^{\frac{1}{\alpha_2} \ln(\pi^*/\pi_0)} \pi_0 e^{\alpha_2 s} e^{-rs} ds \right) d\tau_2. \end{aligned} \quad (25)$$

To see how  $\bar{\mathcal{E}}(\pi_0, \pi^*)$  arises, note that the process will start declining at some random future date  $\tau_1$ , so the firm gets a stream of discounted profits since the current date until then, and also the discounted value of an asset worth  $\underline{\mathcal{E}}(\pi_0 e^{\alpha_1 \tau_1}, \pi^*)$ . As for  $\underline{\mathcal{E}}(\pi_0, \pi^*)$ , we have that the process grows at some random future date  $\tau_2$ , although it may start growing before hitting  $\pi^*$  (from above). If it does not, then the firm reaps a stream of discounted profits until  $\pi^*$  is hit at time  $\frac{\ln(\pi^*/\pi_0)}{\alpha_2}$ . If the process starts growing before hitting  $\pi^*$ , then the firm reaps a discounted profit stream until the process stops declining, and the discounted value of an asset worth  $\bar{\mathcal{E}}(\pi_0 e^{\alpha_2 \tau_2}, \pi^*)$  at such switching date.

Let us suppose that  $\bar{\mathcal{E}}(\pi_0, \pi^*) = E(\pi_0)^{\theta_6} + G(\pi_0)^{\theta_7}$  (where  $\theta_6$  and  $\theta_7$  are constants, whereas  $E$  and  $G$  do not depend on  $\pi_0$  although they may depend on  $\pi^*$ ), so plugging the assumed functional form of  $\bar{\mathcal{E}}(\pi_0, \pi^*)$  into (25) and performing several manipulations yields:

$$\begin{aligned} \underline{\mathcal{E}}(\pi_0, \pi^*) &= \frac{\pi_0 \left(1 - \left(\frac{\pi_0}{\pi^*}\right)^{\frac{(r-\alpha_2)}{\alpha_2}}\right) \left(\frac{\pi_0}{\pi^*}\right)^{\frac{\lambda_2}{\alpha_2}}}{r - \alpha_2} + \frac{\pi_0}{r + \lambda_2 - \alpha_2} - \frac{\pi_0 \left(\frac{\pi_0}{\pi^*}\right)^{\frac{\lambda_2}{\alpha_2}}}{r - \alpha_2} + \\ &\frac{\lambda_2 \pi_0 \left(\frac{\pi_0}{\pi^*}\right)^{\frac{(r+\lambda_2-\alpha_2)}{\alpha_2}}}{(r - \alpha_2)(r + \lambda_2 - \alpha_2)} + \frac{\lambda_2 E \pi_0^{\theta_6} \left(1 - \left(\frac{\pi_0}{\pi^*}\right)^{\frac{(r+\lambda_2-\alpha_2\theta_6)}{\alpha_2}}\right)}{(r + \lambda_2 - \alpha_2\theta_6)} + \\ &\frac{\lambda_2 G \pi_0^{\theta_7} \left(1 - \left(\frac{\pi_0}{\pi^*}\right)^{\frac{(r+\lambda_2-\alpha_2\theta_7)}{\alpha_2}}\right)}{(r + \lambda_2 - \alpha_2\theta_7)} \\ &= \left(\frac{1}{r + \lambda_2 - \alpha_2}\right) \pi_0 + \frac{\lambda_2 E \pi_0^{\theta_6}}{r + \lambda_2 - \alpha_2\theta_6} + \frac{\lambda_2 G \pi_0^{\theta_7}}{r + \lambda_2 - \alpha_2\theta_7} - \\ &\left(\frac{\left(\pi^*\right)^{-\frac{(r+\lambda_2-\alpha_2)}{\alpha_2}}}{r + \lambda_2 - \alpha_2} + \frac{\lambda_2 E \left(\pi^*\right)^{-\frac{(r+\lambda_2-\alpha_2\theta_6)}{\alpha_2}}}{r + \lambda_2 - \alpha_2\theta_6} + \frac{\lambda_2 G \left(\pi^*\right)^{-\frac{(r+\lambda_2-\alpha_2\theta_7)}{\alpha_2}}}{r + \lambda_2 - \alpha_2\theta_7}\right) \left(\pi_0\right)^{\frac{(r+\lambda_2)}{\alpha_2}}. \end{aligned}$$

Substitute this into (24) so as to get the following after some manipulations:

$$\begin{aligned} \bar{\mathcal{E}}(\pi_0, \pi^*) &= \left(1 + \frac{\lambda_1}{r + \lambda_2 - \alpha_2}\right) \frac{\pi_0}{(r + \lambda_1 - \alpha_1)} - \\ &\frac{\lambda_1 \alpha_2 \left(\pi_0\right)^{\frac{(r+\lambda_2)}{\alpha_2}}}{(\alpha_2(r + \lambda_1) - \alpha_1(r + \lambda_2))} \left(\frac{\left(\pi^*\right)^{-\frac{(r+\lambda_2-\alpha_2)}{\alpha_2}}}{(r + \lambda_2 - \alpha_2)} + \frac{\lambda_2 E \left(\pi^*\right)^{-\frac{(r+\lambda_2-\alpha_2\theta_6)}{\alpha_2}}}{(r + \lambda_2 - \alpha_2\theta_6)} + \frac{\lambda_2 G \left(\pi^*\right)^{-\frac{(r+\lambda_2-\alpha_2\theta_7)}{\alpha_2}}}{(r + \lambda_2 - \alpha_2\theta_7)}\right) + \\ &\frac{\lambda_2 E (\pi_0)^{\theta_6} \lambda_1}{(r + \lambda_2 - \alpha_2\theta_6)(r + \lambda_1 - \alpha_1\theta_6)} + \frac{\lambda_2 G (\pi_0)^{\theta_7} \lambda_1}{(r + \lambda_2 - \alpha_2\theta_7)(r + \lambda_1 - \alpha_1\theta_7)}. \end{aligned}$$

Assume that  $\theta_6 = 1$  so that the assumption that  $\bar{\mathcal{E}}(\pi_0, \pi^*) = E(\pi_0)^{\theta_6} + G(\pi_0)^{\theta_7}$  implies that the following must hold:

$$E = \frac{(r + \lambda_2 - \alpha_2)(r + \lambda_1 - \alpha_2) - \lambda_1 \lambda_2}{(r - \alpha_2)((r + \lambda_2 - \alpha_2)(r + \lambda_1 - \alpha_1) - \lambda_1 \lambda_2)} = \rho_1, \quad (26)$$

$$G = -\frac{(r + \lambda_2 - \alpha_2\theta_7)}{\lambda_2 (\pi^*)^{(\theta_7-1)}} \left(\frac{1 + \lambda_2 \rho_1}{r + \lambda_2 - \alpha_2}\right), \quad (27)$$

and

$$(r + \lambda_2 - \alpha_2\theta_7)(r + \lambda_1 - \alpha_1\theta_7) = \lambda_1 \lambda_2.$$

We must clearly have that  $\theta_7 = \beta_2 < 0$ , so using this result as well as expressions (26) and

(27) yields after very laborious manipulations:

$$\begin{aligned}
\underline{\mathcal{E}}(\pi_0, \pi^*) &= \left( \frac{1 + \lambda_2 \rho_1}{r + \lambda_2 - \alpha_2} \right) \pi_0 - \\
&\quad \left( \frac{(\pi^*)^{-\frac{(r+\lambda_2-\alpha_2)}{\alpha_2}}}{r+\lambda_2-\alpha_2} + \frac{\lambda_2 E(\pi^*)^{-\frac{(r+\lambda_2-\alpha_2)}{\alpha_2}}}{r+\lambda_2-\alpha_2} + \frac{\lambda_2 G(\pi^*)^{-\frac{(r+\lambda_2-\alpha_2\beta_2)}{\alpha_2}}}{r+\lambda_2-\alpha_2\beta_2} \right) (\pi_0)^{\frac{(r+\lambda_2)}{\alpha_2}} + \frac{\lambda_2 G \pi_0^{\beta_2}}{r + \lambda_2 - \alpha_2 \beta_2} \\
&= \left( \frac{1 + \lambda_2 \rho_1}{r + \lambda_2 - \alpha_2} \right) \pi_0 - \pi_0 \left( \frac{\pi_0}{\pi^*} \right)^{\beta_2 - 1} \left( \frac{1 + \lambda_2 \rho_1}{r + \lambda_2 - \alpha_2} \right) \\
&= \rho_2 \pi_0 \left[ 1 - \left( \frac{\pi_0}{\pi^*} \right)^{\beta_2 - 1} \right],
\end{aligned}$$

where the second equality makes use of the fact that

$$\frac{(\pi^*)^{-\frac{(r+\lambda_2-\alpha_2)}{\alpha_2}}}{r + \lambda_2 - \alpha_2} + \frac{\lambda_2 E(\pi^*)^{-\frac{(r+\lambda_2-\alpha_2)}{\alpha_2}}}{r + \lambda_2 - \alpha_2} + \frac{\lambda_2 G(\pi^*)^{-\frac{(r+\lambda_2-\alpha_2\beta_2)}{\alpha_2}}}{r + \lambda_2 - \alpha_2 \beta_2} = 0,$$

and the third equality uses

$$\frac{1 + \lambda_2 \rho_1}{r + \lambda_2 - \alpha_2} = \rho_2. \tag{28}$$

In turn, note that

$$\begin{aligned}
\bar{\mathcal{E}}(\pi_0, \pi^*) &= E\pi_0 + G(\pi_0)^{\beta_2} \\
&= \rho_1 \pi_0 - \frac{(r + \lambda_2 - \alpha_2 \beta_2) \pi_0}{\lambda_2} \left( \frac{1 + \lambda_2 \rho_1}{r + \lambda_2 - \alpha_2} \right) \left( \frac{\pi_0}{\pi^*} \right)^{\beta_2 - 1} \\
&= \rho_1 \pi_0 - \rho_2 \delta_1 \pi_0 \left( \frac{\pi_0}{\pi^*} \right)^{\beta_2 - 1},
\end{aligned}$$

where the last equality makes use of (28) and the definition of  $\delta_1$ . ■

**Proof of Theorem 5.** Let  $\bar{\mathcal{E}}'(\pi_0, \pi^*)$  denote the expected value of the stream of discounted profits collected while the process transitions from  $\pi_0$  until it first hits  $\pi^*$  from below, conditional on the process being in a growth phase and on the current state being  $\pi_0 < \pi^*$ . Similarly, let  $\underline{\mathcal{E}}'(\pi_0, \pi^*)$  be the expected stream of discounted profits collected while the process transitions from  $\pi_0$  until it first hits  $\pi^*$  from below, conditional on the process being in a decline phase and on the current state being  $\pi_0 < \pi^*$ . We can relate  $\bar{\mathcal{E}}'(\pi_0, \pi^*)$  and  $\underline{\mathcal{E}}'(\pi_0, \pi^*)$  as follows:

$$\begin{aligned}
\bar{\mathcal{E}}'(\pi_0, \pi^*) &= \int_{\frac{1}{\alpha_1} \ln \frac{\pi^*}{\pi_0}}^{\infty} \lambda_1 e^{-\lambda_1 \tau_1} \left( \int_0^{\frac{1}{\alpha_1} \ln \frac{\pi^*}{\pi_0}} \pi_0 e^{\alpha_1 s} e^{-rs} ds \right) d\tau_1 + \\
&\quad \int_0^{\frac{1}{\alpha_1} \ln \frac{\pi^*}{\pi_0}} \lambda_1 e^{-\lambda_1 \tau_1} \left( \int_0^{\tau_1} \pi_0 e^{\alpha_1 s} e^{-rs} ds + e^{-r\tau_1} \underline{\mathcal{E}}'(\pi_0 e^{\alpha_1 \tau_1}, \pi^*) \right) d\tau_1
\end{aligned}$$

and

$$\underline{\mathcal{E}}'(\pi_0, \pi^*) = \int_0^\infty \lambda_2 e^{-\lambda_2 \tau_2} \left( \int_0^{\tau_2} \pi_0 e^{\alpha_2 s} e^{-rs} ds + e^{-r\tau_2} \bar{\mathcal{E}}'(\pi_0 e^{\alpha_2 \tau_2}, \pi^*) \right) d\tau_2.$$

Performing steps similar to those followed in the proof of Theorem 4 leads to the desired results. ■

## Appendix B

**Lemma 1** *A single firm's optimal investment policy calls for no investment while the market is declining.*

**Proof of Lemma 1.** Let  $\bar{\pi}_E^*$  denote the threshold that triggers investment by a single firm when the process is above or at such state for the first time given that the market is growing. In addition, let  $\underline{\pi}_E^*$  denote the threshold that triggers investment when the process is above or at such state for the first time given that the market is declining. We claim that  $\underline{\pi}_E^* \geq \bar{\pi}_E^*$ ,<sup>15</sup> so suppose to the contrary that  $\underline{\pi}_E^* < \bar{\pi}_E^*$ , and consider states such that the firm does not invest immediately if the market is in growth, but such that the arrival of the next switching date would trigger immediate investment: formally,  $\pi_0 \in [\underline{\pi}_E^*, \bar{\pi}_E^*)$ . In this case, the dynamics of the value of the investment opportunity while the market is growing, denoted by  $\bar{V}_E^*(\pi_0)$ , are given by the following Bellman equation:

$$\bar{V}_E^*(\pi_0) = \max\{\rho_1 \pi_0 - K, (1 - rdt)[\lambda_1 dt(\rho_2(\pi_0 + \alpha_1 \pi_0 dt) - K) + (1 - \lambda_1 dt)\bar{V}_E^*(\pi_0 + \alpha_1 \pi_0 dt)]\}.$$

On the waiting region, a Taylor expansion and straightforward manipulations ignoring terms of order higher than  $dt$  yield

$$(r + \lambda_1)\bar{V}_E^*(\pi_0) = \alpha_1 \pi_0 \frac{d\bar{V}_E^*(\pi_0)}{d\pi_0} + \lambda_1(\rho_2 \pi_0 - K).$$

The solution to this differential equation is

$$\bar{V}_E^*(\pi_0) = \frac{\lambda_1 \rho_2 \pi_0}{r + \lambda_1 - \alpha_1} - \frac{\lambda_1 K}{r + \lambda_1} + B(\pi_0)^{\frac{r + \lambda_1}{\alpha_1}}, \quad (29)$$

where  $B$  is a constant. In particular, we know that  $\bar{V}_E^*(\bar{\pi}_E^*) = \rho_1 \bar{\pi}_E^* - K$ , whence one can get the value of  $B$  and plug it into expression (29) so as to get

$$\bar{V}_E^*(\pi_0 | \bar{\pi}_E^*) = \frac{\lambda_1 \rho_2 \pi_0}{r + \lambda_1 - \alpha_1} - \frac{\lambda_1 K}{r + \lambda_1} + [\rho_1 \bar{\pi}_E^* - K + \frac{\lambda_1 K}{r + \lambda_1} - \frac{\lambda_1 \rho_2 \bar{\pi}_E^*}{r + \lambda_1 - \alpha_1}] \left( \frac{\pi_0}{\bar{\pi}_E^*} \right)^{\frac{r + \lambda_1}{\alpha_1}}.$$

---

<sup>15</sup>Note that this implies that the firm invests only if the market is growing owing to the continuous sample path of the stochastic process.

Using the fact that

$$(r + \lambda_1 - \alpha_1)\rho_1 = 1 + \lambda_1\rho_2 \quad (30)$$

and maximizing  $\bar{V}_E^*(\pi_0 | \bar{\pi}_E^*)$  with respect to  $\bar{\pi}_E^*$  yields that  $\bar{\pi}_E^* = rK$  (since  $\bar{V}_E^*(\pi_0 | \bar{\pi}_E^*)$  is strictly quasi-concave). As a result, we have that the value of an optimally managed investment opportunity if the market is growing at  $\pi_0 \in [\underline{\pi}_E^*, \bar{\pi}_E^*)$  equals

$$\bar{V}_E^*(\pi_0 | rK) = \frac{\lambda_1\rho_2\pi_0}{r + \lambda_1 - \alpha_1} - \frac{\lambda_1K}{r + \lambda_1} + \frac{\alpha_1rK}{(r + \lambda_1 - \alpha_1)(r + \lambda_1)} \left(\frac{\pi_0}{rK}\right)^{\frac{r+\lambda_1}{\alpha_1}}. \quad (31)$$

Once the value of  $\bar{\pi}_E^*$  has been found out, it simply remains to find out that of  $\underline{\pi}_E^*$ . To this end, let us examine the value of the investment opportunity in a growing market for  $\pi_0 < \underline{\pi}_E^*$ . Let  $\underline{V}_E^*(\pi_0)$  denote the value of the investment opportunity if the market is declining, and note that the dynamics of  $\underline{V}_E^*(\pi_0)$  are given by

$$(r + \lambda_2)\underline{V}_E^*(\pi_0) = \alpha_2\pi_0 \frac{d\underline{V}_E^*(\pi_0)}{d\pi_0} + \lambda_2\bar{V}_E^*(\pi_0), \quad (32)$$

since the firm is in the waiting region during the decline phase. Similarly,  $\bar{V}_E^*(\pi_0)$  satisfies the following differential equation for  $\pi_0 < \underline{\pi}_E^*$ :

$$(r + \lambda_1)\bar{V}_E^*(\pi_0) = \alpha_1\pi_0 \frac{d\bar{V}_E^*(\pi_0)}{d\pi_0} + \lambda_1\underline{V}_E^*(\pi_0), \quad (33)$$

since the firm does not invest right away if the market switches from growth to decline. Solving the system of differential equations comprised by (32) and (33), and using the boundary condition that  $\bar{V}_E^*(0) = 0$  leads to the following solution:

$$\bar{V}_E^*(\pi_0) = R(\pi_0)^{\beta_1}, \quad (34)$$

where  $R$  is a constant to be found out, and  $\beta_1 = \frac{\alpha_1(r + \lambda_2) + \alpha_2(r + \lambda_1) - \sqrt{\Delta}}{2\alpha_1\alpha_2} > 1$ . Evaluating (31) at  $\pi_0 = \underline{\pi}_E^*$ , we have that the boundary condition from which  $R$  can be derived is

$$R(\underline{\pi}_E^*)^{\beta_1} = \frac{\lambda_1\rho_2\underline{\pi}_E^*}{r + \lambda_1 - \alpha_1} - \frac{\lambda_1K}{r + \lambda_1} + \frac{\alpha_1rK}{(r + \lambda_1 - \alpha_1)(r + \lambda_1)} \left(\frac{\underline{\pi}_E^*}{rK}\right)^{\frac{r+\lambda_1}{\alpha_1}},$$

so expression (34) becomes

$$\bar{V}_E^*(\pi_0 | \underline{\pi}_E^*) = \left[ \frac{\lambda_1\rho_2\underline{\pi}_E^*}{r + \lambda_1 - \alpha_1} - \frac{\lambda_1K}{r + \lambda_1} + \frac{\alpha_1rK}{(r + \lambda_1 - \alpha_1)(r + \lambda_1)} \left(\frac{\underline{\pi}_E^*}{rK}\right)^{\frac{r+\lambda_1}{\alpha_1}} \right] \left(\frac{\pi_0}{\underline{\pi}_E^*}\right)^{\beta_1}. \quad (35)$$

Performing some manipulations, we have that the derivative of  $\bar{V}_E^*(\pi_0 | \underline{\pi}_E^*)$  with respect to

$\underline{\pi}_E^*$  is

$$\frac{\partial \bar{V}_E^*(\pi_0 | \underline{\pi}_E^*)}{\partial \underline{\pi}_E^*} = \left[ \frac{\lambda_1 \rho_2 (1 - \beta_1) \underline{\pi}_E^*}{r + \lambda_1 - \alpha_1} + \frac{rK(r + \lambda_1 - \alpha_1 \beta_1)}{(r + \lambda_1 - \alpha_1)(r + \lambda_1)} \left( \frac{\underline{\pi}_E^*}{rK} \right)^{\frac{r+\lambda_1}{\alpha_1}} + \frac{\beta_1 \lambda_1 K}{r + \lambda_1} \right] \frac{(\pi_0)^{\beta_1}}{(\underline{\pi}_E^*)^{\beta_1+1}}.$$

We claim that  $\frac{\partial^2 \bar{V}_E^*(\pi_0 | \underline{\pi}_E^*)}{\partial (\underline{\pi}_E^*)^2} < 0$  whenever  $\frac{\partial \bar{V}_E^*(\pi_0 | \underline{\pi}_E^*)}{\partial \underline{\pi}_E^*} = 0$  holds (i.e., we claim that  $\bar{V}_E^*(\pi_0 | \underline{\pi}_E^*)$  is strictly quasi-concave). For the value of  $\underline{\pi}_E^*$  such that  $\frac{\partial \bar{V}_E^*(\pi_0 | \underline{\pi}_E^*)}{\partial \underline{\pi}_E^*} = 0$  holds, we have that

$$\begin{aligned} \frac{\partial^2 \bar{V}_E^*(\pi_0 | \underline{\pi}_E^*)}{\partial (\underline{\pi}_E^*)^2} &= \left[ \frac{\lambda_1 \rho_2 (1 - \beta_1)}{r + \lambda_1 - \alpha_1} + \frac{rK(r + \lambda_1 - \alpha_1 \beta_1)}{\alpha_1 (r + \lambda_1 - \alpha_1) \underline{\pi}_E^*} \left( \frac{\underline{\pi}_E^*}{rK} \right)^{\frac{r+\lambda_1}{\alpha_1}} \right] \frac{(\pi_0)^{\beta_1}}{(\underline{\pi}_E^*)^{\beta_1+1}} \\ &= \frac{\lambda_1 [(\beta_1 - 1) \rho_2 \underline{\pi}_E^* - \beta_1 K] (\pi_0)^{\beta_1}}{\alpha_1 (\underline{\pi}_E^*)^{\beta_1+2}} \\ &< \frac{\lambda_1 \beta_1 (\pi_0)^{\beta_1}}{\alpha_1 (\underline{\pi}_E^*)^{\beta_1+2}} \left( \frac{\underline{\pi}_E^* - rK}{r} \right) < 0, \end{aligned}$$

where the last equality follows because  $\frac{\partial \bar{V}_E^*(\pi_0 | \underline{\pi}_E^*)}{\partial \underline{\pi}_E^*} = 0$ , the first inequality holds because  $\rho_2 < \rho_1 < \frac{\beta_1}{r(\beta_1 - 1)}$ ,<sup>16</sup> and the last inequality follows since  $\underline{\pi}_E^* < rK$ . Evaluate  $\frac{\partial \bar{V}_E^*(\pi_0 | \underline{\pi}_E^*)}{\partial \underline{\pi}_E^*}$  at  $\underline{\pi}_E^* = rK$  so as to get

$$\begin{aligned} \frac{\partial \bar{V}_E^*(\pi_0 | rK)}{\partial \underline{\pi}_E^*} &= \left[ \frac{\lambda_1 \rho_2 (1 - \beta_1) (r + \lambda_1) rK}{(r + \lambda_1)(r + \lambda_1 - \alpha_1)} + \frac{rK(r + \lambda_1 - \alpha_1 \beta_1)}{(r + \lambda_1 - \alpha_1)(r + \lambda_1)} + \frac{\beta_1 \lambda_1 (r + \lambda_1 - \alpha_1) K}{(r + \lambda_1)(r + \lambda_1 - \alpha_1)} \right] \times \\ &\quad \frac{(\pi_0)^{\beta_1}}{(\underline{\pi}_E^*)^{\beta_1+1}} \\ &= \frac{[(1 - \beta_1)r\rho_1 + \beta_1] K (\pi_0)^{\beta_1}}{(\underline{\pi}_E^*)^{\beta_1+1}} > 0, \end{aligned}$$

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<sup>16</sup>To show that  $\rho_1 < \frac{\beta_1}{r(\beta_1 - 1)}$ , note that

$$(r + \lambda_1 - \alpha_1)(r + \lambda_2 - \alpha_2) - \lambda_1 \lambda_2 = \alpha_1 \alpha_2 (1 - \beta_1)(1 - \beta_2),$$

so some algebraic manipulations yield that  $\rho_1$  can be rewritten as follows:

$$\rho_1 = \frac{(r - \alpha_2)(r + \lambda_1 + \lambda_2 - \alpha_2)}{\alpha_1 \alpha_2 (r - \alpha_2)(1 - \beta_1)(1 - \beta_2)} = \frac{1}{r} \left[ \frac{r(r + \lambda_1 + \lambda_2) - r\alpha_2}{\alpha_1 \alpha_2 (1 - \beta_1)(1 - \beta_2)} \right] = \frac{1}{r} \left[ \frac{\beta_1 \beta_2 - \frac{r}{\alpha_1}}{(1 - \beta_1)(1 - \beta_2)} \right],$$

where the last equality makes use of the fact that  $\alpha_1 \alpha_2 \beta_1 \beta_2 = r(r + \lambda_1 + \lambda_2)$ . As a result, it follows that

$$\frac{(\beta_1 - 1)\rho_1 r - \beta_1}{\rho_1 (\beta_1 - 1)} = \frac{\beta_1 \beta_2 - \frac{r}{\alpha_1} - \beta_1 (\beta_2 - 1)}{\rho_1 (\beta_1 - 1) (\beta_2 - 1)} < \frac{\alpha_1 \beta_1 - r}{\alpha_1 \rho_1 (\beta_1 - 1) (\beta_2 - 1)} < 0,$$

since  $\beta_1 > 1$ ,  $\beta_2 < 0$  and  $\alpha_1 \beta_1 - r = \lambda_1 (1 - \delta_2) > 0$ . Hence, we must have that  $\rho_1 < \frac{\beta_1}{r(\beta_1 - 1)}$ .

since  $\lambda_1 \rho_2 = (\lambda_1 + r - \alpha_1) \rho_1 - 1$  by expression (30) and  $\rho_1 < \frac{\beta_1}{r(\beta_1 - 1)}$  was just shown to always hold. This proves that  $\bar{V}_E^*(\pi_0 | \underline{\pi}_E^*)$  is increasing for  $\underline{\pi}_E^* < rK$ , and (35) implies that the payoff expected by the firm would be bounded above by

$$\begin{aligned} \bar{V}_E^*(\pi_0 | rK) &= \left[ \frac{\lambda_1 \rho_2 rK}{r + \lambda_1 - \alpha_1} - \frac{\lambda_1 K}{r + \lambda_1} + \frac{\alpha_1 rK}{(r + \lambda_1 - \alpha_1)(r + \lambda_1)} \right] \left( \frac{\pi_0}{rK} \right)^{\beta_1} \\ &= (\rho_1 rK - K) \left( \frac{\pi_0}{rK} \right)^{\beta_1}, \end{aligned}$$

which is the (expected) payoff if the firm does not invest during the market decline and it invests the first time the market reaches state  $rK$ , conditional upon the current state being  $\pi_0 < rK$ . As we show below (see expression (5)), the firm's maximal payoff conditional upon investing only if the market is growing is  $\max_{\pi_E} (\rho_1 \pi_E - K) (\pi_0 / \pi_E)^{\beta_1}$ . The fact that

$$\bar{V}_E^*(\pi_0 | rK) = (\rho_1 rK - K) \left( \frac{\pi_0}{rK} \right)^{\beta_1} < \max_{\pi_E} (\rho_1 \pi_E - K) \left( \frac{\pi_0}{\pi_E} \right)^{\beta_1}$$

contradicts the optimality of investing during the market decline, which concludes the proof.

■

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