Barcelona Graduate School in Mathematics, Fall 2013

Random Structures and the Probabilistic Method

Session 14: Random walks on graphs with applications

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1. Random walks on graphs

We call a *simple random walk* on a graph the Markov chain with state space the vertices of a graph and transition probability defined as:

$$P_{x,y} = \begin{cases} \frac{1}{\deg(x)}, \text{ if } (x,y) \in E \\ 0 \text{ otherwise} \end{cases}$$

The chain is irreducible and aperiodic if and only if the graph is connected and non-bipartite. We can check that in this case the stationary distribution is $\pi(v) = \frac{\deg(v)}{2m}$, for every $v \in V$ and that the chain is reversible.

The *cover time* of a graph $C(G)$ is the expected time to reach all vertices (the worst case over the starting vertex). Recall that for two vertices u, v, the hitting time $\mathbf{E}_u(\tau_v)$ is the expected time to get from u to v.

Theorem 1. *For any $(u,v) \in E$, $\mathbf{E}_u(\tau_v) + \mathbf{E}_v(\tau_u) \leq 2m$.*

Theorem 2. *For a connected graph $C(G) \leq 2m(n-1)$.*

Corollary 3. *There is a log-space $O(n^3)$-time randomized algorithm for st-connectivity.*

2. Electric networks

If the graph is interpreted as an electric network, where on each edge there is a resister of 1 ohm, for two vertices u, v the *effective resistence* between these vertices $R(u \leftrightarrow v)$ (i.e. the voltage difference that has to be applied to u and v to get a unit current flow) is related to the *escape probability.*

Theorem 4.

$$\mathbf{P}_u\{\tau_v < \tau_u^+\} = \frac{1}{\deg(u)R(u \leftrightarrow v)}$$

The proof uses the concept of *harmonic* function. A function $h : \Omega \to \mathbb{R}$ is harmonic for P at a vertex $x \in \Omega$ if $h(x) = \sum_{y \in \Omega} P(x,y)h(y)$. For example, notice that the hitting time $h_{x,a}$, from x to a fixed vertex $a \in \Omega$ (or to a set $A \subset \Omega$), considered as a function of x, is a harmonic function on $\Omega \backslash \{a\}$ (or $\Omega \backslash A$). Another example is the *voltage*. Suppose a voltage difference of 1 is applied to vertices s and t, then the resulting voltage on all of the vertices is a harmonic function on $\Omega \backslash \{s,t\}$.

The important property of harmonic functions that we use is that if we fix the values of a function on a set $B \subset \Omega$ it has a unique extension which is harmonic on $\Omega \backslash B$. This is because the harmonicity constraints form a system of equations with $|\Omega| - |B|$ linear equations and $|\Omega| - |B|$ unknowns.

We also get an exact relationship between the effective resistance and the commute time.

Theorem 5. $E_u(\tau_v) + E_v(\tau_u) = 2mR(u \leftrightarrow v)$.

This theorem can be used to give a simpler proof of Theorem 1. It also gives us a tighter bound for the cover time.

Theorem 6. *Let* $R(G) = \max_{u,v \in V} R(u \leftrightarrow v)$. *Then* $mR(G) \leq C(G) \leq mR(G) \times 2e^3 \ln n + n$.

3. Applications of random walks

We will look at local search algorithms for the 2-SAT and 3-SAT problems. The problem is to find a satisfying assignment for a logical formula in conjunctive normal form with clauses of 2 or 3 literals respetively. For example

$$\phi(x_1, \ldots, x_n) = (x_2 \vee \overline{x_5}) \wedge (\overline{x_1} \vee \overline{x_2}) \wedge \ldots$$

Consider the following local search algorithm for 2-SAT. We start with an arbitrary assignment to the variables. If it's a satifying assignment, it is output, otherwise we find a clause which is not satisfied and flip the value of one of the variables in the clause uniformly at random. This is a random walk on the space of assignments to the variables which has size 2^n.

Theorem 7. *The local search algorithm for 2SAT finds a satisfying assignment, if one exists, with probability* $1 - \frac{1}{2^b}$ *within* $n^2 b$ *steps.*

To prove this we just need to consider the evolution of the random variable X indicating the Hamming distance between the current assignment and some fixed satisfying assignment. This turns out to be a walk on the line of length n with probability at least $1/2$ of going to the left every time. This walk reaches 0 in expected time n^2 so within $2n^2$ steps it reaches with probability at least $1/2$ (which can be boosted by going bn^2 steps).

The same analysis applied to 3-SAT leads to a random walk on the line with probability of $1/3$ of going left. This, together with the observation that with high probability the search starts with X close to $n/2$, gives an algorithm running in time $O((\sqrt{3})^n)$, which is better than the trivial algorithm which takes 2^n steps.

We can improve the algorithm by letting the search run only $3n$ steps but restarting the search $\Theta(\left(\frac{4}{3}\right)^n n^{O(1)})$ times.

Theorem 8. *The local search algorithm with restarts finds a satisfying assignment with probability* $1 - \frac{1}{2^b}$ *within time* $O((\frac{4}{3})^n n^{O(1)})$.